

# Transposed $\delta$ -Poisson algebra structures on null-filiform associative algebras

*Nigora Daukeyeva, Maqpal Eraliyeva and Feruza Toshtemirova*

**Abstract.** In this paper, we consider transposed  $\delta$ -Poisson algebras, which are a generalization of transposed Poisson algebras. In particular, we classify all transposed  $\delta$ -Poisson structures on null-filiform associative algebras. A complete classification of transposed  $\delta$ -Poisson algebras corresponding to each value of the parameter  $\delta$  is provided. Furthermore, we construct all  $\delta$ -Poisson algebra structures on null-filiform associative algebras, and show that they are trivial  $\delta$ -Poisson algebras.

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*Contact information:*

Nigora Daukeyeva:

*Affiliation:* Chirchik State Pedagogical University, Chirchik, Uzbekistan.

*Email:* [daukeyeva.cspu@gmail.com](mailto:daukeyeva.cspu@gmail.com)

Maqpal Eraliyeva:

*Affiliation:* Chirchik State Pedagogical University, Chirchik, Uzbekistan.

*Email:* [eraliyevamaqpal@gmail.com](mailto:eraliyevamaqpal@gmail.com)

Feruza Toshtemirova:

*Affiliation:* Chirchik State Pedagogical University, Chirchik, Uzbekistan.

*Email:* [feruzaisrailova45@gmail.com](mailto:feruzaisrailova45@gmail.com)

# 1 Introduction

Poisson algebras were first studied in connection with Poisson geometry in the 1970s. Since that time, they have appeared in a wide range of mathematical and physical disciplines, including Poisson manifolds, algebraic geometry, operads, quantization theory, quantum groups, and both classical and quantum mechanics. More recently, the notion of transposed Poisson algebras was introduced in [7], providing a dual perspective on Poisson structures and leading to novel algebraic frameworks. This concept has found applications in various algebraic structures, including Novikov–Poisson algebras and 3-Lie algebras [9]. Further developments include Poisson structures on canonical algebras and finitary incidence algebras of arbitrary posets over commutative unital rings [15, 18, 28, 32], as well as modified double Poisson brackets and mixed double Poisson algebras [11].

The study of  $\delta$ -versions of Poisson algebras has also been advanced. In [2], two new types of  $\delta$ -Poisson and transposed  $\delta$ -Poisson algebras were studied. The  $\delta$ -Poisson algebras emerged as a generalization of both Poisson and anti-Poisson algebras, and are closely related to  $\delta$ -derivations introduced by Filippov [14] (see also [16, 34]). It was shown in that paper that transposed  $\delta$ -Poisson algebras share many similarities with those studied in [7]. However, unlike transposed Poisson algebras, transposed anti-Poisson algebras possess simple algebras in the complex finite-dimensional case. Furthermore, it was proven that the tensor product of two  $\delta$ -Poisson (respectively, transposed  $\delta$ -Poisson) algebras is again a  $\delta$ -Poisson (respectively, transposed  $\delta$ -Poisson) algebra.

A related line of research concerns  $\delta$ -Novikov and  $\delta$ -Novikov–Poisson algebras. In [17], their structure and properties were investigated as a generalization of Novikov and Novikov–Poisson algebras characterized by a scalar parameter  $\delta$ . From these results, a crucial difference between Novikov and anti-Novikov algebras was observed: unlike Novikov algebras, anti-Novikov algebras admit complex non-commutative simple finite-dimensional algebra. The article also provided constructions of  $\delta$ -Novikov algebras, proving that the Kantor product of two multiplications of a  $\delta$ -Novikov–Poisson algebra yields a  $\delta$ -Novikov algebra, and that the tensor product of two  $\delta$ -Novikov–Poisson algebras again admits a  $\delta$ -Novikov–Poisson structure under the standard multiplication. Moreover, relations between  $\delta$ -derivations, (transposed)  $\delta$ -Poisson algebras, and  $\delta$ -Novikov–Poisson algebras were established. In particular, a  $\delta$ -Novikov–Poisson algebra under the commutator product gives rise to a transposed  $(\delta + 1)$ -Poisson algebra, and the notion of  $\delta$ -Gelfand–Dorfman algebras was introduced, with commutative  $\delta$ -Gelfand–Dorfman algebras yielding transposed  $(\delta + 1)$ -Poisson algebras.

Research on transposed Poisson structures themselves has gained significant traction in recent years. A comprehensive algebraic and geometric classification was given in [8], expanding the understanding of their structural properties. Various works have explored transposed Poisson structures on different classes of Lie algebras: Block Lie algebras and superalgebras, and Witt-type algebras [19, 20, 21], as well as quasi-filiform Lie algebras of maximum length [4]. Further investigations considered upper triangular matrix Lie algebras [23], incidence algebras [22], and the connection between transposed Poisson algebras and  $\frac{1}{2}$ -derivations of Lie algebras [13]. Transposed Poisson structures on not-finitely graded

Witt-type and Virasoro-type algebras were also studied in [25, 26], emphasizing their relevance in theoretical physics and representation theory. Descriptions of all transposed Poisson structures on certain Lie algebras were obtained in [3, 24, 27, 29, 30, 31, 33]. More recent contributions include connections to Jordan superalgebras [12], and applications to classification problems in non-associative algebras [16].

Finally, regarding null-filiform associative algebras, all Poisson algebra structures were constructed in [1], and a classification of all transposed Poisson algebra structures on these algebras was given in [5]. This paper continues this direction by providing a complete classification of all transposed  $\delta$ -Poisson algebra structures on null-filiform associative algebras. In Section 2, we introduce the necessary definitions and results that form the basis of our study, and in Section 3, we describe all such structures.

Using the Theorems 3.10 and 3.24, we derive the classification of 4-dimensional complex transposed  $\delta$ -Poisson algebra associated on null-filiform associative algebra  $\mu_0^4$ .

**Main Theorem.** *Let  $(\mu_0^4, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra. Then, this algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned} & \mathbf{TP}_0(1, 0, 0, 0), \quad \mathbf{TP}_1(0, 1, \alpha, 0), \quad \mathbf{TP}_1(0, 1, 0, \alpha), \\ & \mathbf{TP}_\delta(0, 1, 0, 0), \quad \mathbf{TP}_\delta(0, 0, \alpha, 0), \quad \mathbf{TP}_\delta(0, 0, 0, 1), \quad \mathbf{TP}_{-4}(0, 1, 0, \alpha), \quad \mathbf{TP}_{\frac{3}{2}}(0, 0, \alpha, 1) \quad \alpha \in \mathbb{C}. \end{aligned}$$

## 2 Preliminaries

In this section, we introduce the relevant concepts and known results. Unless stated otherwise, all algebras considered here are over the field  $\mathbb{C}$ . All parameters, including  $\delta$ , are complex numbers as well.

**Definition 2.1** ([7]). Let  $\mathfrak{L}$  be a vector space equipped with two bilinear operations

$$\cdot, [-, -] : \mathfrak{L} \otimes \mathfrak{L} \rightarrow \mathfrak{L},$$

where  $(\mathfrak{L}, \cdot)$  is a commutative associative algebra and  $(\mathfrak{L}, [-, -])$  is a Lie algebra.

The triple  $(\mathfrak{L}, \cdot, [-, -])$  is called a  **$\delta$ -Poisson algebra** if

$$[x, y \cdot z] = \delta([x, y] \cdot z + y \cdot [x, z]), \quad \text{for all } x, y, z \in \mathfrak{L}. \quad (1)$$

The triple  $(\mathfrak{L}, \cdot, [-, -])$  is called a **transposed  $\delta$ -Poisson algebra** if:

$$\delta z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y]. \quad (2)$$

If we take  $\delta = 1$  and  $\delta = 2$ , respectively, in identities (1) and (2), then we obtain the definitions of the Poisson and the transposed Poisson algebras, respectively.

A (transposed)  $\delta$ -Poisson algebra  $\mathfrak{L}$  is called *trivial*, if  $\mathfrak{L} \cdot \mathfrak{L} = 0$  or  $[\mathfrak{L}, \mathfrak{L}] = 0$ .

Similar to the results in [7], the following proposition shows that the compatibility relations of the  $\delta$ -Poisson algebra and the transposed  $\delta$ -Poisson algebra are independent.

**Proposition 2.2.** *Let  $(\mathfrak{L}, \cdot)$  be a commutative associative algebra and  $(\mathfrak{L}, [-, -])$  be a Lie algebra. Then for any  $\delta \neq 0$ ,  $(\mathfrak{L}, \cdot, [-, -])$  is both a  $\delta$ -Poisson algebra and a transposed  $\delta$ -Poisson algebra if and only if*

$$x \cdot [y, z] = [x \cdot y, z] = 0.$$

*Proof.* Let  $(\mathfrak{L}, \cdot, [-, -])$  be a  $\delta$ -Poisson and transposed  $\delta$ -Poisson algebra, then according to [2], it satisfies the following identities, respectively:

$$[x, y \cdot z] + [y, z \cdot x] + [z, x \cdot y] = 0, \quad (3)$$

$$x \cdot [y, z] + y \cdot [z, x] + z \cdot [x, y] = 0 \quad (4)$$

for all  $x, y, z \in \mathfrak{L}$

It is easy to see that

$$\begin{aligned} 0 &= \delta z \cdot [x, y] + [y, x \cdot z] - [x, y \cdot z] \stackrel{(1)}{=} \\ &\delta \left( z \cdot [x, y] + [y, x] \cdot z + x \cdot [y, z] - [x, y] \cdot z - y \cdot [x, z] \right) = \\ &\delta \left( x \cdot [y, z] + y \cdot [z, x] - z \cdot [x, y] \right). \end{aligned}$$

Then by Eq. (4), we have  $z \cdot [x, y] = 0$ . By Eq. (1) again, we have  $[x, y \cdot z] = 0$ .  $\square$

**Remark 2.3.** If we take  $\delta = 0$  in Proposition 2, the condition  $[x \cdot y, z] = 0$  is necessary and sufficient for  $(\mathfrak{L}, \cdot, [-, -])$  to be both a 0-Poisson and a transposed 0-Poisson algebra.

