Communications in Mathematics **33** (2025), no. 3, Paper no. 12
DOI: https://doi.org/10.46298/cm.15770

Combinatorial twists in $\mathfrak{gl}_n$ Yangians

Anastasia Doikou

Abstract. We introduce the special set-theoretic Yang-Baxter algebra and show that it is a Hopf algebra subject to certain conditions. The associated universal $\mathcal{R}$-matrix is also obtained via an admissible Drinfel'd twist. The structure of braces emerges naturally in this context by requiring the special set-theoretic Yang-Baxter algebra to be a Hopf algebra and a quasi-triangular bialgebra after twisting. The fundamental representation of the universal $\mathcal{R}$-matrix yields the familiar set-theoretic (combinatorial) solutions of the Yang-Baxter equation. We then apply the same Drinfel'd twist to the $\mathfrak{gl}_n$ Yangian after introducing the *augmented Yangian*. We show that the augmented Yangian is also a Hopf algebra and we also obtain its twisted version.

Contents

MSC 2020: 08A99, 20B30, 20F36, 20G42
Keywords: Yang-Baxter equation, set-theoretic solutions, combinatorial Drinfel'd twists, Hopf algebras, Yangians.
Contact information:
Affiliation: Heriot-Watt University and Maxwell Institute for Mathematical Sciences, UK.
Email: a.doikou@hw.ac.uk.

1 Introduction

The main aim of this study is the use of certain universal Drinfel'd twists [11, 12] in the context of $\mathfrak{gl}_n$ Yangians $\mathcal{Y}(\mathfrak{gl}_n)$. We focus on universal twists that are combinatorial matrices in the fundamental representation [4,8] and generate combinatorial (set-theoretic) solutions of the Yang-Baxter equation (see, for instance, [2, 13, 14, 16–18, 23–25]). In this manuscript the solutions of the Yang-Baxter equation are expressed as $n^2 \times n^2$ matrices [9, 10]. In this spirit, when we say combinatorial solutions we mean that the matrices that represent the solutions of the Yang-Baxter equation are combinatorial, i.e. they have only one nonzero element, which takes the value 1, in every row and column. We consider the linearized version of the set-theoretic Yang-Baxter equation and derive the quasi-triangular bialgebras associated to set-theoretic solutions. By identifying a suitable admissible Drinfel'd twist we are able to extract the general set-theoretic universal $\mathcal{R}$-matrix, which is an element of $\mathcal{A} \otimes \mathcal{A}$ and $\mathcal{A}$ is the underlying bialgebra.

More specifically, in Section 2 we introduce the special set-theoretic Yang-Baxter algebra and show that it is a Hopf algebra. The associated universal $\mathcal{R}$-matrix is also obtained via an admissible Drinfel'd twist, making the special set-theoretic Yang-Baxter algebra a quasi-triangular bialgebra. The fundamental representation of the universal $\mathcal{R}$-matrix gives typical set-theoretic (combinatorial) solutions of the Yang-Baxter equation. Here we only obtain reversible universal $\mathcal{R}$ matrices, i.e. $\mathcal{R}_{12}\mathcal{R}_{21} = 1_{\mathcal{A}\otimes\mathcal{A}}$ (and their representations) as opposed to the general scenario discussed in [8], where rack type solutions of the Yang-Baxter equation and their Drinfel'd twists were discussed. The key novel outcomes of Section 2 are summarized in Theorems 2.6, 2.13 and 2.16, where we show that the algebraic structure of (skew) braces (Definition 2.14) [19, 23, 24], emerge naturally if the special set-theoretic Yang-Baxter algebra is required to be a Hopf algebra and a quasi-triangular bialgebra after twisting. In fact, it turns out that the twisted Hopf algebra is a quasi-triangular Hopf algebra (Theorem 2.16). The more general new results of the present investigation are presented in Section 3, where we extend our analysis to the $\mathfrak{gl}_n$ Yangian and to parametric solutions of the Yang-Baxter equation. Specifically, we introduce the augmented $\mathfrak{gl}_n$ Yangian, we show in Theorem 3.5 that it is Hopf algebra and using the set-theoretic Drinfel'd twist we are able to obtain its twisted version. We basically extend the results of [4,6], where only fundamental representations of the augmented Yangian and the twisted R-matrix were presented.

Before we continue with our analysis and the presentation of the main results, we recall the basic definitions of Hopf and quasi-triangular Hopf algebras, which will be used later in our analysis.

We first recall the definition of the Hopf algebra (see, for instance, [3, 21])

Definition 1.1. A Hopf algebra $(\mathcal{A}, \Delta, \epsilon, s)$ is a unital, associative algebra $\mathcal{A}$ over some field k equipped with the following linear maps:

- multiplication, $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $m(a, b) = ab$, which is associative $(ab)c = a(bc)$ for all $a, b, c \in \mathcal{A}$

- $\eta : k \to \mathcal{A}$, such that it produces the unit element for $\mathcal{A}$, $\eta(1) = 1_{\mathcal{A}}$.
- co-product, $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\Delta(a) = \sum_j \alpha_j \otimes \beta_j$, which is coassociative,
$$(\mathrm{id} \otimes \Delta)\Delta(a) = (\Delta \otimes \mathrm{id})\Delta(a), \quad \text{for all } a \in \mathcal{A}.$$
- co-unit, $\epsilon : \mathcal{A} \to k$, such that $(\epsilon \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \epsilon)\Delta(a) = a$, for all $a \in \mathcal{A}$.
- antipode, $s : \mathcal{A} \to \mathcal{A}$, (bijective map) such that
$$m(s \otimes \mathrm{id})\Delta(a) = m(\mathrm{id} \otimes s)\Delta(a) = \epsilon(a)1_{\mathcal{A}}, \quad \text{for all } a \in \mathcal{A}.$$
- Δ, ϵ are algebra homomorphisms and $\mathcal{A} \otimes \mathcal{A}$ has the structure of a tensor product algebra: $(a \otimes b)(c \otimes d) = ac \otimes bd$, for all $a, b, c, d \in \mathcal{A}$.

If we do not require the existence of an antipode then $(\mathcal{A}, \Delta, \epsilon)$ is called a *bialgebra*. We also recall the definition of a quasi-triangular Hopf algebra [11, 12].

Definition 1.2. Let $\mathcal{A}$ be a Hopf algebra over some field k, then $\mathcal{A}$ is a quasi-triangular Hopf algebra if there exists an invertible element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ (universal $\mathcal{R}$-matrix), such that:

1. $\mathcal{R}\Delta(a) = \Delta^{op}(a)\mathcal{R}$, for all $a \in \mathcal{A}$, where $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the co-product on $\mathcal{A}$ and $\Delta^{op}(a) = \pi \circ \Delta(a)$, $\pi : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, such that $\pi(a \otimes b) = b \otimes a$.
2. $(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$, and $(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$.

Also, the following statements hold:

- The antipode $s : \mathcal{A} \to \mathcal{A}$ satisfies $(\mathrm{id} \otimes s)\mathcal{R}^{-1} = \mathcal{R}$, $(s \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{-1}$.
- The co-unit $\epsilon : \mathcal{A} \to k$ satisfies $(\mathrm{id} \otimes \epsilon)\mathcal{R} = (\epsilon \otimes \mathrm{id})\mathcal{R} = 1_{\mathcal{A}}$.
- Due to Definition 1.2 the universal $\mathcal{R}$-matrix satisfies the Yang-Baxter equation
$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{1}$$
We recall the index notation: let $\mathcal{R} = \sum_j a_j \otimes b_j$, then $\mathcal{R}_{12} = \sum_j a_j \otimes b_j \otimes 1_{\mathcal{A}}$, $\mathcal{R}_{23} = \sum_j 1_{\mathcal{A}} \otimes a_j \otimes b_j$ and $\mathcal{R}_{13} = \sum_j a_j \otimes 1_{\mathcal{A}} \otimes b_j$.

Proofs of the above statements can be found, for instance, in [3, 21].

Remark 1.3. Consider a representation $\rho_\lambda : \mathcal{A} \to \mathrm{End}(\mathbb{C}^n)$, $\lambda \in \mathbb{C}$, such that
$$(\rho_\lambda \otimes \mathrm{id})\mathcal{R} =: L(\lambda) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{A},$$
and $(\rho_{\lambda_1} \otimes \rho_{\lambda_2})\mathcal{R} =: R(\lambda_1, \lambda_2) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n)$, $\lambda_{1,2} \in \mathbb{C}$. Then the Yang-Baxter equation (1) reduces to (we suppress the index 3 in the following equation)
$$R_{12}(\lambda_1, \lambda_2)L_1(\lambda_1)L_2(\lambda_2) = L_2(\lambda_2)L_1(\lambda_1)R_{12}(\lambda_1, \lambda_2) \tag{2}$$

after acting with $(\rho_{\lambda_1} \otimes \rho_{\lambda_2} \otimes \mathrm{id})$ on (1). Moreover, (1) turns into the Yang-Baxter equation on $(\mathbb{C}^n)^{\otimes 3}$,

$$R_{12}(\lambda_1, \lambda_2) R_{13}(\lambda_1, \lambda_3) R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3) R_{13}(\lambda_1, \lambda_3) R_{12}(\lambda_1, \lambda_2)$$

after acting with $(\rho_{\lambda_1} \otimes \rho_{\lambda_2} \otimes \rho_{\lambda_3})$. An equation similar to (2) holds for

$$\hat{L}(\lambda) := (\mathrm{id} \otimes \rho_\lambda)\mathcal{R} \in \mathcal{A} \otimes \mathrm{End}(\mathbb{C}^n),$$

and a mixed equation for both L, $\hat{L}$ also follows from (1).