For an algebra  $\mathcal{A}$ , we consider the series

$$\mathcal{A}^1 = \mathcal{A}, \quad \mathcal{A}^{i+1} = \sum_{k=1}^i \mathcal{A}^k \mathcal{A}^{i+1-k}, \quad i \geq 1.$$

We say that an algebra  $\mathcal{A}$  is *nilpotent* if  $\mathcal{A}^i = 0$  for some  $i \in \mathbb{N}$ . The smallest integer  $i$  satisfying  $\mathcal{A}^i = 0$  is called the *nilpotency index* of  $\mathcal{A}$ .

**Definition 2.4.** An  $n$ -dimensional algebra  $\mathcal{A}$  is called null-filiform if  $\dim \mathcal{A}^i = (n+1) - i$ ,  $1 \leq i \leq n+1$ .

All null-filiform associative algebras were classified in [10, Proposition 5.3].

**Theorem 2.5** (see [10]). *An arbitrary  $n$ -dimensional null-filiform associative algebra is isomorphic to the algebra:*

$$\mu_0^n: \quad e_i \cdot e_j = e_{i+j}, \quad 2 \leq i+j \leq n,$$

where  $\{e_1, e_2, \dots, e_n\}$  is a basis of the algebra  $\mu_0^n$  and all other products are defined to be zero.

**Theorem 2.6** ([6]). *A linear map  $\varphi : \mu_0^n \rightarrow \mu_0^n$  is an automorphism of the algebra  $\mu_0^n$  if and only if the map  $\varphi$  has the following form:*

$$\varphi(e_1) = \sum_{i=1}^n A_i e_i, \quad \varphi(e_i) = \sum_{j=i}^n \sum_{k_1+k_2+\dots+k_i=j} A_{k_1} A_{k_2} \dots A_{k_i} e_j, \quad 2 \leq i \leq n,$$

where  $A_i \in \mathbb{C}$ , and  $A_1 \neq 0$ .

## 2.1 Transposed Poisson algebra structures on null-filiform associative algebras

For the case  $\delta = 2$ , all transposed Poisson algebra structures on null-filiform associative algebras were completely classified in [5], where the following three theorems were obtained.

**Theorem 2.7.** *Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed Poisson algebra defined on the associative algebra  $(\mu_0^n, \cdot)$ . Then the multiplication of  $(\mu_0^n, \cdot, [-, -])$  has the following form:*

$$\mathbf{TP}_2(\alpha_2, \dots, \alpha_n) : \begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_i, e_j] = (j-i) \sum_{t=i+j-1}^n \alpha_{t-i-j+3} e_t, & 3 \leq i+j \leq n+1. \end{cases}$$

The following theorem establishes a necessary and sufficient condition for two algebras in the family  $\mathbf{TP}_2(\alpha_2, \dots, \alpha_n)$  to be isomorphic.

**Theorem 2.8.** *Let  $\mathbf{TP}_2(\alpha_2, \dots, \alpha_n)$  and  $\mathbf{TP}'_2(\alpha'_2, \dots, \alpha'_n)$  be isomorphic algebras. Then there exists an automorphism  $\varphi$  between these algebras such that the following relation holds for  $2 \leq t \leq n$ :*

$$\sum_{i=2}^t \sum_{k_1+\dots+k_i=t} A_{k_1} \dots A_{k_i} \alpha'_i = \sum_{j=2}^t \sum_{i=1}^{t-j+1} \sum_{k_1+k_2=t-i-j+3} (t-2i-j+3) A_i A_{k_1} A_{k_2} \alpha_j. \quad (5)$$

**Theorem 2.9.** *Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed Poisson algebra and  $n \geq 5$ . Then this algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathbf{TP}_2(1, 0, \dots, 0), \quad \mathbf{TP}_2(0, \alpha, 0, \dots, 0),$$

$$\mathbf{TP}_2(0, \dots, 0, 1_s, 0, \dots, 0, \alpha_{2s-3}, 0, \dots, 0), \quad 4 \leq s \leq n, \quad \alpha \in \mathbb{C}.$$

The cases for dimensions 2, 3, and 4 are presented below. In the 2-dimensional associative algebra with multiplication given by  $e_1 \cdot e_1 = e_2$ , [7] proved that every 2-dimensional complex transposed Poisson algebra is isomorphic to one of the following transposed Poisson algebras:

$$\mathbf{TP}_2(0) : e_1 \cdot e_1 = e_2; \quad \mathbf{TP}_2(1) : e_1 \cdot e_1 = e_2, \quad [e_1, e_2] = e_2.$$

In the 3-dimensional associative algebra, every 3-dimensional complex transposed Poisson algebra is isomorphic to one of the following transposed Poisson algebras:

$$\mathbf{TP}_2(1, 0), \mathbf{TP}_2(0, \alpha), \alpha \in \mathbb{C}.$$

Finally, in the 4-dimensional associative algebra, every 4-dimensional complex transposed Poisson algebra is isomorphic to one of the following transposed Poisson algebras:

$$\mathbf{TP}_2(1, 0, 0), \mathbf{TP}_2(0, \alpha, 0), \mathbf{TP}_2(0, 0, 1), \alpha \in \mathbb{C}.$$

### 3 Transposed $\delta$ -Poisson algebra structures on null-filiform associative algebras

Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra. To determine the Lie multiplication table of the transposed  $\delta$ -Poisson algebra structure, we set

$$[e_1, e_2] = \sum_{t=1}^n \alpha_t e_t.$$

**Theorem 3.1.** *Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra. Then, for  $n \geq 5$ , the following restriction holds:*

$$(\delta^3 - 3\delta^2 + 2\delta)e_3 \cdot [e_1, e_2] = 0.$$

*Proof.* By considering the identity (2) for triples  $\{e_1, e_1, e_i\}$  and  $\{e_{i-1}, e_1, e_2\}$ :

$$\begin{aligned} \delta e_1 \cdot [e_1, e_i] &= [e_2, e_i] + [e_1, e_{i+1}], \\ \delta e_{i-1} \cdot [e_1, e_2] &= [e_i, e_2] + [e_1, e_{i+1}], \end{aligned}$$

we derive the following recurrence relation

$$[e_1, e_{i+1}] = \frac{\delta}{2} (e_1 \cdot [e_1, e_i] + e_{i-1} \cdot [e_1, e_2]).$$

From this, we derive the following products

$$[e_1, e_3] = \delta e_1 \cdot [e_1, e_2], [e_1, e_4] = \frac{\delta^2 + \delta}{2} e_2 \cdot [e_1, e_2], [e_1, e_5] = \frac{\delta^3 + \delta^2 + 2\delta}{4} e_3 \cdot [e_1, e_2]. \quad (6)$$

Now we consider Eq. (2) for the triple  $\{e_2, e_1, e_3\}$ :

$$\delta e_2 \cdot [e_1, e_3] = [e_3, e_3] + [e_1, e_5].$$

From this we obtain

$$[e_1, e_5] = \delta^2 e_3 \cdot [e_1, e_2]. \quad (7)$$

From the Eqs. (6) and (7) we get the following

$$(\delta^3 - 3\delta^2 + 2\delta)e_3 \cdot [e_1, e_2] = 0.$$

Thus, the theorem is proved.  $\square$

**Corollary 3.2.** *Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra and  $n \geq 5$ . Then*

- *If  $\delta = 0$ , then  $[e_1, e_2] = \sum_{t=1}^n \alpha_t e_t$  and  $[e_1, e_i] = 0$  for  $3 \leq i \leq n$ . Note that this result holds when  $n = 3$  and  $n = 4$  as well;*
- *If  $\delta = 1$ , then  $[e_1, e_i] = \sum_{t=i-1}^n \alpha_{t-i+2} e_t$  for  $2 \leq i \leq n$ ;*
- *If  $\delta = 2$ , then  $[e_1, e_i] = (i-1) \sum_{t=i-1}^n \alpha_{t-i+2} e_t$  for  $2 \leq i \leq n$ ;*
- *If  $\delta^3 - 3\delta^2 + 2\delta \neq 0$ , then  $\alpha_t = 0$  for  $1 \leq t \leq n-3$  and*

$$\begin{aligned} [e_1, e_2] &= \alpha_{n-2} e_{n-2} + \alpha_{n-1} e_{n-1} + \alpha_n e_n, & [e_1, e_4] &= \frac{\delta^2 + \delta}{2} \alpha_{n-2} e_n, \\ [e_1, e_3] &= \delta(\alpha_{n-2} e_{n-1} + \alpha_{n-1} e_n), & [e_1, e_i] &= 0, \end{aligned} \quad 5 \leq i \leq n.$$

### 3.1 Transposed 0-Poisson algebra structures on null-filiform associative algebras

**Theorem 3.3.** *Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed 0-Poisson algebra. Then the multiplication of  $(\mu_0^n, \cdot, [-, -])$  has the following form:*

$$\mathbf{TP}_0(\alpha_1, \dots, \alpha_n) : \begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_1, e_2] = \sum_{t=1}^n \alpha_t e_t. \end{cases}$$

*Proof.* Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed 0-Poisson algebra. We consider the identity (2) for the triple  $\{e_{i-1}, e_1, e_2\}$ :

$$0 = [e_{i-1} \cdot e_1, e_2] + [e_1, e_{i-1} \cdot e_2],$$

or

$$[e_i, e_2] + [e_1, e_{i+1}] = 0. \quad (8)$$

From Eq. (8) and Corollary 3.2, we deduce  $[e_i, e_2] = 0$  for  $2 \leq i \leq n$ .

Applying induction and the identity (2) for  $3 < i+j$ , we establish:

$$[e_i, e_j] = 0. \quad (9)$$

We can write

$$0 = [e_i \cdot e_1, e_j] + [e_1, e_i \cdot e_j] = [e_{i+1}, e_j] + [e_1, e_{i+j}] = [e_{i+1}, e_j].$$

It is known that the Jacobi identity is observed for 3 different elements  $\{x, y, z\}$ . The identity is satisfied if any 2 of these elements are equal. In our case, the elements  $e_3, \dots, e_n$  lie in the center of the Lie algebra. Therefore, the Jacobi identity is satisfied for all elements. Hence, we obtain the transposed 0-Poisson algebras  $\mathbf{TP}_0(\alpha_1, \dots, \alpha_n)$ .  $\square$

The following theorem establishes a necessary and sufficient condition for two algebras in the family  $\mathbf{TP}_0(\alpha_1, \dots, \alpha_n)$  to be isomorphic.

**Theorem 3.4.** *Let  $\mathbf{TP}_0(\alpha_1, \dots, \alpha_n)$  and  $\mathbf{TP}'_0(\alpha'_1, \dots, \alpha'_n)$  be isomorphic algebras. Then there exists an automorphism  $\varphi$  between these algebras such that the following relation holds for  $1 \leq t \leq n$ :*

$$\sum_{i=1}^t \sum_{k_1+\dots+k_i=t} A_{k_1} \dots A_{k_i} \alpha'_i = A_1^3 \alpha_t. \quad (10)$$

*Proof.* Using the automorphism of the algebra  $\mu_0^n$  from Theorem 2.6, we introduce the following notations:

$$e'_i = \varphi(e_i), \quad 1 \leq i \leq n.$$

Thus, we consider

$$\begin{aligned} [e'_1, e'_2] &= \sum_{i=1}^n \alpha'_i e'_i = \sum_{i=1}^n \alpha'_i \sum_{j=i}^n \sum_{k_1+\dots+k_i=j} A_{k_1} \dots A_{k_i} e_j \\ &= \sum_{i=1}^n \sum_{j=i}^n \sum_{k_1+\dots+k_i=j} \alpha'_i A_{k_1} \dots A_{k_i} e_j = \sum_{t=1}^n \sum_{i=1}^t \sum_{k_1+\dots+k_i=t} \alpha'_i A_{k_1} \dots A_{k_i} e_t. \end{aligned}$$

On the other hand, we have

$$[e'_1, e'_2] = \left[ \sum_{i=1}^n A_i e_i, \sum_{j=2}^n \sum_{k_1+k_2=j} A_{k_1} A_{k_2} e_j \right] = \sum_{k_1+k_2=2}^n A_1 A_{k_1} A_{k_2} \sum_{t=1}^n \alpha_t e_t = \sum_{t=1}^n A_1^3 \alpha_t e_t.$$

Comparing the coefficients of the obtained expressions for the basis elements for  $1 \leq t \leq n$ , we get the following restrictions:

$$\sum_{i=1}^t \sum_{k_1+\dots+k_i=t} A_{k_1} \dots A_{k_i} \alpha'_i = A_1^3 \alpha_t.$$

□

**Lemma 3.5.** *Let  $\mathbf{TP}_0(0, \dots, 0, \alpha_s, \dots, \alpha_n)$ , with  $\alpha_s \neq 0$ , be a transposed Poisson algebra defined above. Then, there exists  $A \in \mathbb{C}$  such that the relation  $\alpha'_s = \alpha_s A^{3-s}$  holds for any  $1 \leq s \leq n$ .*

*Proof.* Let  $\mathbf{TP}_0(0, \dots, 0, \alpha_s, \dots, \alpha_n)$  be a transposed 0-Poisson algebra and consider a general change of basis. Then, we have the following restriction (10):

$$\sum_{i=1}^s \sum_{k_1+\dots+k_i=s} A_{k_1} \dots A_{k_i} \alpha'_i = A_1^3 \alpha_s.$$

Now, considering that  $\alpha_i = 0$ ,  $\alpha'_i = 0$  for  $1 \leq i < s - 1$ , this reduces to

$$\sum_{k_1 + \dots + k_i = s} A_{k_1} \dots A_{k_i} \alpha'_s = A_1^3 \alpha_s,$$

which simplifies further to  $A_1^s \alpha'_s = A_1^3 \alpha_s$ . From this equality, we obtain  $\alpha'_s = \alpha_s A_1^{3-s}$ .  $\square$

**Lemma 3.6.** *If  $\alpha_1 \neq 0$ , then  $\mathbf{TP}_0(\alpha_1, \dots, \alpha_n)$  is isomorphic to the algebra  $\mathbf{TP}_0(1, 0, \dots, 0)$ .*

*Proof.* We begin by outlining the strategy of the proof.

**Step 1.** Consider the algebra  $\mathbf{TP}_0(\alpha_1, \alpha_2, \dots, \alpha_n)$ . If  $\alpha_1 \neq 0$ , we perform the basis change  $e'_1 = A e_1$ . Since the multiplication is given by  $e_i \cdot e_j = e_{i+j}$ , it reduces the algebra to the form

$$\mathbf{TP}_0(1, \alpha'_2, \dots, \alpha'_n).$$

**Step 2.** Next, we apply the basis change  $e'_1 = e_1 + B e_2$ . With respect to the product  $e_i \cdot e_j$ , this eliminates the coefficient  $\alpha_2$  in the relation

$$[e_1, e_2] = e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n,$$

yielding the algebra

$$\mathbf{TP}_0(1, 0, \alpha'_3, \dots, \alpha'_n).$$

By continuing this procedure, we can analyze the subsequent coefficients in the same manner.

Let  $\alpha_1 \neq 0$  and consider the following change of basis:  $e'_i = \frac{1}{\sqrt{\alpha_1^i}} e_i$ ,  $1 \leq i \leq n$ , then we have  $\alpha'_1 = 1$ .

Next, consider another change of basis  $e'_1 = e_1 + \alpha_2 e_2$ . From the relations  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i + j \leq n$ , we obtain

$$e'_i = \sum_{t=0}^i \binom{i}{t} \alpha_2^t e_{i+t}, \quad 1 \leq i \leq n,$$

We conclude that  $\alpha'_2 = 0$ .

Now, we prove by induction that it is possible to set  $\alpha_j = 0$  for all  $2 \leq j \leq n$ . The base case  $j = 2$  is already established. Assuming the claim holds for some  $j$ , we show it also holds for  $j + 1$ . Consider the change of basis:

$$e'_1 = e_1 + \alpha_{j+1} e_{j+1}.$$

Using  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i + j \leq n$ , we derive

$$e'_i = \sum_{t=0}^i \binom{i}{t} \alpha_{j+1}^t e_{i+tj}, \quad 1 \leq i \leq n,$$

where we assume that  $e_t = 0$  for  $t > n$  in this sum. We obtain that  $\alpha_{j+1} = 0$ .

By induction, we have  $\alpha_j = 0$  for all  $2 \leq j \leq n$ . Hence, we get the algebra  $\mathbf{TP}_0(1, 0, \dots, 0)$ , with the following multiplication rules:

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_1, e_2] = e_1. \end{cases}$$

□

The strategy outlined in this Lemma will be used in several lemmas below.

**Lemma 3.7.** *If  $\alpha_2 \neq 0$ , then  $\mathbf{TP}_0(0, \alpha_2, \dots, \alpha_n) \cong \mathbf{TP}_0(0, 1, 0, \dots, 0)$ .*

*Proof.* Let  $\alpha_2 \neq 0$  and consider the following change of basis:  $e'_i = \alpha_2^{-i} e_i$ ,  $1 \leq i \leq n$ . Then, we have  $\alpha'_2 = 1$ .

Next, consider the following change of basis:

$$e'_1 = e_1 + \frac{\alpha_3}{2} e_2.$$

From the products  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we obtain

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left(\frac{\alpha_3}{2}\right)^t e_{i+t}, \quad 1 \leq i \leq n.$$

Here we assume that  $e_t = 0$  for  $t > n$ . A direct computation shows that  $\alpha'_3 = 0$ .

Now, we prove by induction that we can eliminate  $\alpha_j$  for all  $3 \leq j \leq n$ . The base case  $j = 3$  follows from the above step. Assuming that  $\alpha_j = 0$  holds for some  $j \geq 3$ , we prove it for  $j + 1$ . Consider the change of basis:

$$e'_1 = e_1 + \frac{\alpha_{j+1}}{2} e_j.$$

From the products  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we derive

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left(\frac{\alpha_{j+1}}{2}\right)^t e_{i+t(j-1)}, \quad 1 \leq i \leq n.$$

Similarly, we assume that  $e_t = 0$  for  $t > n$ . One can verify that  $\alpha_{j+1} = 0$ .

As a result, we have shown that  $\alpha_j = 0$  for  $3 \leq j \leq n$ , and obtained the algebra  $\mathbf{TP}_0(0, 1, 0, \dots, 0)$ :

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_1, e_2] = e_2. \end{cases}$$

□

**Lemma 3.8.** *If  $\alpha_3 \neq 0$ , then  $\mathbf{TP}_0(0, 0, \alpha_3, \dots, \alpha_n) \cong \mathbf{TP}_0(0, 0, \alpha, 0, 0, \dots, 0)$ .*

*Proof.* From the Eq. (10), we obtain  $\alpha'_3 = \alpha_3$ .

Now, we define a new basis via:  $e'_1 = e_1 + \frac{\alpha_4}{3\alpha_3}e_2$ . From the products  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we derive

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left( \frac{\alpha_4}{3\alpha_3} \right)^t e_{i+t}, \quad 1 \leq i \leq n.$$

Here we assume that  $e_t = 0$  for  $t > n$ . Then, a direct computation shows that  $\alpha'_4 = 0$ .