2 Set-theoretic Hopf algebras

In this section, we introduce the *special set-theoretic Yang-Baxter algebra* (or special set-theoretic YB algebra for the sake of brevity) and show that it is a Hopf algebra. Then by introducing a suitable Drinfel'd twist we derive the universal $\mathcal{R}$-matrix associated to the special set-theoretic YB algebra (see a general analysis and definitions in [4, 8], see also [20, 25]). We also show that the special set-theoretic Hopf algebra becomes a quasi-triangular bialgebra after twisting, subject to certain conditions that naturally lead to the structure of (skew) braces.

2.1 Special set-theoretic YB algebra as a Hopf algebra

We first define the basic set-theoretic YB algebra as follows (see also [8]).

Definition 2.1. Let X be a non-empty set, and for all $a, b \in X$, $\sigma_a,\ \tau_b : X \to X$. We say that the unital, associative algebra $\mathcal{A}$ over k, generated by indeterminates $1_\mathcal{A}$ (unit element), h_a, w_a, w_a^{-1}, for $a \in X$, and relations for all $a, b \in X$:

$$h_a h_b = \delta_{a,b} h_a, \quad w_a^{-1} w_a = w_a w_a^{-1} = 1_\mathcal{A}, \quad w_a w_b = w_{\sigma_a(b)} w_{\tau_b(a)} \quad w_a h_b = h_{\sigma_a(b)} w_a, \tag{3}$$

is a *basic set-theoretic YB algebra.*

Proposition 2.2. *Let $\mathcal{A}$ be the basic set-theoretic YB algebra then*

$$h_{\sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c))} = h_{\sigma_a(\sigma_b(c))}.$$

If, in addition, for all $a, b \in X$, $h_a = h_b \Rightarrow a = b$, then for all $a, b, c \in X$,

$$\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c)) \tag{4}$$

and $\sigma_a : X \to X$ is an injection.

Proof. We compute $w_a w_b h_c$ using the associativity of the algebra, relations (3) and the invertibility of w_a, for all $a \in X$ we conclude for all $a, b, c \in X$ (see also [8])

$$h_{\sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c))} = h_{\sigma_a(\sigma_b(c))} \Rightarrow \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c)) = \sigma_a(\sigma_b(c)).$$

If in addition for all $a, b \in X$, $h_a = h_b \Rightarrow a = b$, then for all $a, b, c \in X$, (4) holds.

Also, assume that $\sigma_a(b) = \sigma_a(c)$, then $h_{\sigma_a(b)} w_a = h_{\sigma_a(c)} w_a$ and by the last relation in (3) we obtain $w_a h_b = w_a h_c$, which due to the invertibility of w_a leads to $h_b = h_c$ and hence $b = c$. □

Definition 2.3. The basic set-theoretic YB algebra $\mathcal{A}$ is called a special set-theoretic Yang-Baxter algebra if for all $a \in X$, $\sigma_a : X \to X$ is a bijection.

Notice that the element $c = \sum_{a \in X} h_a$, is a central element of the special set-theoretic YB algebra $\mathcal{A}$. This can be immediately shown by means of the definition of the algebra $\mathcal{A}$. We consider henceforth, without loss of generality, $c = 1_{\mathcal{A}}$.

Remark 2.4. Throughout this manuscript we will be considering the following notation. Let $X = \{x_1, x_2, \dots, x_n\}$ and consider the vector space $V = \mathbb{C}X$ of dimension equal to the cardinality of X. Also $\mathbb{B} = \{e_x\}$, $x \in X$ is the standard canonical basis of the n-dimensional vector space $\mathbb{C}^n$, that is e_{x_j} is the n-dimensional column vector with 1 in the j^{th} row and zeros elsewhere. Let also $\mathbb{B}^* = \{e_x^T\}$, $x \in X$ (^{T} denotes transposition) be the dual basis: $e_x^T e_y = \delta_{x,y}$, also $e_{x,y} := e_x e_y^T$ ($n \times n$ matrices), $x, y \in X$ and they form a basis of $\text{End}(\mathbb{C}^n)$. That is, to each finite set X we associate a vector space of dimension equal to the cardinality of X, so that each element of the set is represented by a vector of the basis and each map within the set is represented as an $n \times n$ matrix.

Remark 2.5 (Fundamental representation of the special set-theoretic YB algebra:). Let $\mathcal{A}$ be the special set-theoretic YB algebra and $\rho : \mathcal{A} \to \text{End}(\mathbb{C}^n)$, such that

$$h_a \mapsto e_{a,a}, \quad w_a \mapsto \sum_{b \in X} e_{\sigma_a(b),b}. \tag{5}$$

Indeed, it can be verified that the above represented elements satisfy the algebraic relations of the special set-theoretic YB algebra (3) if and only if $\sigma_a(\sigma_b(c)) = \sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c))$ for all $a, b, c \in X$.

Theorem 2.6 (Hopf algebra)**.** *Let $\mathcal{A}$ be the special set-theoretic YB algebra and*

$$+ : X \times X \to X, \quad (a, b) \mapsto a + b.$$

If $(X, +, 0)$ is a group and for all $a, b, x \in X$,

$$\sigma_x(a) + \sigma_x(b) = \sigma_x(a + b), \tag{6}$$

then, $(\mathcal{A}, \Delta, \epsilon, s)$ is a Hopf algebra with:

1. ***Co-product:*** $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$,

$$\Delta(w_a^{\pm 1}) = w_a^{\pm 1} \otimes w_a^{\pm 1} \quad \textit{and} \quad \Delta(h_a) = \sum_{b,c\in X} h_b \otimes h_c\big|_{b+c=a}.$$

2. ***Co-unit:*** $\epsilon : \mathcal{A} \to k$, $\epsilon(w_a^{\pm 1}) = 1$ *and* $\epsilon(h_a) = \delta_{a,0}$

3. ***Antipode:*** $s : \mathcal{A} \to \mathcal{A}$, $s(w_a^{\pm 1}) = w_a^{\mp 1}$ *and* $s(h_a) = h_{-a}$, *where* $-a$ *is the inverse of* $a \in X$, *in* $(X, +)$.

The opposite is also true, i.e. if $(\mathcal{A}, \Delta, \epsilon, s)$ *is a Hopf algebra with coproducts given above (part 1 of the Theorem), then* $(X, +, 0)$ *is a group and (6) holds.*

Proof. We first assume that $(X, +, 0)$ is a group and (6) holds. To prove that $(\mathcal{A}, \Delta, \epsilon, s)$ is a Hopf algebra, we show that all the axioms of Definition 1.1 are satisfied (see also [8] for a general proof).

- The coproduct Δ is an algebra homomorphism. Indeed, the coproducts satisfy the algebraic relations (3). Specifically, we show that $\Delta(w_a)\Delta(h_b) = \Delta(h_{\sigma_a(b)})\Delta(w_a)$ for all $a, b \in X$, by using (6).

- The coproducts are coassociative: for all $a \in X$, w_a is a group-like element, so co-associativity obviously holds. For h_a co-associativity holds, due to the associativity of $+$, recall $(X, +)$ is a group.

- It immediately follows for the group-like elements,

$$(\mathrm{id} \otimes \epsilon)\Delta(w_a^{\pm 1}) = (\epsilon \otimes \mathrm{id})\Delta(w_a^{\pm 1}) = w_a^{\pm 1}.$$

  Also, $(\epsilon \otimes \mathrm{id})\Delta(h_a) = \sum_{b,c\in X} \delta_{b,0} h_c\big|_{b+c=a} = h_a$, similarly $(\mathrm{id} \otimes \epsilon)\Delta(h_a) = h_a$, for all $a \in X$.

- For the group-like elements, $m(s \otimes \mathrm{id})\Delta(w_a^{\pm 1}) = m(\mathrm{id} \otimes s)\Delta(w_a^{\pm 1}) = 1_{\mathcal{A}}$. Moreover,

$$m(s \otimes \mathrm{id})\Delta(h_a) = \sum_{b,c\in X} h_{-b} h_c\big|_{b+c=a} = \delta_{a,0} 1_{\mathcal{A}} = \epsilon(h_a) 1_{\mathcal{A}},$$

  where we have used that $h_b h_c = \delta_{b,c} h_b$ and $\sum_{b\in X} h_b = 1_{\mathcal{A}}$. Similarly,

$$m(\mathrm{id} \otimes s)\Delta(h_a) = \epsilon(h_a) 1_{\mathcal{A}}.$$

To prove the opposite, i.e. if $(\mathcal{A}, \Delta, \epsilon, s)$ is a Hopf algebra then $(X, +, 0)$ is a group and (6) holds is straightforward; we follow the logic of the proof above backwards. We also recall that given the coproduct in a Hopf algebra the counit and antipode can be uniquely derived by the axioms of a Hopf algebra (see for instance [21]). □

Remark 2.7. From the proof of Theorem 2.6 it follows: by requiring $(\mathcal{A}, \Delta, \epsilon)$ to be a bialgebra with coproducts given by (2) in Theorem 2.6, we conclude that $(X, +, 0)$ is a monoid and also $\sigma_x(a) + \sigma_x(b) = \sigma_x(a + b)$, for all $a, b, x \in X$. If we further require $(\mathcal{A}, \Delta, \epsilon, s)$ to be a Hopf algebra then $(X, +, 0)$ is a group.

Note that whenever $(X, +)$ forms an abelian group, the corresponding Hopf algebra $(\mathcal{A}, \Delta, \epsilon, s)$ is co-commutative, which means that the opposite coproduct coincides with the coproduct itself, i.e. $\Delta^{(op)} = \Delta$.

2.2 Set-theoretic Drinfel'd twist

In this subsection we introduce the set-theoretic (or combinatorial) Drinfel'd twist (see [4, 6, 8] and a relevant construction in [25]). Using the twist, we will be able to obtain the universal set-theoretic $\mathcal{R}$-matrix associated with the special set-theoretic YB algebra.

Before we introduce the set-theoretic twist, we recall a general statement [11].