Now we prove by induction that it is possible to set  $\alpha_j = 0$  for  $4 \leq j \leq n$ . The base case  $j = 4$  holds by the above argument. Now, assuming it holds for some  $j$ , we prove it for  $j+1$ . Consider basis change:  $e'_1 = e_1 + \frac{\alpha_{j+1}}{3\alpha_3}e_{j-1}$ . From the products  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we derive

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left( \frac{\alpha_{j+1}}{3\alpha_3} \right)^t e_{i+t(j-2)}, \quad 1 \leq i \leq n,$$

where we assume that  $e_t = 0$  for  $t > n$  in this sum. We obtain that  $\alpha_{j+1} = 0$ .

As a result, we have shown that  $\alpha_j = 0$  for  $4 \leq j \leq n$ , and obtained the algebra  $\mathbf{TP}_0(0, 0, \alpha, 0, \dots, 0)$ :

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_1, e_2] = \alpha e_3 \end{cases}$$

□

**Lemma 3.9.** *If  $\alpha_s \neq 0$ ,  $s \geq 4$  then  $\mathbf{TP}_0(0, \dots, 0, \alpha_s, \dots, \alpha_n) \cong \mathbf{TP}_0(0, \dots, 0, 1_s, 0, \dots, 0)$ .*

*Proof.* Let  $\alpha_s \neq 0$  and consider the change of basis as follows  $e'_i = \alpha_s^{\frac{i}{s-3}} e_i$ ,  $1 \leq i \leq n$ . Then, we have  $\alpha'_s = 1$ .

Now consider the change of basis as follows:  $e'_1 = e_1 + \frac{\alpha_{s+1}}{s}e_2$ . From the products  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we derive

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left( \frac{\alpha_{s+1}}{s} \right)^t e_{i+t}, \quad 1 \leq i \leq n.$$

Here we assume that  $e_t = 0$  for  $t > n$ . A direct computation shows that  $\alpha_{s+1} = 0$ .

We proceed by induction on  $j$  to show that a sequence of basis transformations can set  $\alpha_j = 0$  for all  $s+1 \leq j \leq n$ . If  $j = i$ , the relation holds according to the above equalities. Now, assuming that it holds for some  $j$ , we prove it for  $j+1$ . We consider the change of the basis element  $e_1$  as  $e'_1 = e_1 + \frac{\alpha_{j+1}}{s}e_{j-s+2}$ . From the products  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we derive

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left( \frac{\alpha_{j+1}}{s} \right)^t e_{i+t(j-s+1)}, \quad 1 \leq i \leq n.$$

Here we assume that  $e_t = 0$  for  $t > n$ . Thus, one can verify that  $\alpha'_{j+1} = 0$ .

Thus, we have shown that  $\alpha_j = 0$  for  $s + 1 \leq j \leq n$ , and obtained the algebra  $\mathbf{TP}_0(0, \dots, 0, 1_s, 0, \dots, 0)$ :

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i + j \leq n, \\ [e_1, e_2] = e_s. \end{cases}$$

□

**Theorem 3.10.** *Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed 0-Poisson algebra and  $n \geq 4$ . Then, this algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathbf{TP}_0(1, 0, \dots, 0), \quad \mathbf{TP}_0(0, 1, 0, \dots, 0),$$

$$\mathbf{TP}(0, 0, \alpha, 0, \dots, 0), \quad \mathbf{TP}_0(0, \dots, 0, 1_s, 0, \dots, 0), \quad 4 \leq s \leq n, \quad \alpha \in \mathbb{C}.$$

*Proof.* Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed Poisson algebra. Then Theorem 3.3 implies that it is isomorphic to the algebra  $\mathbf{TP}_0(\alpha_1, \dots, \alpha_n)$ . Moreover,

- If  $\alpha_1 \neq 0$ , then by Lemma 3.6, we obtain the algebra  $\mathbf{TP}_0(1, 0, \dots, 0)$ .
- If  $\alpha_1 = 0$ ,  $\alpha_2 \neq 0$ , then from Lemma 3.7, we get the algebra  $\mathbf{TP}_0(0, 1, 0, \dots, 0)$ , where  $\alpha \neq 0$ ;
- If  $\alpha_1 = \alpha_2 = 0$ ,  $\alpha_3 \neq 0$ , then it follows from Lemma 3.8 that the resulting algebra is  $\mathbf{TP}_0(0, 0, \alpha, 0, \dots, 0)$ ;
- If  $\alpha_1 = \dots = \alpha_{s-1} = 0$ ,  $\alpha_s \neq 0$ ,  $s \geq 4$ , then from Lemma 3.9, we have the algebra

$$\mathbf{TP}_0(0, \dots, 0, 1_s, 0, \dots, 0);$$

- If  $\alpha_i = 0$ ,  $1 \leq i \leq n$ , then we have the algebra  $\mathbf{TP}_0(0, \dots, 0)$ .

According to Lemma 3.5, for the first non-zero parameter  $\alpha_s$ , the relation  $\alpha_s \neq 0$  is an invariant of the algebra  $\mathbf{TP}_0(\alpha_1, \dots, \alpha_n)$ . Therefore, an algebra in which  $\alpha_s = 0$  cannot be isomorphic to one in which  $\alpha_s \neq 0$ .

□

### 3.2 Transposed 1-Poisson algebra structures on null-filiform associative algebras

Now we consider the transposed 1-Poisson algebra structures on null-filiform associative algebras.

**Theorem 3.11.** *Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed 1-Poisson algebra. Then, the multiplication of  $(\mu_0^n, \cdot, [-, -])$  has the following form:*

$$\mathbf{TP}_1(\alpha_2, \dots, \alpha_n) : \begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i + j \leq n, \\ [e_1, e_i] = \sum_{t=i}^n \alpha_{t-i+2} e_t, & 2 \leq i \leq n. \end{cases}$$

*Proof.* Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed 1-Poisson algebra. We consider the identity (2) for the triple  $\{e_{i-1}, e_1, e_2\}$ ,  $2 \leq i \leq n+1$ :

$$e_{i-1} \cdot [e_1, e_2] = [e_{i-1} \cdot e_1, e_2] + [e_1, e_{i-1} \cdot e_2],$$

or

$$[e_i, e_2] + [e_1, e_{i+1}] = \sum_{t=i}^n \alpha_{t-i+1} e_t. \quad (11)$$

From Corollary (3.2), we deduce  $[e_i, e_2] = 0$  for  $2 \leq i \leq n$ .

Applying induction and the identity (2) for  $2 \leq i, j \leq n$ , we establish:

$$[e_i, e_j] = 0. \quad (12)$$

We will prove this equality by induction on the value of  $j$ . For  $j = 2$ , we have  $[e_i, e_2] = 0$  for all  $i$  with  $2 \leq i \leq n$ . Assuming that  $[e_i, e_j] = 0$  holds for  $2 \leq i \leq n$ , we will prove the equality for  $j+1$ . We can write

$$0 = e_1 \cdot [e_i, e_j] = [e_1 \cdot e_i, e_j] + [e_i, e_1 \cdot e_j] = [e_{i+1}, e_j] + [e_i, e_{j+1}] = [e_i, e_{j+1}].$$

Thus,  $[e_i, e_j] = 0$  for  $2 \leq j \leq n$ . Thus we have proven the Eq. (12).

Next, applying the Jacobi identity to  $\{e_1, e_2, e_n\}$ , we get:

$$\begin{aligned} 0 &= [[e_1, e_2], e_n] + [[e_2, e_n], e_1] + [[e_n, e_1], e_2] = \\ &= \left[ \sum_{t=1}^n \alpha_t e_t, e_n \right] - \left[ \sum_{t=n-1}^n \alpha_{t-n+2} e_t, e_2 \right] = \alpha_1 (\alpha_1 e_{n-1} + \alpha_2 e_n). \end{aligned}$$

From the last equality, we obtain the relation  $\alpha_1 = 0$ .

Hence, we obtain the transposed 1-Poisson algebras  $\mathbf{TP}_1(\alpha_2, \dots, \alpha_n)$  given by the following multiplications:

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_1, e_i] = \sum_{t=i}^n \alpha_{t-i+2} e_t, & 2 \leq i \leq n. \end{cases}$$

□

The following theorem establishes a necessary and sufficient condition for two algebras in the family  $\mathbf{TP}_1(\alpha_2, \dots, \alpha_n)$  to be isomorphic.

**Theorem 3.12.** *Let  $\mathbf{TP}_1(\alpha_2, \dots, \alpha_n)$  and  $\mathbf{TP}'_1(\alpha'_2, \dots, \alpha'_n)$  be isomorphic algebras. Then there exists an automorphism  $\varphi$  between these algebras such that the following relation holds for  $2 \leq t \leq n$ :*

$$\sum_{i=2}^t \sum_{k_1+\dots+k_i=t} A_{k_1} \dots A_{k_i} \alpha'_i = \sum_{j=2}^t \sum_{k_1+k_2=t-j+2} A_1 A_{k_1} A_{k_2} \alpha_j. \quad (13)$$

*Proof.* Using the automorphism of the algebra  $\mu_0^n$  from Theorem 2.6, we introduce the following notations:

$$e'_i = \varphi(e_i), \quad 1 \leq i \leq n.$$

Thus, we consider

$$\begin{aligned} [e'_1, e'_2] &= \sum_{i=2}^n \alpha'_i e'_i = \sum_{i=2}^n \alpha'_i \sum_{j=i}^n \sum_{k_1+\dots+k_i=j} A_{k_1} \dots A_{k_i} e_j \\ &= \sum_{i=2}^n \sum_{j=i}^n \sum_{k_1+\dots+k_i=j} \alpha'_i A_{k_1} \dots A_{k_i} e_j = \sum_{t=2}^n \sum_{i=2}^t \sum_{k_1+\dots+k_i=t} \alpha'_i A_{k_1} \dots A_{k_i} e_t. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [e'_1, e'_2] &= \left[ \sum_{i=1}^n A_i e_i, \sum_{j=2}^n \sum_{k_1+k_2=j} A_{k_1} A_{k_2} e_j \right] = [A_1 e_1, \sum_{j=2}^n \sum_{k_1+k_2=j} A_{k_1} A_{k_2} e_j] \\ &= \sum_{j=2}^n \sum_{k_1+k_2=j} A_1 A_{k_1} A_{k_2} [e_1, e_j] \\ &= \sum_{j=2}^n \sum_{k_1+k_2=j} A_1 A_{k_1} A_{k_2} \sum_{t=j}^n \alpha_{t-j+2} e_t \\ &= \sum_{t=2}^n \sum_{j=2}^t \sum_{k_1+k_2=j} A_1 A_{k_1} A_{k_2} \alpha_{t-j+2} e_t. \end{aligned}$$

By comparing the coefficients at the basis elements for  $2 \leq t \leq n$ , we obtain the following restrictions.

$$\sum_{i=2}^t \sum_{k_1+\dots+k_i=t} A_{k_1} \dots A_{k_i} \alpha'_i = \sum_{j=2}^t \sum_{k_1+k_2=j} A_1 A_{k_1} A_{k_2} \alpha_{t-j+2}.$$

If we denote  $t - j + 2$  as  $j$  on the right hand side of this relation, we can rewrite the sum as follows:

$$\sum_{i=2}^t \sum_{k_1+\dots+k_i=t} A_{k_1} \dots A_{k_i} \alpha'_i = \sum_{j=2}^t \sum_{k_1+k_2=t-j+2} A_1 A_{k_1} A_{k_2} \alpha_j.$$

□

**Lemma 3.13.** *Let  $\mathbf{TP}_1(0, \dots, 0, \alpha_s, \dots, \alpha_n)$ , with  $\alpha_s \neq 0$ , be a transposed 1-Poisson algebra defined above. Then, there exists  $A \in \mathbb{C}$  such that the relation  $\alpha'_s = \alpha_s A^{3-s}$  holds for any  $2 \leq s \leq n$ .*

*Proof.* Let  $\mathbf{TP}_1(\alpha_2, \dots, \alpha_n)$  be a transposed 1-Poisson algebra, and consider a general change of basis. Let  $\alpha_i = 0$  for  $2 \leq i < s - 1$  then  $\alpha'_i = 0$ . Then, we have the following restriction (13):

$$\sum_{i=2}^s \sum_{k_1+\dots+k_i=s} A_{k_1} \dots A_{k_i} \alpha'_i = \sum_{j=2}^s \sum_{k_1+k_2=j} A_1 A_{k_1} A_{k_2} \alpha_{s-j+2}$$

or

$$\sum_{k_1+k_2=2} A_{k_1}A_{k_2}\alpha'_2 + \sum_{k_1+\dots+k_s=s} A_{k_1}\dots A_{k_s}\alpha'_s = \sum_{k_1+k_2=2} A_1A_{k_1}A_{k_2}\alpha_2 + \sum_{k_1+k_2=s} A_1A_{k_1}A_{k_2}\alpha_s.$$

For  $s = 2$ , we derive  $\alpha'_2 = A_1\alpha_2$ . In the next steps we will have the following

$$\sum_{k_1+\dots+k_s=s} A_{k_1}\dots A_{k_s}\alpha'_s = \sum_{k_1+k_2=2} A_1A_{k_1}A_{k_2}\alpha_s.$$

From this equality, we obtain the relation  $\alpha'_s = \alpha_s A_1^{3-s}$ .  $\square$

**Lemma 3.14.** *If the parameters  $\alpha_2 \neq 0$  and  $\alpha_s \neq 0$  for  $3 \leq s \leq n$ , then the algebra  $\mathbf{TP}_1(\alpha_2, 0, \dots, 0, \alpha_s, \dots, \alpha_n)$  is isomorphic to the algebra  $\mathbf{TP}_1(1, 0, \dots, 0, \alpha_s, 0, \dots, 0)$ .*

*Proof.* By general substitution and according to Lemma 3.13, we obtain the following equality.

$$\alpha'_2 = A_1\alpha_2, \quad \alpha'_s = \alpha_s A_1^{3-s}.$$

Since  $\alpha_2 \neq 0$ , making the substitution  $e'_i = \alpha_2^{-i} e_i$ , for  $1 \leq i \leq n$ , we obtain the relation  $\alpha'_2 = 1$ .

Next, consider another change of basis for  $e_1$  :

$$e'_1 = e_1 + \frac{\alpha_{s+1}}{(s-2)\alpha_s} e_2.$$

From the relations  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we obtain

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left( \frac{\alpha_{s+1}}{(s-2)\alpha_s} \right)^t e_{i+t}, \quad 1 \leq i \leq n.$$

Here we assume that  $e_t = 0$  for  $t > n$ . Thus, a direct computation shows that  $\alpha_{s+1} = 0$ .

Now, we prove by induction that it is possible to set  $\alpha_j = 0$  for all  $s+1 \leq j \leq n$ . The base case  $j = s+1$  is already established. Assuming that the claim holds for some  $j$ , we show that it also holds for  $j+1$ . Consider the change of basis:

$$e'_1 = e_1 + \frac{\alpha_{j+1}}{(s-2)\alpha_s} e_{j-s+2}.$$

Using  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we derive

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left( \frac{\alpha_{j+1}}{(s-2)\alpha_s} \right)^t e_{i+t(j-s+1)}, \quad 1 \leq i \leq n.$$

Here we assume that  $e_t = 0$  for  $t > n$ . One can verify that  $\alpha_{j+1} = 0$ .

By induction, we obtain  $\alpha_j = 0$  for all  $s + 1 \leq j \leq n$ , and hence the algebra  $\mathbf{TP}_1(1, 0, \dots, 0, \alpha_s, 0, \dots, 0)$  with the following multiplication rules:

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_1, e_i] = e_i + \alpha e_{s+i-2}, & 2 \leq i \leq n-s+2. \end{cases}$$

□

**Lemma 3.15.** *If  $\alpha_3 \neq 0$ , then  $\mathbf{TP}_1(0, \alpha_3, \dots, \alpha_n) \cong \mathbf{TP}_1(0, \alpha, 0, \dots, 0)$ .*

*Proof.* By general substitution and according to Lemma 3.13, we obtain  $\alpha'_3 = \alpha_3$ . Next, consider another change of basis  $e'_1 = e_1 + \frac{\alpha_4}{\alpha_3} e_2$ . From the relations  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we obtain

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left( \frac{\alpha_4}{\alpha_3} \right)^t e_{i+t}, \quad 1 \leq i \leq n.$$

Here we assume that  $e_t = 0$  for  $t > n$ . It follows that  $\alpha'_4 = 0$ .

Now, we prove by induction that  $\alpha_j = 0$  for all  $4 \leq j \leq n$ . The base case  $j = 4$  is already established. Assuming that the claim holds for some  $j$ , we show that it also holds for  $j + 1$ . Consider the change of basis:

$$e'_1 = e_1 + \frac{\alpha_{j+1}}{\alpha_3} e_{j-1}.$$

Using  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we derive

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left( \frac{\alpha_{j+1}}{\alpha_3} \right)^t e_{i+t(j-2)}, \quad 1 \leq i \leq n.$$

Here we assume that  $e_t = 0$  for  $t > n$ , which yields  $\alpha_{j+1} = 0$ .

By induction, we have  $\alpha_j = 0$  for all  $4 \leq j \leq n$ , and get the algebra  $\mathbf{TP}_1(0, \alpha, 0, \dots, 0)$ , with the following multiplication rules:

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_1, e_i] = \alpha e_{i+1}, & 2 \leq i \leq n-1. \end{cases}$$

□

**Lemma 3.16.** *If  $\alpha_s \neq 0, s \geq 4$ , then  $\mathbf{TP}_1(0, \dots, 0, \alpha_s, \dots, \alpha_n) \cong \mathbf{TP}_1(0, \dots, 0, 1_s, 0, \dots, 0)$ .*

*Proof.* By general substitution and according to Lemma 3.13, we obtain the following equality

$$\alpha'_s = \alpha_s A_1^{3-s}.$$

If we perform the substitution  $e'_i = \alpha_s^{\frac{i}{s-3}} e_i$ , for  $1 \leq i \leq n$ , then we obtain the relation  $\alpha'_s = 1$ .

Next, consider another change of basis for  $e_1$  :

$$e'_1 = e_1 + \frac{\alpha_{s+1}}{(s-2)} e_2.$$

From the relations  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we obtain

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left( \frac{\alpha_{s+1}}{(s-2)} \right)^t e_{i+t}, \quad 1 \leq i \leq n.$$

Here we assume that  $e_t = 0$  for  $t > n$ . A direct computation shows that  $\alpha_{s+1} = 0$ .

Now, we prove by induction that  $\alpha_j = 0$  for all  $s+1 \leq j \leq n$ . The base case  $j = s+1$  is already established. Assuming that the claim holds for some  $j$ , we show that it also holds for  $j+1$ . Consider the change of basis:

$$e'_1 = e_1 + \frac{\alpha_{j+1}}{(s-2)} e_{j-s+2}.$$

Using  $e_i \cdot e_j = e_{i+j}$  for  $2 \leq i+j \leq n$ , we derive

$$e'_i = \sum_{t=0}^i \binom{i}{t} \left( \frac{\alpha_{j+1}}{(s-2)} \right)^t e_{i+t(j-s+1)}, \quad 1 \leq i \leq n.$$

Here we assume that  $e_t = 0$  for  $t > n$ . It follows that  $\alpha_{j+1} = 0$ .