Proposition 2.8 (Drinfel'd). *Let $\mathcal{A}$ be a unital, associative algebra, $\mathcal{F}, \mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ be invertible elements, $\mathcal{R}$ satisfy the Yang-Baxter equation (1), and $\mathcal{F}_{1,23}, \mathcal{F}_{12,3} \in \mathcal{A}^{\otimes 3}$ be such that*

1. $\mathcal{F}_{23}\mathcal{F}_{1,23} = \mathcal{F}_{12}\mathcal{F}_{12,3}$, *where recall* $\mathcal{F}_{12} = \mathcal{F} \otimes 1_{\mathcal{A}}$ *and* $\mathcal{F}_{23} = 1_{\mathcal{A}} \otimes \mathcal{F}$.
2. $\mathcal{F}_{1,32}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{F}_{1,23}$ *and* $\mathcal{F}_{21,3}\mathcal{R}_{12} = \mathcal{R}_{12}\mathcal{F}_{12,3}$.

That is, $\mathcal{F}$ is an admissible Drinfel'd twist. Define also $\mathcal{R}^F := \mathcal{F}^{(op)}\mathcal{R}\mathcal{F}^{-1}$, $\mathcal{F}^{(op)} = \pi(\mathcal{F})$ where $\pi : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ is the flip map. Then $\mathcal{R}^F$ also satisfies the Yang-Baxter equation.

Proof. It is convenient to introduce some handy notation that can be used in the following. First, let $\mathcal{F}_{123} := \mathcal{F}_{12}\mathcal{F}_{12,3} = \mathcal{F}_{23}\mathcal{F}_{1,23}$. Let also $i, j, k \in \{1, 2, 3\}$, then $\mathcal{F}_{jik} = \pi_{ij}(\mathcal{F}_{ijk})$ and $\mathcal{F}_{ikj} = \pi_{jk}(\mathcal{F}_{ijk})$, where π is the flip map. This notation describes all possible permutations of the indices 1, 2, 3.

The proof is quite straightforward, [11], we just give a brief outline here. We first prove that $\mathcal{F}_{jik}\mathcal{R}_{ij}\mathcal{F}_{ijk}^{-1} = \mathcal{R}^F_{ij}$, indeed via condition (2) of the proposition the definition of $\mathcal{R}^F$ and the notation introduced above we have

$$\mathcal{F}_{jik}\mathcal{R}_{ij}\mathcal{F}_{ijk}^{-1} = \mathcal{F}_{ji}\mathcal{F}_{ji,k}\mathcal{R}_{ij}\mathcal{F}_{ijk}^{-1} = \mathcal{F}_{ji}\mathcal{R}_{ij}\mathcal{F}_{ij,k}\mathcal{F}_{ijk}^{-1} = \mathcal{R}^F_{ij}\mathcal{F}_{ij}\mathcal{F}_{ij,k}\mathcal{F}_{ijk}^{-1} = \mathcal{R}^F_{ij}. \tag{7}$$

Similarly, it is shown that $\mathcal{F}_{ikj}\mathcal{R}_{jk}\mathcal{F}_{ijk}^{-1} = \mathcal{R}^F_{jk}$.

Then from the YBE we have (see also [4]),

$$\mathcal{F}_{321}\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{F}_{321}\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \;\Rightarrow\; \mathcal{R}^F_{12}\mathcal{R}^F_{13}\mathcal{R}^F_{23}\mathcal{F}_{123} = \mathcal{R}^F_{23}\mathcal{R}^F_{13}\mathcal{R}^F_{12}\mathcal{F}_{123}.$$

But $\mathcal{F}_{123}$ is invertible, hence $\mathcal{R}^F$ indeed satisfies the Yang-Baxter equation. □

Theorem 2.9 (Set-theoretic twist [4, 6, 8]). *Let $\mathcal{A}$ be the special set-theoretic YB algebra and $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$, such that $\mathcal{F} = \sum_{b\in X} h_b \otimes w_b^{-1}$, $\mathcal{R}^F_{ij} := \mathcal{F}_{ji}\mathcal{F}_{ij}^{-1}$, $i, j \in \{1, 2, 3\}$. We also define:*

$$\mathcal{F}_{1,23} := \sum_{a\in X} h_a \otimes w_a^{-1} \otimes w_a^{-1}, \quad \mathcal{F}_{12,3} := \sum_{a,b\in X} h_a \otimes h_{\sigma_a(b)} \otimes w_b^{-1}w_a^{-1}. \tag{8}$$

Let $\sigma_a, \tau_b : X \to X$, be such that $\sigma_{\sigma_a(b)}(\tau_b(a)) = a$ for every $a, b \in X$. Then, the following statements are true:

1. $\mathcal{F}_{12}\mathcal{F}_{12,3} = \mathcal{F}_{23}\mathcal{F}_{1,23} =: \mathcal{F}_{123}$.
2. *For* $i, j, k \in \{1, 2, 3\}$*: (i)* $\mathcal{F}_{ikj}\mathcal{F}^{-1}_{ijk} = \mathcal{R}^F_{jk}$ *and (ii)* $\mathcal{F}_{jik}\mathcal{F}^{-1}_{ijk} = \mathcal{R}^F_{ij}$.

That is, $\mathcal{F}$ is an admissible Drinfel'd twist.

Proof. The proof is straightforward based on the underlying algebra $\mathcal{A}$ (the detailed proof can also be found in [4, 8]).

1. Indeed, this is proven by a direct computation and use of the special set-theoretic YB algebra. In fact, $\mathcal{F}_{123} = \sum_{a,b\in X} h_a \otimes h_b w_a^{-1} \otimes w_b^{-1} w_a^{-1}$.

2. Given the notation introduced before in the proof of Proposition 2.8 it suffices to show that $\mathcal{F}_{132}\mathcal{F}^{-1}_{123} = \mathcal{R}^F_{23}$ and $\mathcal{F}_{213}\mathcal{F}^{-1}_{123} = \mathcal{R}^F_{12}$.

   We first show that $\mathcal{F}_{1,23} = \mathcal{F}_{1,32}$, which is straightforward from the definition in (8); notice that $\mathcal{F}_{1,23} = (\mathrm{id} \otimes \Delta)\mathcal{F}$. Also,

$$\begin{aligned}\mathcal{F}_{12,3} &= \sum_{a,b\in X} h_a \otimes h_{\sigma_a(b)} \otimes (w_a w_b)^{-1} = \sum_{a,b\in X} h_a \otimes h_{\sigma_a(b)} \otimes (w_{\sigma_a(b)} w_{\tau_b(a)})^{-1} \\ &= \sum_{\hat{a},\hat{b}\in X} h_{\sigma_{\hat{a}}(\hat{b})} \otimes h_{\hat{a}} \otimes (w_{\hat{a}} w_{\hat{b}})^{-1} = \mathcal{F}_{21,3},\end{aligned} \tag{9}$$

   where we have set in the equation above $\hat{a} := \sigma_a(b)$ and $\hat{b} := \tau_b(a)$ which leads to $a = \sigma_{\hat{a}}(\hat{b})$ due to $\sigma_{\sigma_a(b)}(\tau_b(a)) = a$.

   It then immediately follows (see also Proposition 2.8):

$$\mathcal{F}_{132}\mathcal{F}^{-1}_{123} = \mathcal{F}_{32}\mathcal{F}_{1,32}\mathcal{F}^{-1}_{1,23}\mathcal{F}^{-1}_{23} = \mathcal{F}_{32}\mathcal{F}^{-1}_{23} = \mathcal{R}^F_{23}.$$

$$\mathcal{F}_{213}\mathcal{F}^{-1}_{123} = \mathcal{F}_{21}\mathcal{F}_{21,3}\mathcal{F}^{-1}_{12,3}\mathcal{F}^{-1}_{12} = \mathcal{F}_{21}\mathcal{F}^{-1}_{12} = \mathcal{R}^F_{12}.$$

Due to Proposition 2.8, we also deduce that $\mathcal{R}^F$ is a solution of the Yang-Baxter equation. ◻

Remark 2.10 (Twisted universal $\mathcal{R}$-matrix)**.** We derive explicit expressions of the twisted universal $\mathcal{R}$-matrix and the twisted coproducts of the algebra. We recall the admissible twist $\mathcal{F} = \sum_{b\in X} h_b \otimes w_b^{-1}$.

- The twisted $\mathcal{R}$-matrix:

$$\mathcal{R}^F = \mathcal{F}^{(op)}\mathcal{F}^{-1} = \sum_{a,b\in X} h_b w_a^{-1} \otimes h_a w_{\sigma_a(b)}.$$

- The twisted coproducts: $\Delta_F(y) = \mathcal{F}\Delta(y)\mathcal{F}^{-1}$, $y \in \mathcal{A}$ and recall from Theorem 2.6, that $(X, +)$ is a group and for all $a \in X$,

$$\Delta(w_a) = w_a \otimes w_a, \quad \Delta(h_a) = \sum_{b,c\in X} h_b \otimes h_c\big|_{b+c=a}.$$

  Then, the twisted coproducts read as:

$$\Delta_F(w_a) = \sum_{b\in X} w_a h_b \otimes w_{\tau_b(a)}, \quad \Delta_F(h_a) = \sum_{b,c\in X} h_b \otimes h_c\big|_{b+\sigma_b(c)=a}, \tag{10}$$

and recall $\tau_b(a) := \sigma^{-1}_{\sigma_a(b)}(a)$, hence $\mathcal{R}^F_{12}\mathcal{R}^F_{21} = 1_{\mathcal{A}\otimes\mathcal{A}}$. It also follows that

$$\mathcal{R}^F\Delta_F(Y) = \Delta_F^{(op)}(Y)\mathcal{R}^F, Y \in \mathcal{A}$$

if $(X, +)$ is an abelian group (see a detailed proof in Theorem 2.16).