By induction, we obtain  $\alpha_j = 0$  for all  $s+1 \leq j \leq n$ , and hence the algebra  $\mathbf{TP}_1(0, 0, \dots, 0, 1_s, 0, \dots, 0)$  with the following multiplication rules:

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_1, e_i] = e_{s+i-2}, & 2 \leq i \leq n+2-s. \end{cases}$$

□

**Theorem 3.17.** *Let  $(\mu_0^n, \cdot, [-, -])$  be a 1-transposed Poisson algebra. Then this algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathbf{TP}_1(1, 0, \dots, 0, \alpha_s, 0, \dots, 0), \quad \mathbf{TP}_1(0, \alpha, 0, \dots, 0),$$

$$\mathbf{TP}_1(0, \dots, 0, 1_p, 0, \dots, 0), \quad 3 \leq s \leq n, \quad 4 \leq p \leq n, \quad \alpha \in \mathbb{C}.$$

*Proof.* Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed 1-Poisson algebra. Then Theorem 3.11 implies that it is isomorphic to the algebra  $\mathbf{TP}_1(\alpha_2, \dots, \alpha_n)$ . Moreover,

- If  $\alpha_2 \neq 0$ ,  $\alpha_3 = \dots = \alpha_{s-1} = 0$  and  $\alpha_s \neq 0$  for  $s \geq 3$ , then from Lemma 3.14, we have the algebra  $\mathbf{TP}_1(1, 0, \dots, 0, \alpha_s, 0, \dots, 0)$ .

- If  $\alpha_2 = 0, \alpha_3 \neq 0$  then from Lemma 3.15, we have the algebra  $\mathbf{TP}_1(0, \alpha, 0, \dots, 0)$ .
- If  $\alpha_2 = \dots = \alpha_{p-1} = 0, \alpha_p \neq 0, p \geq 4$  then from Lemma 3.16, we have the algebra  $\mathbf{TP}_1(0, \dots, 0, 1_p, 0, \dots, 0), 4 \leq p \leq n$ .
- If  $\alpha_i = 0, 2 \leq i \leq n$  then we have the algebra  $\mathbf{TP}_1(0, \dots, 0)$ .

□

### 3.3 Transposed $\delta$ -Poisson algebra structures on null-filiform associative algebras

Now we consider transposed  $\delta$ -Poisson algebra structures on null-filiform associative algebras for  $\delta \neq 0, 1, 2$ .

**Theorem 3.18.** *Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra and  $\delta \neq 0, 1, 2$ . Then the multiplication of  $(\mu_0^n, \cdot, [-, -])$  has the following form:*

$$\mathbf{TP}_\delta(\alpha_{n-2}, \alpha_{n-1}, \alpha_n) : \begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_1, e_2] = \alpha_{n-2}e_{n-2} + \alpha_{n-1}e_{n-1} + \alpha_n e_n, \\ [e_1, e_3] = \delta(\alpha_{n-2}e_{n-1} + \alpha_{n-1}e_n), \\ [e_2, e_3] = \frac{\delta^2 - \delta}{2}\alpha_{n-2}e_n, \\ [e_1, e_4] = \frac{\delta^2 + \delta}{2}\alpha_{n-2}e_n. \end{cases}$$

*Proof.* Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra. Then, according to Corollary 3.2, we have the multiplications

$$\begin{aligned} [e_1, e_2] &= \alpha_{n-2}e_{n-2} + \alpha_{n-1}e_{n-1} + \alpha_n e_n, & [e_1, e_4] &= \frac{\delta^2 + \delta}{2}\alpha_{n-2}e_n, \\ [e_1, e_3] &= \delta(\alpha_{n-2}e_{n-1} + \alpha_{n-1}e_n), & [e_1, e_i] &= 0, & 5 \leq i \leq n. \end{aligned}$$

Now we check the identity (2) for the triple  $\{e_1, e_1, e_3\}$ :

$$\delta e_1 \cdot [e_1, e_3] = [e_2, e_3] + [e_1, e_4].$$

From this, we derive the following product

$$[e_2, e_3] = \frac{\delta^2 - \delta}{2}\alpha_{n-2}e_n.$$

Applying induction by  $i + j$ , we prove  $[e_i, e_j] = 0$  for  $i + j \geq 6$ .

Now, we consider the condition (2) for the triple  $\{e_1, e_i, e_j\}$ :

$$0 = \delta e_i \cdot [e_1, e_j] = [e_i \cdot e_1, e_j] + [e_1, e_i \cdot e_j] = [e_{i+1}, e_j].$$

So we get  $[e_i, e_j] = 0$  for  $i + j \geq 6$ . Thus, we get the proof of the theorem.

□

The following theorem establishes a necessary and sufficient condition for two algebras in the family  $\mathbf{TP}_\delta(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)$  to be isomorphic.

**Theorem 3.19.** *Let  $\mathbf{TP}_\delta(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)$  and  $\mathbf{TP}'_\delta(\alpha'_{n-2}, \alpha'_{n-1}, \alpha'_n)$  be isomorphic algebras. Then there exists an automorphism  $\varphi$  between these algebras such that:*

$$\begin{aligned}\alpha'_{n-2} &= \frac{\alpha_{n-2}}{A_1^{n-5}}, & \alpha'_{n-1} &= \frac{A_1\alpha_{n-1} + A_2\alpha_{n-2}(2\delta - n + 2)}{A_1^{n-3}}, \\ \alpha'_n &= \frac{2\alpha_n A_1^2 + 2\alpha_{n-1} A_1 A_2(2\delta - n + 1) + \alpha_{n-2}(A_1 A_3(\delta^2 + 3\delta - 2n + 4) + A_2^2(3\delta^2 + \delta(3 - 4n) + n^2 - n - 2))}{2A_1^{n-1}}.\end{aligned}$$

*Proof.* Using the automorphism of the algebra  $\mu_0^n$  from Theorem 2.6, we introduce the following notations:

$$e'_1 = \sum_{i=1}^n A_i e_i, \quad e'_2 = \sum_{i=2}^n \sum_{k_1+k_2=i} A_{k_1} A_{k_2} e_i,$$

$$\begin{aligned}e'_{n-2} &= A_1^{n-2} e_{n-2} + (n-2)A_1^{n-3} A_2 e_{n-1} + \left( \frac{(n-2)(n-3)}{2} A_1^{n-4} A_2^2 + (n-2)A_1^{n-3} A_3 \right) e_n, \\ e'_{n-1} &= A_1^{n-1} e_{n-1} + (n-1)A_1^{n-2} A_2 e_n, \quad e'_n = A_1^n e_n.\end{aligned}$$

Thus, we consider

$$\begin{aligned}[e'_1, e'_2] &= \alpha'_{n-2} e'_{n-2} + \alpha'_{n-1} e'_{n-1} + \alpha'_n e'_n \\ &= \alpha'_{n-2} \left( A_1^{n-2} e_{n-2} + (n-2)A_1^{n-3} A_2 e_{n-1} + \right. \\ &\quad \left. \left( \frac{(n-2)(n-3)}{2} A_1^{n-4} A_2^2 + (n-2)A_1^{n-3} A_3 \right) e_n \right) \\ &\quad + \alpha'_{n-1} (A_1^{n-1} e_{n-1} + (n-1)A_1^{n-2} A_2 e_n) + \alpha'_n A_1^n e_n.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}[e'_1, e'_2] &= \left[ \sum_{i=1}^n A_i e_i, \sum_{i=2}^n \sum_{k_1+k_2=i} A_{k_1} A_{k_2} e_i \right] \\ &= A_1^3 (\alpha_{n-2} e_{n-2} + \alpha_{n-1} e_{n-1} + \alpha_n e_n) + 2\delta A_1^2 A_2 (\alpha_{n-2} e_{n-1} + \alpha_{n-1} e_n) \\ &\quad + \frac{\delta^2 + \delta}{2} (2A_1^2 A_3 + A_1 A_2^2) \alpha_{n-2} e_n + \frac{\delta^2 - \delta}{2} (2A_1 A_2^2 - A_1^2 A_3) \alpha_{n-2} e_n \\ &= A_1^3 \alpha_{n-2} e_{n-2} + (A_1^3 \alpha_{n-1} + 2\delta A_1^2 A_2 \alpha_{n-2}) e_{n-1} \\ &\quad + \left( A_1^3 \alpha_n + 2\delta A_1^2 A_2 \alpha_{n-1} + \frac{\delta^2 + \delta}{2} (2A_1^2 A_3 + A_1 A_2^2) \alpha_{n-2} + \frac{\delta^2 - \delta}{2} (2A_1 A_2^2 - A_1^2 A_3) \alpha_{n-2} \right) e_n.\end{aligned}$$

By comparing the coefficients at the basis elements, we obtain the expressions given in the lemma for  $\alpha'_{n-2}$ ,  $\alpha'_{n-1}$ , and  $\alpha'_n$ .  $\square$

**Theorem 3.20.** *Let  $(\mu_0^n, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra, and  $n \geq 5$ . Then this algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathbf{TP}_\delta(1, 0, 0), \quad \mathbf{TP}_\delta(0, 1, 0), \quad \mathbf{TP}_\delta(0, 0, 1),$$

$$\mathbf{TP}_{\frac{n-2}{2}}(1, \alpha, 0), \quad \mathbf{TP}_{\delta^2+3\delta=2n-4}(1, 0, \alpha), \quad \mathbf{TP}_{\frac{n-1}{2}}(0, 1, \alpha), \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

*Proof.* If we make a general substitution on the  $\delta$ -Poisson algebra  $\mathbf{TP}_\delta(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)$ , then according to Theorem 3.19, for two algebras in this family to be isomorphic, the following conditions must be satisfied.

$$\begin{aligned} \alpha'_{n-2} &= \frac{\alpha_{n-2}}{A_1^{n-5}}, \quad \alpha'_{n-1} = \frac{A_1\alpha_{n-1} + A_2\alpha_{n-2}(2\delta - n + 2)}{A_1^{n-3}}, \\ \alpha'_n &= \frac{2\alpha_n A_1^2 + 2\alpha_{n-1} A_1 A_2(2\delta - n + 1) + \alpha_{n-2}(A_1 A_3(\delta^2 + 3\delta - 2n + 4) + A_2^2(3\delta^2 + \delta(3 - 4n) + n^2 - n - 2))}{2A_1^{n-1}}. \end{aligned}$$

One can notice that the relation  $\alpha_{n-2} = 0$  is an invariant. Thus we consider the following two cases.

- (1) Let  $\alpha_{n-2} \neq 0$ . Then we choose  $A_1 = \sqrt[n-5]{\alpha_{n-2}}$  and we have  $\alpha'_{n-2} = 1$ . We apply the automorphism in Theorem 2.6 to obtain the following relations:

$$\begin{aligned} \alpha'_{n-1} &= \alpha_{n-1} + A_2(2\delta - n + 2), \\ \alpha'_n &= \alpha_n + \alpha_{n-1} A_2(2\delta - n + 1) + \\ &\quad \frac{A_3(\delta^2 + 3\delta - 2n + 4) + A_2^2(3\delta^2 + \delta(3 - 4n) + n^2 - n - 2)}{2}. \end{aligned}$$

We obtain the following possible cases:

- (a) If  $\delta = \frac{n-2}{2}$ , then we derive

$$\alpha'_{n-1} = \alpha_{n-1}, \quad \alpha'_n = \frac{8\alpha_n - 8\alpha_{n-1}A_2 + A_3(n^2 - 6n + 8) - A_2(n^2 - 6n + 8)}{8}.$$

Then, by choosing  $A_2 = 0$ ,  $A_3 = -\frac{8\alpha_n}{n^2 - 6n + 8}$ , we have the algebra  $\mathbf{TP}_{\frac{n-2}{2}}(1, \alpha, 0)$ .

- (b) If  $\delta \neq \frac{n-2}{2}$ , then, by putting  $A_2 = \frac{\alpha_{n-1}}{n-2-2\delta}$ , we conclude that  $\alpha'_{n-1} = 0$ . We use the automorphism in Theorem 2.6 to obtain the following relation:

$$\alpha'_n = \frac{2\alpha_n + A_3(\delta^2 + 3\delta - 2n + 4)}{2}.$$

If  $\delta^2 + 3\delta = 2n - 4$ , then we have the algebra  $\mathbf{TP}_{\delta^2+3\delta=2n-4}(1, 0, \alpha)$ .

If  $\delta^2 + 3\delta \neq 2n - 4$ , then we obtain the algebra  $\mathbf{TP}_{\delta^2+3\delta \neq 2n-4}(1, 0, 0)$ .

(2) Let  $\alpha_{n-2} = 0$ . Then, we get

$$\alpha'_{n-2} = 0, \quad \alpha'_{n-1} = \frac{\alpha_{n-1}}{A_1^{n-4}}, \quad \alpha'_n = \frac{2\alpha_n A_1 + 2\alpha_{n-1} A_2 (2\delta - n + 1)}{2A_1^{n-2}}.$$

In this case, the relation  $\alpha_{n-1} = 0$  is an invariant. Hence, we consider the following two cases.

(a)  $\alpha_{n-1} \neq 0$ . Then, we choose  $A_1 = \sqrt[n-4]{\alpha_{n-1}}$  and we have  $\alpha'_{n-1} = 1$ . We again apply the automorphism in Theorem 2.6 to obtain the following expression:

$$\alpha'_n = \alpha_n + A_2(2\delta - n + 1).$$

If  $\delta = \frac{n-1}{2}$ , then we have  $\mathbf{TP}_{\frac{n-1}{2}}(0, 1, \alpha)$ .

If  $\delta \neq \frac{n-1}{2}$ , then we have  $\mathbf{TP}_{\delta \neq \frac{n-1}{2}}(0, 1, 0)$ .

(b) If  $\alpha_{n-1} = 0$ , then we derive  $\alpha'_n = \frac{\alpha_n}{A_1^{n-3}}$  and choosing  $A_1 = \sqrt[n-3]{\alpha_n}$ , we obtain the algebra  $\mathbf{TP}_\delta(0, 0, 1)$ .

□

In the following theorems, we consider 2, 3 and 4-dimensional cases. For  $\delta \neq 0$ , in [7] it was proved that any 2-dimensional complex transposed  $\delta$ -Poisson algebra on the 2-dimensional associative algebra whose product is given by  $e_1 \cdot e_1 = e_2$  is isomorphic to one of the following transposed Poisson algebras:

$$\mathbf{TP}_\delta(0, 0) : e_1 \cdot e_1 = e_2; \quad \mathbf{TP}_\delta(0, 1) : e_1 \cdot e_1 = e_2, [e_1, e_2] = e_2, \quad \delta \in \mathbb{C} \setminus \{0\}.$$

If  $\delta = 0$ , then we have a transposed 0-Poisson algebra with the multiplication

$$\mathbf{TP}_0 : e_1 \cdot e_1 = e_2, [e_1, e_2] = \alpha_1 e_1 + \alpha_2 e_2.$$

Then, by applying the Theorem 2.6, we obtain the following expressions:

$$\alpha'_1 = A_1^2 \alpha_1, \quad \alpha'_2 = A_1 \alpha_2 - A_2 \alpha_1.$$

By considering the possible values of the parameters  $\alpha_1$  and  $\alpha_2$ , we obtain the following transposed 0-Poisson algebras:

$$\mathbf{TP}_0(0, 0) : e_1 \cdot e_1 = e_2; \quad \mathbf{TP}_0(0, 1) : e_1 \cdot e_1 = e_2, [e_1, e_2] = e_2;$$

$$\mathbf{TP}_0(1, 0) : e_1 \cdot e_1 = e_2, [e_1, e_2] = e_1.$$

Thus, we have obtained the following 2-dimensional transposed  $\delta$ -Poisson algebras.

$$\mathbf{TP}_\delta(0, 0), \mathbf{TP}_\delta(0, 1), \mathbf{TP}_0(1, 0), \quad \delta \in \mathbb{C}.$$

Now we consider the transposed  $\delta$ -Poisson algebra structures on the 3-dimensional null-filiform associative algebra  $\mu_0^3$ . In this case, we set

$$[e_1, e_2] = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3.$$

By analyzing the identity (2) for the triples  $\{e_1, e_1, e_2\}$ ,  $\{e_1, e_1, e_3\}$ ,  $\{e_2, e_1, e_2\}$  and also by considering the Jacobi identity for triple  $\{e_1, e_2, e_3\}$  we have

$$[e_1, e_2] = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, [e_1, e_3] = \delta \alpha_2 e_3, [e_2, e_3] = 0, \delta \alpha_1 = 0. \quad (14)$$

**Theorem 3.21.** *Let  $(\mu_0^3, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra and  $\delta \neq 0$ . Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathbf{TP}_\delta(0, 1, 0), \mathbf{TP}_\delta(0, 0, \alpha), \mathbf{TP}_1(0, 1, \alpha), \alpha \in \mathbb{C}.$$

*Proof.* Let  $(\mu_0^3, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra and  $\delta \neq 0$ . Then, according to (14), we have the following

$$\mathbf{TP}(0, \alpha_2, \alpha_3) : \begin{cases} e_1 \cdot e_1 = e_2, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = e_3, \\ [e_1, e_2] = \alpha_2 e_2 + \alpha_3 e_3, [e_1, e_3] = \delta \alpha_2 e_3. \end{cases}$$

By applying the automorphism in Theorem 2.6, we get the following relations:

$$\alpha'_2 = A_1 \alpha_2, \alpha'_3 = \frac{A_1 \alpha_3 + (2\delta - 2)A_2 \alpha_2}{A_1}.$$

(1) Let  $\delta = 1$ . Then, we have  $\alpha'_2 = A_1 \alpha_2$ ,  $\alpha'_3 = \alpha_3$ . In this case we obtain the algebras  $\mathbf{TP}_1(0, 1, \alpha)$  and  $\mathbf{TP}_1(0, 0, \alpha)$ , where  $\alpha \in \mathbb{C}$ .

(2) Let  $\delta \neq 1$ . Then:

If  $\alpha_2 \neq 0$ , then by choosing  $A_1 = \frac{1}{\alpha_2}$ ,  $A_2 = -\frac{\alpha_3}{(2\delta-2)\alpha_2^2}$ , we have the algebra  $\mathbf{TP}_\delta(0, 1, 0)$ .

If  $\alpha_2 = 0$ , then  $\alpha'_3 = \alpha_3$  and we derive the algebra  $\mathbf{TP}_\delta(0, 0, \alpha)$ .