Remark 2.11 (Fundamental representation & the set-theoretic solution:). Let $\mathcal{A}$ be the special set-theoretic algebra and $\rho : \mathcal{A} \to \mathrm{End}(\mathbb{C}^n)$, such that

$$h_a \mapsto e_{a,a}, \quad w_a \mapsto \sum_{b\in X} e_{\sigma_a(b),b}. \tag{11}$$

Moreover, $\mathcal{F} \mapsto F := \sum_{a,b\in X} e_{a,a} \otimes e_{b,\sigma_a(b)}$ and $\mathcal{R}^F \mapsto R^F := \sum_{a,b\in X} e_{b,\sigma_a(b)} \otimes e_{a,\tau_b(a)}$, where we recall that for all $a, b, c \in X$, $\sigma_{\sigma_a(b)}(\sigma_{\tau_b(a)}(c)) = \sigma_a(\sigma_b(c))$ (see also Proposition 2.2), $\tau_b(a) := \sigma^{-1}_{\sigma_a(b)}(a)$ and $R^F_{12}R^F_{21} = 1_{n^2}$, where 1_{n^2} is the n^2 dimensional identity matrix. Then, F is a combinatorial twist and R^F is a combinatorial (set-theoretic) solution of the Yang-Baxter equation.

We present below the n-fold twist (see also [4, 5]).

Lemma 2.12 (The n-fold twist). *Let $\mathcal{A}$ be the special set-theoretic YB algebra and let $\mathcal{F} \in \mathcal{A}\otimes\mathcal{A}$ be such that $\mathcal{F} = \sum_{a\in X} h_a \otimes w_a^{-1}$. Define also,*

$$\begin{aligned}
\mathcal{F}_{1,23\ldots n} &:= \sum_{a\in X} h_a \otimes \Delta^{(n-1)}(w_a^{-1}) = \sum_{a\in X} h_a \otimes w_a^{-1} \otimes w_a^{-1} \otimes \ldots \otimes w_a^{-1}, \\
\mathcal{F}_{12\ldots n-1,n} &:= \sum_{a_1,a_2,\ldots,a_{n-1}\in X} h_{a_1} \otimes h_{\sigma_{a_1}(a_2)} \otimes h_{\sigma_{a_1}(\sigma_{a_2}(a_3))} \otimes \ldots \\
&\otimes \; h_{\sigma_{a_1}(\sigma_{a_2}(\ldots\sigma_{a_{n-2}}(a_{n-1}))\ldots)} \otimes w^{-1}_{a_{n-1}} w^{-1}_{a_{n-2}} \ldots w^{-1}_{a_1}.
\end{aligned}$$

Then,

1. $\mathcal{F}_{2\ldots n}\mathcal{F}_{1,2\ldots n} = \mathcal{F}_{12\ldots n-1}\mathcal{F}_{12\ldots n-1,n} =: \mathcal{F}_{12\ldots n}$.

2. *The explicit expression of the n-fold twist is given as*

$$\mathcal{F}_{12\dots n} = \sum_{a_1,a_2,\dots,a_{n-1}\in X} h_{a_1}\otimes h_{a_2}w_{a_1}^{-1}\otimes h_{a_3}w_{a_2}^{-1}w_{a_1}^{-1}\otimes\dots\otimes h_{a_{n-1}}w_{a_{n-2}}^{-1}\dots w_{a_1}^{-1}\otimes w_{a_{n-1}}^{-1}w_{a_{n-2}}^{-1}\dots w_{a_1}^{-1}. \tag{12}$$

3. $\mathcal{F}_{1,23\dots j+1j\dots n} = \mathcal{F}_{1,23\dots jj+1\dots n}$, $n-1\geq j>1$, $\mathcal{F}_{12\dots j+1j\dots n-1,n} = \mathcal{F}_{12\dots jj+1\dots n-1,n}$, $n-1>j\geq 1$, $\mathcal{F}_{12\dots j+1j\dots n} = \mathcal{R}^F_{jj+1}\mathcal{F}_{12\dots jj+1\dots n}$, $n-1\geq j\geq 1$.

Proof. These statements are proven by iteration and direct computation using the $\mathcal{A}$ algebra relations. Part (2) of Theorem 2.9 is also used in proving (3) (see also [4,5]). □

2.3 The twisted Hopf algebra

Motivated by Theorem 2.6 on the conditions that make the special set-theoretic YB algebra a Hopf algebra and by the twisted coproducts (10) in Remark 2.10 we prove the following Theorem (see also [7] for relevant results). Notice in particular the condition for all $a,b,c\in X$, $b+\sigma_b(c)=a$ that appears in $\Delta_F(h_a)=\sum_{b,c\in X}h_b\otimes h_c\big|_{b+\sigma_b(c)=a}$. It is thus natural to introduce a new binary operation, $\circ: X\times X\to X$, such that $a\circ b:=a+\sigma_a(b)$, for all $a,b\in X$.

Theorem 2.13. *Let $\mathcal{A}$ be the special set-theoretic YB algebra, $(X,+,0)$ a group and for all $a,b\in X$, $\sigma_a,\tau_b: X\to X$, such that $\sigma_{\sigma_a(b)}(\tau_b(a))=a$. Let also for all $a,b\in X$, $a\circ b:=a+\sigma_a(b)$.*

1. *Then for all $a,b\in X$, $\sigma_a(b)\circ\tau_b(a)=-a+a\circ b+a$.*
2. *If in addition $(X,\circ)$ is a semigroup and for all $a,b,c\in X$, $\sigma_a(b+c)=\sigma_a(b)+\sigma_a(c)$, then for all $a,b,c\in X$,*
   - *(a) $\sigma_a(\sigma_b(c))=\sigma_{a\circ b}(c)$.*
   - *(b) $(X,\circ,0)$ is a group.*
   - *(c) $a\circ(b+c)=a\circ b-a+a\circ c$.*
   - *(d) $\sigma_a(0)=0$, $\tau_0(a)=a$ and $\sigma_0(a)=a$, $\tau_a(0)=0$.*

Proof.

1. Recall $\sigma_{\sigma_a(b)}(\tau_b(a))=a$, and

$$a\circ b = a+\sigma_a(b) \quad\text{and}\quad \sigma_a(b)\circ\tau_b(a)=\sigma_a(b)+\sigma_{\sigma_a(b)}(\tau_b(a))=\sigma_a(b)+a.$$

   The two equations above lead to $-a+a\circ b+a=\sigma_a(b)\circ\tau_b(a)$.

2. We now assume that $(X,\circ)$ is a semigroup and σ_a is a $(X,+)$ group homomorphism for all $a\in X$.

(a) From associativity in $(X, \circ)$:

$$(a \circ b) \circ c = a \circ b + \sigma_{a \circ b}(c) \qquad \text{and}$$
$$a \circ (b \circ c) = a + \sigma_a(b \circ c) = a + \sigma_a(b + \sigma_b(c)) =$$
$$a + \sigma_a(b) + \sigma_a(\sigma_b(c)) = a \circ b + \sigma_a(\sigma_b(c)).$$

From the two equations above we conclude that $\sigma_a(\sigma_b(c)) = \sigma_{a \circ b}(c)$.

(b) From $\sigma_a(\sigma_b(c)) = \sigma_{a \circ b}(c)$ we obtain for all $a, b, c \in X$

$$-a + a \circ \sigma_b(c) = -a \circ b + a \circ b \circ c \Rightarrow$$
$$a \circ (-b + b \circ c) = a - a \circ b + a \circ b \circ c.$$

That is for all $a, b, c \in X$,

$$a \circ (-b + c) = a - a \circ b + a \circ c. \tag{13}$$

There is a right neutral element. From the distributivity condition above,

$$a \circ (-0 + b) = a \circ b \Rightarrow a - a \circ 0 + a \circ b = a \circ b \Rightarrow a \circ 0 = a.$$

Also, from the bijectivity of σ_a for all $a \in X$:

$$\sigma_a(b) = \sigma_b(c) \Rightarrow b = c$$

which leads to

$$a \circ b = a \circ c \Rightarrow b = c,$$

i.e. left cancellation holds.

There is a unique right inverse in $(X, \circ)$, indeed, $a^{-1} := \sigma_a^{-1}(-a)$ for all $a \in X$, then

$$a \circ a^{-1} = a + \sigma_a(\sigma_a^{-1}(-a)) = a - a = 0,$$

which is also a left inverse. Indeed, from the definition of the inverse, $a^{-1} \in X$, so there is a unique right inverse element for a^{-1} denoted as $(a^{-1})^{-1}$, then

$$a^{-1} \circ (a^{-1})^{-1} = 0 \;\Rightarrow (a^{-1})^{-1} = a \circ 0 = a \;\Rightarrow a^{-1} \circ a = 0.$$

Also,

$$a \circ 0 \circ a^{-1} = 0 \Rightarrow a \circ 0 = 0 \circ a,$$

i.e. 0 is also a left neutral element in $(X, \circ)$. And we conclude that $(X, \circ)$ is a group.

(c) From the distributivity condition (13),

$$a \circ (-b + 0) = a \circ (-b) \Rightarrow a - a \circ b + a = a \circ (-b). \tag{14}$$

Then, from (13), (14):

$$a \circ (b + c) = a - a \circ (-b) + a \circ c = a \circ b - a + a \circ c.$$

(d) These equalities follow from expressions

$$\sigma_a(b) = -a + a \circ b, -a + a \circ b + a = \sigma_a(b) \circ \tau_b(a)$$

and the fact that both $(X, +, 0)$, $(X, \circ, 0)$ are groups. □

Notice that if $(X, +)$ is an abelian group, then $a \circ b = \sigma_a(b) \circ \tau_b(a)$.

Algebraic structures as the one derived in Theorem 2.13, where X is a non-empty set equipped with two group operation $+, \circ$, such that $a \circ (b + c) = a \circ b - a + b \circ c$, for all $a, b, c \in X$ are known as skew *left braces* [19, 23, 24]. If $(X, +)$ is abelian then the structure is called a left brace. Braces were introduced by Rump [2, 23, 24] in the context of finding involutive set-theoretic solutions of the Yang-Baxter equation. The precise definition of (skew) braces is given below.