□

**Theorem 3.22.** *Let  $(\mu_0^3, \cdot, [-, -])$  be a transposed 0-Poisson algebra. Then, it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathbf{TP}_0(1, 0, 0), \mathbf{TP}_0(0, 1, 0), \mathbf{TP}_0(0, 0, \alpha), \alpha \in \mathbb{C}.$$

*Proof.* Let  $(\mu_0^3, \cdot, [-, -])$  be a transposed 0-Poisson algebra. Then, according to (14), we have the following

$$\mathbf{TP}(\alpha_1, \alpha_2, \alpha_3) : \begin{cases} e_1 \cdot e_1 = e_2, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = e_3, \\ [e_1, e_2] = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3. \end{cases}$$

By applying the automorphism in Theorem 2.6, we get the following relations:

$$\alpha'_1 = A_1^2 \alpha_1, \quad \alpha'_2 = A_1 \alpha_2 - A_2 \alpha_1, \quad \alpha'_3 = \frac{A_1^2 \alpha_3 + (2A_2^2 - A_1 A_3) \alpha_1 - 2A_1 A_2 \alpha_2}{A_1^2}.$$

By checking all possible cases of the parameters  $\alpha_1, \alpha_2$  and  $\alpha_3$ , we obtain the above algebras.  $\square$

**Corollary 3.23.** *Let  $(\mu_0^3, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra. Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathbf{TP}_0(1, 0, 0), \quad \mathbf{TP}_\delta(0, 1, 0), \quad \mathbf{TP}_\delta(0, 0, \alpha), \quad \mathbf{TP}_1(0, 1, \alpha), \quad \alpha \in \mathbb{C}.$$

Now we consider the transposed  $\delta$ -Poisson algebra structures on the 4-dimensional null-filiform associative algebra  $\mu_0^4$ . In this case, we set

$$[e_1, e_2] = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4.$$

By analyzing the identity (2) for the triples

$$\{e_1, e_1, e_2\}, \quad \{e_1, e_1, e_3\}, \quad \{e_2, e_1, e_2\}, \quad \{e_1, e_1, e_4\}, \quad \{e_2, e_1, e_3\}$$

we have  $\delta \alpha_1 = 0$  and

$$\mathbf{TP}_\delta(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq 4, \\ [e_1, e_2] = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4, \\ [e_1, e_3] = \delta(\alpha_2 e_3 + \alpha_3 e_4), \\ [e_2, e_3] = \frac{\delta^2 - \delta}{2} \alpha_2 e_4, \\ [e_1, e_4] = \frac{\delta^2 + \delta}{2} \alpha_2 e_4. \end{cases}$$

**Theorem 3.24.** *Let  $(\mu_0^4, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra, and  $\delta \neq 0$ . Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathbf{TP}_1(0, 1, \alpha, 0), \quad \mathbf{TP}_1(0, 1, 0, \alpha),$$

$$\mathbf{TP}_\delta(0, 1, 0, 0), \quad \mathbf{TP}_\delta(0, 0, \alpha, 0), \quad \mathbf{TP}_\delta(0, 0, 0, 1), \quad \mathbf{TP}_{-4}(0, 1, 0, \alpha), \quad \mathbf{TP}_{\frac{3}{2}}(0, 0, \alpha, 1) \quad \alpha \in \mathbb{C}.$$

*Proof.* Let  $(\mu_0^4, \cdot, [-, -])$  be a transposed Poisson algebra. According to the above, this algebra is isomorphic to the algebra  $\mathbf{TP}_\delta(0, \alpha_2, \alpha_3, \alpha_4)$ . Applying a change of basis of Theorem 2.6, we get the following relations:

$$\alpha'_2 = A_1 \alpha_2, \quad \alpha'_3 = \frac{A_1 \alpha_3 + (2\delta - 2) A_2 \alpha_2}{A_1},$$

$$\alpha'_4 = \frac{2A_1^2 \alpha_4 + 2(2\delta - 3) A_1 A_2 \alpha_3 + ((\delta^2 + 3\delta - 4) A_1 A_3 + (3\delta^2 - 13\delta + 10) A_2^2) \alpha_2}{2A_1^3}.$$

(1) Let  $\delta = 1$ . Then

$$\alpha'_2 = A_1\alpha_2, \alpha'_3 = \alpha_3, \alpha'_4 = \frac{A_1\alpha_4 - A_2\alpha_3}{A_1^2}.$$

- (a) If  $\alpha_2 \neq 0$  and  $\alpha_3 \neq 0$ , then by putting  $A_1 = \frac{1}{\alpha_2}$ ,  $A_2 = \frac{\alpha_4}{\alpha_2\alpha_3}$ , we have the algebra  $\mathbf{TP}_1(0, 1, \alpha, 0)$ .
- (b) If  $\alpha_2 \neq 0$  and  $\alpha_3 = 0$ , then by putting  $A_1 = \frac{1}{\alpha_2}$ , we obtain the algebra  $\mathbf{TP}_1(0, 1, 0, \alpha)$ .
- (c) If  $\alpha_2 = 0$  and  $\alpha_3 \neq 0$ , then by choosing  $A_2 = \frac{A_1\alpha_4}{\alpha_3}$ , we obtain the algebra  $\mathbf{TP}_1(0, 0, \alpha, 0)$ .
- (d) If  $\alpha_2 = \alpha_3 = 0$  and  $\alpha_4 \neq 0$ , then by choosing  $A_1 = \alpha_4$ , we obtain the algebra  $\mathbf{TP}_1(0, 0, 0, 1)$ .
- (e) If  $\alpha_2 = \alpha_3 = \alpha_4 = 0$ , then we have the algebra  $\mathbf{TP}_1(0, 0, 0, 0)$ .

(2) Let  $\delta \neq 1$ . Then:

- (a)  $\alpha_2 \neq 0$ , then by choosing  $A_1 = \frac{1}{\alpha_2}$ ,  $A_2 = -\frac{\alpha_3}{(2\delta-2)\alpha_2^2}$ , we have  $\alpha'_2 = 1$ ,  $\alpha'_3 = 0$ . We apply the automorphism in Theorem 2.6 to obtain the following relation.

$$\alpha'_4 = \frac{2\alpha_4 + (\delta^2 + 3\delta - 4)A_3}{2}.$$

If  $\delta \neq -4$ , then we put  $A_3 = -\frac{2\alpha_4}{\delta^2+3\delta-4}$  and obtain the algebra  $\mathbf{TP}_{\delta \neq -4}(0, 1, 0, 0)$ .

If  $\delta = -4$ , then we derive the algebra  $\mathbf{TP}_{-4}(0, 1, 0, \alpha)$ .

- (b) Let  $\alpha_2 = 0$  and  $\alpha_3 \neq 0$ . Then, we have

$$\alpha'_3 = \alpha_3, \alpha'_4 = \frac{A_1\alpha_4 + (2\delta - 3)A_2\alpha_3}{A_1^2}.$$

If  $\delta \neq \frac{3}{2}$ , choosing  $A_1 = 1$ ,  $A_2 = \frac{\alpha_4}{(3-2\delta)\alpha_3}$ , we derive the algebra  $\mathbf{TP}_{\delta \neq \frac{3}{2}}(0, 0, \alpha, 0)$ .

If  $\delta = \frac{3}{2}$ , then we have

$$\alpha'_3 = \alpha_3, \alpha'_4 = \frac{\alpha_4}{A_1}.$$

Depending on whether  $\alpha_4 = 0$  or not, we derive the algebras  $\mathbf{TP}_{\frac{3}{2}}(0, 0, \alpha, 0)$  and  $\mathbf{TP}_{\frac{3}{2}}(0, 0, \alpha, 1)$ , respectively.

- (c) Let  $\alpha_2 = \alpha_3 = 0$  and  $\alpha_4 \neq 0$ . Then, by setting  $A_1 = \alpha_4$ , we obtain the algebra  $\mathbf{TP}_{\delta}(0, 0, 0, 1)$ .
- (d) If  $\alpha_2 = \alpha_3 = \alpha_4 = 0$ , then we obtain the algebra  $\mathbf{TP}_{\delta}(0, 0, 0, 0)$ .

□

Using the Theorems 3.10 and 3.24, we derive the classification of 4-dimensional complex transposed  $\delta$ -Poisson algebra associated on  $\mu_0^4$ .

**Theorem 3.25.** *Let  $(\mu_0^4, \cdot, [-, -])$  be a transposed  $\delta$ -Poisson algebra. Then, this algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned} & \mathbf{TP}_0(1, 0, 0, 0), \mathbf{TP}_1(0, 1, \alpha, 0), \mathbf{TP}_1(0, 1, 0, \alpha), \\ & \mathbf{TP}_\delta(0, 1, 0, 0), \mathbf{TP}_\delta(0, 0, \alpha, 0), \mathbf{TP}_\delta(0, 0, 0, 1), \mathbf{TP}_{-4}(0, 1, 0, \alpha), \mathbf{TP}_{\frac{3}{2}}(0, 0, \alpha, 1) \quad \alpha \in \mathbb{C}. \end{aligned}$$

## 4 $\delta$ -Poisson algebra structures on null-filiform associative algebras

All Poisson algebra structures and the trivial Poisson algebra on null-filiform associative algebras have been obtained in [1]. Below, we construct all  $\delta$ -Poisson algebra structures on null-filiform associative algebras.

**Theorem 4.1.** *Let  $(\mu_0^n, \cdot, [-, -])$  be a  $\delta$ -Poisson algebra. Then,  $(\mu_0^n, \cdot, [-, -])$  is a trivial  $\delta$ -Poisson algebra.*

*Proof.* Let  $(\mu_0^n, \cdot, [-, -])$  be a  $\delta$ -Poisson algebra. To establish the table of multiplications for the operation  $[-, -]$  in this algebra, we consider the following computation for  $1 \leq i \leq n-1$ :

$$[e_1, e_{i+1}] = [e_1, e_1 \cdot e_i] = \delta([e_1, e_1] \cdot e_i + e_1 \cdot [e_1, e_i]) = \delta e_1 \cdot [e_1, e_i].$$

From this we get  $[e_1, e_i] = 0$  for  $2 \leq i \leq n$ .

Next, considering the following equalities

$$[e_i, e_2] = [e_i, e_1 \cdot e_1] = \delta([e_i, e_1] \cdot e_1 + e_1 \cdot [e_i, e_1]) = 0, \quad 3 \leq i \leq n-1,$$

$$[e_i, e_j] = [e_i, e_1 \cdot e_{j-1}] = \delta([e_i, e_1] \cdot e_{j-1} + e_1 \cdot [e_i, e_{j-1}]) = 0, \quad 3 \leq i, j \leq n,$$

we obtain

$$[e_i, e_j] = 0, \quad 1 \leq i, j \leq n.$$

□

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