Definition 2.14. A skew left brace is a set X together with two group operations

$$+, \circ : X \times X \to X.$$

The $+$ operation is called addition and $\circ$ is called multiplication, such that for all $a, b, c \in B$,

$$a \circ (b + c) = a \circ b - a + a \circ c. \tag{15}$$

If $(X, +)$ is an abelian group, then $(X, +, \circ)$ is called a left brace. In this paper, whenever we say (skew) brace, we mean a (skew) left brace. Recall also that for every (skew) brace $0 = 1$, where 0 is the neutral element in $(X, +)$ and 1 is the neutral element in $(X, \circ)$.

Lemma 2.15. *Let $\mathcal{A}$ be the special set-theoretic YB algebra. Let also $(X, +, \circ)$ be a skew brace and for all $a, b \in X$, $\sigma_a, \tau_b : X \to X$, such that*

$$\sigma_a(b) = -a + a \circ b, \qquad \sigma_{\sigma_a(b)}(\tau_b(a)) = a. \tag{16}$$

Then w_0 is a central element in $\mathcal{A}$, where 0 is the neutral element in $(X, +)$ and $(X, \circ)$.

Proof. Note that $\sigma_0(b) = b$, $\tau_b(0) = 0$, then from (3) for all $b \in X$,

$$w_0 w_b = w_{\sigma_0(b)} w_{\tau_b(0)} = w_b w_0 \ \text{ and } \ w_0 h_b = h_{\sigma_0(b)} w_0 = h_b w_0.$$ □

Theorem 2.16. *Let $\mathcal{A}$ be the special set-theoretic YB algebra. If $(X, +, \circ)$ is a brace and for all $a, b \in X$, σ_a, $\tau_b : X \to X$, such that $\sigma_a(b) = -a + a \circ b$ and $\sigma_{\sigma_a(b)}(\tau_b(a)) = a$, then $(\mathcal{A}, \Delta_F, \epsilon, \tilde{s})$ is a Hopf algebra, where the twisted coproducts are given in Remark 2.10, ϵ is given in Theorem 2.6 and $\tilde{s} : \mathcal{A} \to \mathcal{A}$, such that for all $a \in X$,*

$$\tilde{s}(h_a) = h_{a^{-1}}, \quad \tilde{s}(w_a) = \sum_{b \in X} h_b w^{-1}_{\tau_{b^{-1}}(a)}, \tag{17}$$

a^{-1} is the inverse of $a \in X$, in the group $(X, \circ)$. If in addition for all $a, b \in X$, $w_a w_b = w_{a \circ b}$ and $\mathcal{R}^F$ is given in Remark 2.10, then $(\mathcal{A}, \Delta_F, \epsilon, \tilde{s}, \mathcal{R}^F)$ is a quasi-triangular Hopf algebra.

Proof. This is a consequence of Theorem 2.6, Remark 2.10 and Lemma 2.15.

We first prove the coassociativity of the twisted coproducts; indeed, due to the associativity in $(X, \circ)$, for all $a \in X$,

$$(\Delta_F \otimes \mathrm{id})\Delta_F(h_a) = (\mathrm{id} \otimes \Delta_F)\Delta_F(h_a) = \sum_{b,c,d\in X} h_b \otimes h_c \otimes h_d|_{b\circ c\circ d=a}.$$

Also, due to $\tau_b(a) = \sigma^{-1}_{\sigma_a(b)}(a)$ and $\sigma_a(b) = -a + a \circ b$,

$$(\Delta_F \otimes \mathrm{id})\Delta_F(w_a) = (\mathrm{id} \otimes \Delta_F)\Delta_F(w_a) = \sum_{b,c\in X} w_a h_b \otimes w_{\tau_b(a)} h_c \otimes w_{\tau_{b\circ c}(a)}.$$

Moreover, we observe that $(\epsilon \otimes \mathrm{id})\mathcal{F} = w_0$, recall from Lemma 2.15 that w_0 is central in $\mathcal{A}$; also $(\mathrm{id} \otimes \epsilon)\mathcal{F} = 1_{\mathcal{A}}$, which lead to:

$$(\epsilon \otimes \mathrm{id})\Delta_F(x) = (\mathrm{id} \otimes \epsilon)\Delta_F(x) = x, \quad x \in \mathcal{A}.$$

This concludes our proof that $(\mathcal{A}, \Delta_F, \epsilon)$ is a bialgebra. Moreover, from the form of the antipode $\tilde{s}$ (17), we show that

$$m(\tilde{s} \otimes \mathrm{id})\Delta_F(x) = m(\mathrm{id} \otimes \tilde{s})\Delta_F(x) = \epsilon(x)1_{\mathcal{A}}, \quad x \in \mathcal{A}.$$

And this concludes that proof that $(\mathcal{A}, \Delta_F, \epsilon, \tilde{s})$ is a Hopf algebra.

To show that $(\mathcal{A}, \Delta_F, \epsilon, \tilde{s}, \mathcal{R}^F)$ is a quasi-triangular Hopf algebra we also need to show conditions (1) and (2) of Definition 1.2. The Hopf algebra $(\mathcal{A}, \Delta, \epsilon, s)$, is cocommutative due the fact that $(X, +)$ is an abelian group, i.e. for $x \in \mathcal{A}$, $\Delta^{(op)}(x) = \Delta(x)$ and for $\mathcal{F}$ being the admissible twist of Theorem 2.9:

$$\mathcal{F}^{(op)}\Delta^{(op)}(x)(\mathcal{F}^{(op)})^{-1}\mathcal{F}^{(op)}\mathcal{F}^{-1} = \mathcal{F}^{(op)}\mathcal{F}^{-1}\mathcal{F}\Delta(x)\mathcal{F}^{-1} \Rightarrow \Delta_F^{(op)}(x)\mathcal{R}^F = \mathcal{R}^F\Delta_F(x).$$

From the algebraic relations of the special set-theoretic YB algebra and recalling that

$$\mathcal{R}^F = \sum_{a,b\in X} h_b w_a^{-1} \otimes h_a w_{\sigma_a(b)}, a \circ b = \sigma_a(b) \circ \tau_b(a), \sigma_a(b) = -a + a \circ b,$$

and $w_a w_b = w_{a\circ b}$, we deduce

$$\begin{aligned}\mathcal{R}^F_{13}\mathcal{R}^F_{23} &= \sum_{a,b,c\in X} h_b w_a^{-1} \otimes h_c w^{-1}_{\tau_b(a)} \otimes h_a w_{\sigma_a(b)} w_{\sigma_{\tau_b(a)}}(c) \\ &= \sum_{a,b,c\in X} h_b w_a^{-1} \otimes h_c w^{-1}_{\tau_b(a)} \otimes h_a w_{\sigma_a(b\circ c)} = (\Delta_F \otimes \mathrm{id})\mathcal{R}^F.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{R}^F_{13}\mathcal{R}^F_{12} &= \sum_{a,b,\hat{a},\hat{b}\in X} h_b w_a^{-1} h_{\sigma_a(\hat{b})} w_{\hat{a}}^{-1} \otimes h_{\hat{a}} w_{\sigma_{\hat{a}}(\sigma_a(\hat{b}))} \otimes h_a w_{\sigma_a(b)} \\ &= \sum_{a,\hat{a},b\in X} h_b w^{-1}_{\hat{a}\circ a} \otimes h_{\hat{a}} w_{\sigma_{\hat{a}\circ a}(b)} \otimes h_a w_{\sigma_a(b)}. \qquad (18)\end{aligned}$$

Also,

$$
\begin{aligned}
(\mathrm{id} \otimes \Delta_F)\mathcal{R}^F &= \sum_{a,b\in X} h_b w_a^{-1} \otimes \Delta(h_a w_{\sigma_a(b)}) \\
&= \sum_{a_1\circ a_2=a,b,c\in X} h_b w_a^{-1} \otimes h_{a_1} w_{\sigma_a(b)} \otimes h_{a_2} w_{\tau_c(\sigma_a(b))}\big|_{a_1=\sigma_{\sigma_a(b)}(c)}. \qquad (19)
\end{aligned}
$$

From the condition $a_1 = \sigma_{\sigma_a(b)}(c)$ we deduce $c = \sigma^{-1}_{\sigma_a(b)}(a_1) = \sigma_{(\sigma_a(b))^{-1}}(a_1)$, we also recall $a \circ b = \sigma_a(b) \circ \tau_b(a)$, and $\sigma_a(b) = -a + a \circ b$ and $a_1 \circ a_2 = a$ (19), which lead to

$$
\begin{aligned}
\tau_c(\sigma_a(b)) &= (\sigma_{\sigma_a(b)}(c))^{-1} \circ \sigma_a(b) \circ c = a_1^{-1} \circ \sigma_a(b) \circ \sigma_{(\sigma_a(b))^{-1}}(a_1) \\
&= a_1^{-1} \circ (\sigma_a(b) + a_1) = a_1^{-1} \circ \sigma_a(b) - a_1^{-1} = -a_2 + a_2 \circ b = \sigma_{a_2}(b). \qquad (20)
\end{aligned}
$$

From equations (19) and (20) we conclude,

$$
(\mathrm{id} \otimes \Delta_F)\mathcal{R}^F = \sum_{a_1,a_2,b\in X} h_b w^{-1}_{a_1\circ a_2} \otimes h_{a_1} w_{\sigma_{a_1\circ a_2}(b)} \otimes h_{a_2} w_{\sigma_{a_2}(b)}. \qquad (21)
$$

Comparing equation (21) with (18) we arrive at $\mathcal{R}^F_{13}\mathcal{R}^F_{12} = (\mathrm{id} \otimes \Delta_F)\mathcal{R}^F$. And this concludes the second part of our proof that $(\mathcal{A}, \Delta_F, \epsilon, \tilde{s}, \mathcal{R}^F)$ is a quasi-triangular Hopf algebra. □

Remark 2.17. Following the proof of Theorem 2.16 we also conclude:

1. Assuming $(\mathcal{A}, \Delta, \epsilon, s)$ is a Hopf algebra and requiring $(\mathcal{A}, \Delta_F, \epsilon)$ to be a bialgebra with coproducts given in Remark 2.10, we deduce that $(X, \circ)$ is a semigroup. And via Theorem 2.13 we obtain that $(X, +, \circ)$ is a skew brace. Hence, we can define an antipode (17) and $(\mathcal{A}, \Delta_F, \epsilon, \tilde{s})$ is a Hopf algebra.

2. Requiring also $(\mathcal{A}, \Delta_F, \epsilon, \mathcal{R}^F)$ to be a quasi-triangular bialgebra we deduce that $(X, +, \circ)$ is a brace.

Lemma 2.18. *Let $\mathcal{A}$ be the special set-theoretic YB algebra and $\mathcal{F}_{12...n} \in \mathcal{A}^{\otimes n}$ be the n-fold twist (12) and $\mathcal{F}_{12...n-1,n} \in \mathcal{A}^{\otimes n}$ is given in Lemma 2.12. Let also $(X, +, \circ)$ be a brace, $\sigma_a, \tau_b : X \to X$, such that $\sigma_a(b) = -a + a \circ b$, $\sigma_{\sigma_a(b)}(\tau_b(a)) = a$ and $w_a w_b = w_{a\circ b}$, for all $a, b \in X$. Then the following statements are true:*

1. *$\mathcal{F}_{12...n-1,n} = (\Delta^{(n-1)} \otimes \mathrm{id})\mathcal{F}$.*

2. *The n-fold twist is given as*

$$
\mathcal{F}_{12...n} = \sum_{a_1,...,a_n\in X} h_{a_1} \otimes h_{a_2} w^{-1}_{a_1} \otimes \ldots \otimes h_{a_{n-1}} w^{-1}_{a_1\circ a_2\circ...\circ a_{n-2}} \otimes w^{-1}_{a_1\circ a_2\circ...\circ a_{n-1}}.
$$

Proof. 1. Recall the definition of $\mathcal{F}_{1,23}$ (8), then

$$\mathcal{F}_{12,3} = \sum_{a,b\in X} h_a \otimes h_{\sigma_a(b)} \otimes \left(w_{a+\sigma_a(b)}\right)^{-1} = \sum_{c\in X} \Delta(h_c) \otimes w_c^{-1} = (\Delta \otimes \mathrm{id})\mathcal{F}$$

and due to co-associativity $\mathcal{F}_{12\dots n-1,n} = (\Delta^{(n-1)} \otimes \mathrm{id})\mathcal{F}$.

2. This is a consequence of the form of the n-twist (12), relation $w_a w_b = w_{a\circ b}$ for all $a, b \in X$ and the associativity in $(X, \circ)$. □

3 Twisting the $\mathfrak{gl}_n$ Yangian

3.1 Preliminaries: a review on the $\mathfrak{gl}_n$ Yangian

We first recall the derivation of quantum groups (or quantum algebras) associated with any given solution $R : V \otimes V \to V \otimes V$ of the (parametric) Yang-Baxter equation (YBE) [1, 26] (in this manuscript $V = \mathbb{C}^n$)

$$R_{12}(\lambda_1, \lambda_2)R_{13}(\lambda_1, \lambda_3)R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3)R_{13}(\lambda_1, \lambda_3)R_{12}(\lambda_1, \lambda_2), \tag{22}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$. Let $R = \sum_x \mathrm{a}_x \otimes \mathrm{b}_x$, $\mathrm{a}_x, \mathrm{b}_x \in \mathrm{End}(\mathbb{C}^n)$, then in the "index notation":

$$R_{12} = \sum_x \mathrm{a}_x \otimes \mathrm{b}_x \otimes 1_V,\ R_{23} = 1_V \otimes \sum_x \mathrm{a}_x \otimes \mathrm{b}_x,\ \text{ and } R_{13} = \sum_x \mathrm{a}_x \otimes 1_V \otimes \mathrm{b}_x.$$

For the derivation of a quantum algebra associated with a given R-matrix we employ the FRT (Faddeev-Reshetikhin-Takhtajan) construction [15]. We recall the standard $n \times n$ matrices $e_{x,y}$, with entries $(e_{x,y})_{z,w} = \delta_{x,z}\delta_{y,w}$, $x, y, z, w \in X$, and recall $X = \{x_1, x_2, \dots, x_n\}$ (see also Remark 2.4).

Definition 3.1. Let $R(\lambda_1, \lambda_2) \in \mathrm{End}(V \otimes V)$ be a solution of the Yang-Baxter equation (22), $\lambda_1, \lambda_2 \in \mathbb{C}$, $(V = \mathbb{C}^n)$. Let also

$$L(\lambda) := \sum_{x,y\in X} e_{x,y} \otimes L_{x,y}(\lambda) \in \mathrm{End}(V) \otimes \mathfrak{A},$$

where $\lambda \in \mathbb{C}$ and $L_{x,y}(\lambda) = \sum_{m=0}^{\infty} \lambda^{-m} L_{x,y}^{(m)} \in \mathfrak{A}$. The quantum algebra $\mathfrak{A}$, associated to R, is defined as the quotient of the free unital, associative $\mathbb{C}$-algebra, generated by $\left\{L_{x,y}^{(m)} | x, y \in X,\ m \in \{0, 1, 2, \dots\}\right\}$, and relations

$$R_{12}(\lambda_1, \lambda_2)\ L_1(\lambda_1)\ L_2(\lambda_2) = L_2(\lambda_2)\ L_1(\lambda_1)\ R_{12}(\lambda_1, \lambda_2), \tag{23}$$

where $R_{12} = R \otimes 1_{\mathfrak{A}}$ and $L_1 = \sum_{x,y\in X} e_{x,y} \otimes 1_V \otimes L_{x,y}$[1], $L_2 = \sum_{x,y\in X} 1_V \otimes e_{x,y} \otimes L_{x,y}$.

We note that if equation (23) holds, then R is a solution of the Yang-Baxter equation (22) (see, e.g., [21] for a proof; see also relevant Remark 1.3). Definition 22 states that different choices of solutions of the Yang-Baxter equation yield distinct quantum algebras.

[1]Notice that in L in addition to the indices 1 and 2 in (23) there is also an implicit "quantum index" 3 associated to $\mathfrak{A}$, which for now is omitted, i.e. one writes L_{13}, L_{23}.

3.2 The Yangian $\mathcal{Y}(\mathfrak{gl}_n)$

We give a brief review of a special example of a quantum algebra, the $\mathfrak{gl}_n$ Yangian $\mathcal{Y}(\mathfrak{gl}_n)$ (or sometimes $\mathcal{Y}$ in this manuscript for brevity; for a more detailed exposition, the interested reader is referred for instance to [3, 22]). We consider the FRT point of view (Definition 3.1). Specifically, in the case of the Yangian, the R-matrix is given by $R(\lambda_1, \lambda_2) = 1_{V\otimes V} + (\lambda_1 - \lambda_2)^{-1}\mathcal{P}$, where $\mathcal{P} = \sum_{i,j\in X} e_{i,j} \otimes e_{j,i}$ is the permutation operator, such that $\mathcal{P}(a \otimes b) = b \otimes a$, $a, b \in V$ and

$$L(\lambda) = 1 + \sum_{m=1}^{\infty} \lambda^{-m} L^{(m)},\ L^{(m)} = \sum_{x,y\in X} e_{x,y} \otimes L^{(m)}_{x,y}.$$

Then, by the fundamental relation (23) the algebraic relations among the generators $L^{(m)}_{x,y}$ of the $\mathfrak{gl}_n$ Yangian are deduced and are given in the following definition (the interested reader is referred to [22] for a more detailed discussion on Yangians).

Definition 3.2. Let X be some finite set of cardinality $n \in \mathbb{Z}^+$. The $\mathfrak{gl}_n$ Yangian $\mathcal{Y}(\mathfrak{gl}_n)$ (or $\mathcal{Y}$ for brevity) is a unital, associative algebra generated by indeterminates $1_{\mathcal{Y}}$ (unit element) and $L^{(m)}_{i,j}$, $i, j \in X$, $m \in \{0, 1, 2, \dots\}$ ($L^{(0)}_{i,j} = \delta_{i,j} 1_{\mathcal{Y}}$) and relations:

$$\Big[L^{(p+1)}_{i,j},\ L^{(m)}_{k,l}\Big] - \Big[L^{(p)}_{i,j},\ L^{(m+1)}_{k,l}\Big] = L^{(m)}_{k,j} L^{(p)}_{i,l} - L^{(p)}_{k,j} L^{(m)}_{i,l}, \tag{24}$$

where $[\ ,\ ] : \mathcal{Y}(\mathfrak{gl}_n) \times \mathcal{Y}(\mathfrak{gl}_n) \to \mathcal{Y}(\mathfrak{gl}_n)$, such that $[a, b] = ab - ba$, for all $a, b \in \mathcal{Y}$.

Let us focus on the first few explicit exchange relations from (24)

1. $p = 0$, $m = 1$ ($L^{(0)}_{i,j} = \delta_{i,j}$):
$$\big[L^{(1)}_{i,j},\ L^{(1)}_{k,l}\big] = \delta_{i,l} L^{(1)}_{k,j} - \delta_{k,j} L^{(1)}_{i,l}$$
the latter are the familiar $\mathfrak{g}l_n$ exchange relations.

2. $p = 2$, $m = 0$:
$$\big[L^{(2)}_{i,j},\ L^{(1)}_{k,l}\big] = \delta_{i,l} L^{(2)}_{k,j} - \delta_{k,j} L^{(2)}_{i,l}$$

3. $p = 2$, $m = 1$:
$$\big[L^{(3)}_{i,j},\ L^{(1)}_{k,l}\big] - \big[L^{(2)}_{i,j},\ L^{(2)}_{k,l}\big] = L^{(1)}_{k,j} L^{(2)}_{i,l} - L^{(2)}_{k,j} L^{(1)}_{i,l}$$

4. $p = 3$, $m = 0$
$$\big[L^{(3)}_{i,j},\ L^{(1)}_{k,l}\big] = \delta_{i,l} L^{(3)}_{k,j} - \delta_{k,j} L^{(3)}_{i,l}$$

Remark 3.3. The Yangian is a quasi-triangular Hopf algebra on $\mathbb{C}$ [11] equipped with (recall Definitions 1.1, 1.2 and Remark 1.3):

1. A co-product $\Delta : \mathcal{Y}(\mathfrak{gl}_n) \to \mathcal{Y}(\mathfrak{gl}_n) \otimes \mathcal{Y}(\mathfrak{gl}_n)$ such that $(\mathrm{id} \otimes \Delta) L(\lambda) = L_{13}(\lambda) L_{12}(\lambda)$.

2. A counit $\epsilon : \mathcal{Y}(\mathfrak{gl}_n) \to \mathbb{C}$, such that $(\mathrm{id} \otimes \epsilon) L(\lambda) = 1_V$.
3. An antipode $s : \mathcal{Y}(\mathfrak{gl}_n) \to \mathcal{Y}(\mathfrak{gl}_n) : (\mathrm{id} \otimes s) L^{-1}(\lambda) = L(\lambda)$.

We recall that $L(\lambda) = \sum_{m=0}^{\infty} \lambda^{-m} L^{(m)} = \sum_{m=0}^{\infty} \sum_{a,b \in X} \lambda^{-m} e_{a,b} \otimes L^{(m)}_{a,b}$, then the coproducts of the Yangian generators $L^{(n)}_{a,b}$ are given as ($L^{(0)}_{a,b} = \delta_{a,b} 1_{\mathcal{Y}}$)

$$\Delta(L^{(m)}_{a,b}) = \sum_{c \in X} \sum_{k=0}^{m} L^{(k)}_{c,b} \otimes L^{(m-k)}_{a,c} \tag{25}$$

For instance, the first couple of generators of the Yangian are given for $a, b \in X$, as

$$\begin{aligned}
\Delta(L^{(1)}_{a,b}) &= L^{(1)}_{a,b} \otimes 1_{\mathcal{Y}} + 1_{\mathcal{Y}} \otimes L^{(1)}_{a,b} \\
\Delta(L^{(2)}_{a,b}) &= L^{(2)}_{a,b} \otimes 1_{\mathcal{Y}} + 1_{\mathcal{Y}} \otimes L^{(2)}_{a,b} + \sum_{c \in X}^{n} L^{(1)}_{c,b} \otimes L^{(1)}_{a,c}, \\
\Delta(L^{(3)}_{a,b}) &= L^{(3)}_{a,b} \otimes 1_{\mathcal{Y}} + 1_{\mathcal{Y}} \otimes L^{(3)}_{a,b} + \sum_{c \in X} L^{(1)}_{c,b} \otimes L^{(2)}_{a,c} + \sum_{c \in X} L^{(2)}_{c,b} \otimes L^{(1)}_{a,c}, \ \ldots
\end{aligned} \tag{26}$$

The Yangian as a Hopf algebra is co-associative, and the n-coproducts can be derived by iteration via $\Delta^{(n+1)} = (\mathrm{id} \otimes \Delta^{(n)}) \Delta = (\Delta^{(n)} \otimes \mathrm{id}) \Delta$.

Moreover, the counit exists $\epsilon : \mathcal{Y}(\mathfrak{gl}_n) \to \mathbb{C}$, such that

$$(\epsilon \otimes \mathrm{id}) \Delta(L^{(m)}_{a,b}) = (\mathrm{id} \otimes \epsilon) \Delta(L^{(m)}_{a,b}) = L^{(m)}_{a,b},$$

and hence we obtain by iteration that $\epsilon(L^{(m)}_{a,b}) = 0$, for all $a, b \in X$ and $m \in \mathbb{Z}^+$. The antipode $s : \mathcal{Y}(\mathfrak{gl}_n) \to \mathcal{Y}(\mathfrak{gl}_n)$ exists, such that

$$m(s \otimes \mathrm{id}) \Delta(L^{(m)}_{a,b}) = m(\mathrm{id} \otimes s) \Delta(L^{(m)}_{a,b}) = \epsilon(L^{(m)}_{a,b}) 1_{\mathcal{Y}}$$

and recalling that $\epsilon(L^{(m)}_{a,b}) = 0$, we obtain the antipode for each generator via:

$$\sum_{c \in X} \sum_{k=0}^{m} s(L^{(k)}_{c,b}) L^{(m-k)}_{a,c} = \sum_{c \in X} \sum_{k=0}^{m} L^{(k)}_{c,b} s(L^{(m-k)}_{a,c}) = 0. \tag{27}$$

For example, the antipode for the first couple of generators is given as:

$$\begin{aligned}
&s(L^{(1)}_{a,b}) = -L^{(1)}_{a,b} \\
&s(L^{(2)}_{a,b}) = -L^{(2)}_{a,b} + \sum_{c \in X} L^{(1)}_{c,b} L^{(1)}_{a,c}, \\
&s(L^{(3)}_{a,b}) = -L^{(3)}_{a,b} + \sum_{c \in X} L^{(1)}_{c,b} L^{(2)}_{a,c} + \sum_{c \in X} L^{(2)}_{c,b} L^{(1)}_{a,c} - \sum_{c,d \in X} L^{(1)}_{d,b} L^{(1)}_{c,d} L^{(1)}_{a,c}, \ \ldots
\end{aligned} \tag{28}$$

3.3 Twisting the Yangian

Before we present the main findings regarding the twisting of the $\mathfrak{gl}_n$ Yangian we give the definition of the *augmented* $\mathfrak{gl}_n$ *Yangian.*

Definition 3.4. Let X be a finite non-empty set and for all $a \in X$, $\sigma_a, \tau_a : X \to X$, such that σ_a is bijecitve. The augmented $\mathfrak{gl}_n$ Yangian, denoted as $\mathcal{Y}_n^+$, is a unital, associative algebra generated by indeterminates $1_{\mathcal{Y}}$, $L_{a,b}^{(m)}$, $w_a^{\pm 1}$, h_a, $a, b \in X$, $m \in \{0, 1, 2, \dots\}$ ($L_{a,b}^{(0)} = \delta_{a,b} 1_{\mathcal{Y}}$) and relations

$$\begin{aligned}
&\Big[L_{a,b}^{(p+1)},\ L_{c,d}^{(m)}\Big] - \Big[L_{a,b}^{(p)},\ L_{c,d}^{(m+1)}\Big] = L_{c,b}^{(m)} L_{a,d}^{(p)} - L_{c,b}^{(p)} L_{a,d}^{(m)},\\
&h_a h_b = \delta_{a,b} h_a, \quad w_a^{-1} w_a = w_a w_a^{-1} = 1_{\mathcal{Y}}, \quad w_a w_b = w_{\sigma_a(b)} w_{\tau_b(a)} \quad w_a h_b = h_{\sigma_a(b)} w_a,\\
&w_a L_{b,c}^{(p)} = L_{\sigma_a(b),\sigma_a(c)}^{(p)} w_a, \quad h_b L_{a,b}^{(p)} = L_{a,b}^{(p)} h_a, \quad h_c L_{a,b}^{(p)} = L_{a,b}^{(p)} h_c = 0 \ \text{ if } \ c \neq a, b. \qquad (29)
\end{aligned}$$

Theorem 3.5. *Let $\mathcal{Y}_n^+$ be the augmented $\mathfrak{gl}_n$ Yangian and $+ : X \times X \to X$, such that $(a, b) \mapsto a + b$. If $(X, +, 0)$ is a group and for all $a, b, x \in X$,*

$$\sigma_x(a) + \sigma_x(b) = \sigma_x(a + b), \qquad (30)$$

then $\mathcal{Y}_n^+$ is a Hopf algebra, with co-product $\Delta : \mathcal{Y}_n^+ \to \mathcal{Y}_n^+ \otimes \mathcal{Y}_n^+$, such that

$$\Delta(w_a^{\pm 1}) = w_a^{\pm 1} \otimes w_a^{\pm 1},\ \Delta(h_a) = \sum_{b,c \in X} h_b \otimes h_c\big|_{b+c=a}$$

and

$$\Delta(L_{a,b}^{(m)}) = \sum_{k=1}^{m} \sum_{c \in X} L_{c,b}^{(k)} \otimes L_{a,c}^{(m-k)}$$

for all $a, b \in X$ and $m \in \mathbb{Z}^+$.

Proof. The proof is based on the fact that $\mathcal{Y}(\mathfrak{gl}_n)$ is a Hopf algebra (see Remark 3.3) and on Theorem 2.6. Co-associativity holds (Theorem 2.6) and it is straightforward to show that $\Delta : \mathcal{Y}_n^+ \to \mathcal{Y}_n^+ \otimes \mathcal{Y}_n^+$ is an algebra homomorphism. The counits and antipodes of the algebra generators are uniquely defined from the basic axioms of the Hopf algebra (see also Theorem 2.6 and Remark 3.3). ∎

Proposition 3.6. *Consider the representation $\rho : \mathcal{Y}_n^+ \to End(\mathbb{C}^n)$, such that for all $a, b \in X$, $m \in \mathbb{Z}^+$*

$$\rho(L_{a,b}^{(m)}) = e_{b,a}, \quad \rho(w_a) = \sum_{c \in X} e_{\sigma_a(c),c}, \quad \rho(h_a) = e_{a,a}. \qquad (31)$$

Let also the Yangian R-matrix, $R(\lambda) \in End(\mathbb{C}^n \otimes \mathbb{C}^n)$, $R(\lambda) = 1 + \frac{1}{\lambda}\mathcal{P}$, where $\lambda \in \mathbb{C}$, $\mathcal{P} = \sum_{a,b \in X} e_{a,b} \otimes e_{b,a}$ is the permutation operator and $L(\lambda) = 1 + \sum_{m=1}^{\infty} \lambda^{-m} L^{(m)}$, where $L^{(m)} = \sum_{a,b \in X} e_{a,b} \otimes L_{a,b}^{(m)}$, $L_{a,b}^{(m)} \in \mathcal{Y}(\mathfrak{gl}_n)$. Then,

1. *For all* $f \in \mathcal{Y}_n^+$,

$$\big((\rho\otimes\rho)\Delta^{(op)}(f)\big)R(\lambda) = R(\lambda)\big((\rho\otimes\rho)\Delta(f)\big),$$

$$\big((\rho\otimes \mathrm{id})\Delta^{(op)}(f)\big)L(\lambda) = L(\lambda)\big((\rho\otimes \mathrm{id})\Delta(f)\big).$$

2. *Let also* $\mathcal{F} := \sum_{a\in X} h_a \otimes w_a$ *and* $\mathcal{F}_{123}, \mathcal{F}_{12,3}, \mathcal{F}_{1,23}$ *are defined in Theorem 2.9. Moreover,* $F := (\rho\otimes\rho)\mathcal{F}$, $\mathrm{F} := (\rho\otimes\mathrm{id})\mathcal{F}$, $R^F(\lambda) = F^{(op)}R(\lambda)F^{-1}$, $L^F(\lambda) = \mathrm{F}^{(op)}L(\lambda)\mathrm{F}^{-1}$, *(*$\mathrm{F}^{(op)} = (\rho\otimes\mathrm{id})\mathcal{F}^{(op)}$*) and* $\mathrm{F}_{123} := (\rho\otimes\rho\otimes\mathrm{id})\mathcal{F}_{123}$, *then*

$$R_{12}^F(\lambda_1-\lambda_2)L_1^F(\lambda_1)L_2^F(\lambda_2) = L_2^F(\lambda_2)L_1^F(\lambda_1)R_{12}^F(\lambda_1-\lambda_2). \tag{32}$$

Proof. 1. The proof is based on the algebraic relations of $\mathcal{Y}_n^+$ and the expressions of the co-products of the algebra generators are given in Theorem 3.5.

2. In the proof of the second part we use part one of the proposition as well as Theorem 2.9. Specifically, we start from equation (23) for the Yangian and as in the proof of Theorem 2.9 and using the fundamental representation (31):

$$\mathrm{F}_{321}R_{12}(\lambda_1-\lambda_2)L_{13}(\lambda_1)L_{23}(\lambda_2) = \mathrm{F}_{321}L_{23}(\lambda_2)L_{13}(\lambda_1)R_{12}(\lambda_1-\lambda_2) \;\Rightarrow$$
$$R_{12}^F(\lambda_1-\lambda_2)L_{13}^F(\lambda_1)L_{23}^F(\lambda_2)\mathrm{F}_{123} = L_{23}^F(\lambda_2)L_{13}^F(\lambda_1)R_{12}^F(\lambda_1-\lambda_2)\mathrm{F}_{123},$$

which leads to (32), due to that fact that F_{123} is invertible. □

Remark 3.7. According to Proposition 3.6 $R^F = r + \frac{1}{\lambda}\mathcal{P}$, where $\mathcal{P} = \sum_{a,b\in X} e_{a,b}\otimes e_{b,a}$ and $r = \sum_{a,b\in X} e_{b,\sigma_a(b)}\otimes e_{a,\tau_b(a)}$. Moreover, $L^F(\lambda) = L'^{(0)} + \sum_{m=1}^{\infty}\lambda^{-m}L'^{(m)}$, where

$$L'^{(0)} = \sum_{a,b\in X} e_{b,\sigma_a(b)}\otimes h_a w_{\sigma_a(b)}, \qquad L'^{(m)} = \sum_{a,b,c\in X} e_{a,b}\otimes h_c L^{(m)}_{\sigma_c(a),b} w_b.$$

Moreover, the twisted coproducts for the augmented $\mathfrak{gl}_n$ Yangian are $\Delta_F(x) = \mathcal{F}\Delta(x)\mathcal{F}^{-1}$, $x\in\mathcal{Y}_n^+$. Recall first the twisted coproducts of the special set-theoretic YB algebra, for $a\in X$, $(X,+)$ is a group,

$$\Delta_F(w_a) = \sum_{b\in X} w_a h_b \otimes w_{\tau_b(a)},\ \Delta_F(h_a) = \sum_{b\in X} h_b\otimes h_c\big|_{b+\sigma_b(c)=a}$$

(recall Remark 2.10). Also, for $a,b\in X$,

$$\Delta_F(L_{a,b}^{(m)}) = \sum_{k=1}^{m}\sum_{c\in X} L_{c,b}^{(k)}h_c\otimes w_b^{-1}L_{a,c}^{(m-k)}w_c.$$

Acknowledgments

Support from the EPSRC research grant EP/V008129/1 is acknowledged.

References

[1] R. J. Baxter. *Exactly solved models in statistical mechanics.* London, Academic Press, paperback ed. edition, 1989.

[2] F. Cedó, E. Jespers, and J. Okniński. Braces and the Yang-Baxter equation. *Commun. Math. Phys.*, 327(1):101–116, 2014.

[3] V. Chari and A. Pressley. *A guide to quantum groups.* Cambridge University Press, 1995.

[4] A. Doikou. Set-theoretic Yang-Baxter equation, braces and Drinfeld twists. *J. Phys. A, Math. Theor.*, 54(41):21, 2021.

[5] A. Doikou. Parametric set-theoretic Yang-Baxter equation: p-racks, solutions & quantum algebras. Preprint, arXiv:2405.04088 [math-ph] (2025), 2025.

[6] A. Doikou, A. Ghionis, and B. Vlaar. Quasi-bialgebras from set-theoretic type solutions of the Yang-Baxter equation. *Lett. Math. Phys.*, 112(4):29, 2022.

[7] A. Doikou and B. Rybołowicz. Near braces and p-deformed braided groups. *Bull. Lond. Math. Soc.*, 56(1):124–139, 2024.

[8] A. Doikou, B. Rybołowicz, and P. Stefanelli. Quandles as pre-Lie skew braces, set-theoretic Hopf algebras & universal $\mathcal{R}$-matrices. *J. Phys. A, Math. Theor.*, 57(40):35, 2024. Id/No 405203.

[9] A. Doikou and A. Smoktunowicz. Set-theoretic Yang-Baxter & reflection equations and quantum group symmetries. *Lett. Math. Phys.*, 111(4):40, 2021.

[10] A. Doikou and A. Smoktunowicz. From braces to Hecke algebras and quantum groups. *J. Algebra Appl.*, 22(8):28, 2023.

[11] V. G. Drinfel'd. Hopf algebras and the quantum Yang-Baxter equation. *Sov. Math., Dokl.*, 32:256–258, 1985.

[12] V. G. Drinfel'd. Quasi-hopf algebras. *Leningr. Math. J.*, 1(6):1419–1457, 1990.

[13] V. G. Drinfel'd. On some unsolved problems in quantum group theory. In *Quantum groups. Proceedings of workshops, held in the Euler International Mathematical Institute, Leningrad, USSR, Fall 1990*, pages 1–8. Berlin etc.: Springer-Verlag, 1992.

[14] P. Etingof, T. Schedler, and A. Soloviev. Set-theoretical solutions to the quantum Yang-Baxter equation. *Duke Math. J.*, 100(2):169–209, 1999.

[15] L. D. Faddeev, N. Y. Reshetikhin, and L. A. Takhtadzhyan. Quantization of Lie groups and Lie algebras. *Leningr. Math. J.*, 1(1):193–225, 1990.

[16] T. Gateva-Ivanova. Quadratic algebras, Yang-Baxter equation, and Artin-Schelter regularity. *Adv. Math.*, 230(4-6):2152–2175, 2012.

[17] T. Gateva-Ivanova. Set-theoretic solutions of the Yang-Baxter equation, braces and symmetric groups. *Adv. Math.*, 338:649–701, 2018.

[18] T. Gateva-Ivanova and S. Majid. Matched pairs approach to set theoretic solutions of the Yang-Baxter equation. *J. Algebra*, 319(4):1462–1529, 2008.

[19] L. Guarnieri and L. Vendramin. Skew braces and the Yang-Baxter equation. *Math. Comput.*, 86(307):2519–2534, 2017.

[20] V. Lebed and L. Vendramin. Reflection equation as a tool for studying solutions to the Yang-Baxter equation. *J. Algebra*, 607:360–380, 2022.

[21] S. Majid. *Foundations of quantum group theory.* Cambridge: Cambridge Univ. Press, 1995.

[22] A. Molev, M. Nazarov, and G. Ol'shanskij. Yangians and classical Lie algebras. *Russ. Math. Surv.*, 51(2):205–282, 1996.

[23] W. Rump. A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation. *Adv. Math.*, 193(1):40–55, 2005.

[24] W. Rump. Braces, radical rings, and the quantum Yang-Baxter equation. *J. Algebra*, 307(1):153–170, 2007.

[25] A. Soloviev. Non-unitary set-theoretical solutions to the quantum Yang-Baxter equation. *Math. Res. Lett.*, 7(5-6):577–596, 2000.

[26] C. N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.*, 19:1312–1315, 1967.

Received: May 29, 2025
Accepted for publication: September 8, 2025
Communicated by: Ivan Kaygorodov and David Towers