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Description of DG-algebra structures via characters on a groupoid

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Abstract. We construct a description of graded derivations in group algebras. Using this result for arbitrary grading of the group algebra, we describe all possible DG-algebra structures. Examples are given. The description is given in terms of characters on a groupoid analogous to the groupoid of conjugate action.

Contents

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1 Introduction

In several previous papers by A.S. Mishchenko and others [1,3,4], a method for studying the derivations of group algebras using the character space on the groupoid of conjugacy action was presented. In the present paper, a similar groupoid and characters on it are constructed in order to describe graded derivations on graded group algebras and all possible DG-algebra structures on group algebras.

DG-algebras and graded derivations occur frequently. See e.g. [10], where they are studied in the context of algebraic geometry. Graded derivations in isolation are an even better-known subject, see e.g. [9]. The classification of possible DG-structures has also been a focus of study [7,8].

The main object of our study will be the graded group algebra $\mathbb{C}[G] \cong \bigoplus A_i$ and its graded derivations, that is, linear maps $d : \mathbb{C}[G] \to \mathbb{C}[G]$ satisfying the graded Leibniz rule

$$d(uv) = d(u)v + (-1)^{|u|}ud(v), \quad \forall u, v \in \cup A_i.$$

In Section 2, we present the structure of a groupoid which is connected to the graded algebra structure. We will give a description of graded derivations as locally finite characters on this groupoid (see Theorem 2.9). In Section 3, we describe the inner, quasi-inner, and central derivations in the graded case. Using this construction in the case of DG-algebras, in Section 4, we obtain a description of DG-algebra structures in terms of characters and an isomorphism criterion for DG-algebras. In addition, at the end of Section 4, we give several examples of gradings on group algebras and derivations that define a DG-algebra structure on them. The definitions and properties from combinatorial group theory that are necessary to understand the present paper can be found in [6].

2 The groupoid, its characters, and graded derivations

Let's start with the definitions and fix the notation for future reference. We will study derivations of group algebras with a DG-structure. The standard definition of a group algebra is

Definition 2.1. Let G be a group and let K be a field. The group algebra $K[G]$ is the associative algebra over K the elements of which are all possible formal finite sums of the form $\sum_{g\in G} k_g g$, $g \in G$, $k_g \in K$, and the operations are defined by:

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g,$$

$$(\sum_{g\in G} a_g g)(\sum_{g\in G} b_g g) = \sum_{h\in G} \Big(\sum_{xy=h\in G} (a_x b_y) h \Big).$$

The elements of the group G form a basis of $K[G]$. Multiplication of the basis elements in the group algebra is induced by the group multiplication.

Furthermore, we need the notion of a groupoid. Usually, it is defined as a category in which every morphism is invertible. We will work with the following construction associated with a finitely generated group G, so this category is small (the classes of objects and morphisms are sets).

Definition 2.2. A groupoid Γ has a set of objects $\mathrm{Obj}(\Gamma)$ and a set of morphisms $\mathrm{Hom}(\Gamma)$. Each morphism has a source $s(\phi)$ and a target $t(\phi)$, i.e. each morphism is in the set $\mathrm{Hom}(s(\phi), t(\phi))$. Composable morphisms have an associative operation (composition). The following conditions hold:

1. For each object $a \in \mathrm{Obj}(\Gamma)$, there is a neutral endomorphism $1_a \in \mathrm{Hom}(a,a)$ such that $\forall \phi \in \mathrm{Hom}(b,a)$: $1_a \circ \phi = \phi$ and $\forall \phi \in \mathrm{Hom}(a,b)$ $\phi \circ 1_a = \phi$.

2. Each morphism has an inverse: for every $\phi \in \mathrm{Hom}(a,b)$ there exists $\psi \in \mathrm{Hom}(b,a)$ such that $\phi \circ \psi = 1_b$ and $\psi \circ \phi = 1_a$.

2.1 Groupoid of conjugacy action

The following construction generalizes the one from [4]. Suppose we have a $\mathbb{C}$–graded group algebra $\mathbb{C}[G] \cong \bigoplus A_i$. The group G is assumed to be finitely generated. We will define the groupoid of conjugacy action Γ for the group G (in the sense of Definition 2.2) as follows:

$$\Gamma = \begin{cases} \mathrm{Obj}(\Gamma) = \{\pm g \mid g \in G\}, \\ \mathrm{Hom}(\Gamma) = \{(u,v) \in \pm G \times \pm G \ \mid s((u,v)) = (-1)^{|v|} v^{-1} u,\ t((u,v)) = uv^{-1}\}. \end{cases}$$

Consider two morphisms $\phi = (u_1, v_1)$ and $\psi = (u_2, v_2)$ such that $t(\phi) = s(\psi)$, that is, the composition $\psi \circ \phi$ should be defined. Define it as follows:

$$(u_2, v_2) \circ (u_1, v_1) := \big((-1)^{|v_2|} v_2 u_1, v_2 v_1\big).$$

This formula is conveniently depicted as a diagram.

$$\begin{tikzcd} & (-1)^{|v_1|} v_1^{-1} u_1 = u_2 v_2^{-1} \arrow[dr, "{(u_2,v_2)}"] & \\ u_1 v_1^{-1} \arrow[ur, "{(u_1,v_1)}"] \arrow[rr, "{(u_2,v_2)\circ(u_1,v_1)}"'] & & (-1)^{|v_2|} v_2^{-1} u_2 \end{tikzcd}$$

Proposition 2.3. *Γ is a groupoid.*

Proof. First of all, we note that $\exists g \to h \Leftrightarrow \exists t : h = (-1)^{|t|} t g t^{-1}$.

Consider $\phi = (u_1, v_1)$ and $\psi = (u_2, v_2)$ such that $t(\phi) = s(\psi)$. Define $s(\phi) = a$, $t(\phi) = s(\psi) = b$ and $t(\psi) = c$.

We need to check that

$$s(\psi \circ \phi) = a \text{ and } t(\psi \circ \phi) = c.$$

By definition,

$$s(\psi \circ \phi) = (-1)^{|v_2 v_1|} v_1^{-1} v_2^{-1} (-1)^{|v_2|} v_2 u_1 = (-1)^{|v_1|} v_1^{-1} u_1 = a.$$

Calculate that

$$t(\psi \circ \phi) = (-1)^{|v_2|} v_2 u_1 v_1^{-1} v_2^{-1} = u_2 v_2^{-1} = c, \text{ because } u_1 v_1^{-1} = (-1)^{|v_2|} v_2^{-1} u_2.$$

Here we shall prove that the composition is associative. Let ϕ_1, ϕ_2, ϕ_3 be morphisms such that $s(\phi_1) = t(\phi_2)$ and $s(\phi_2) = t(\phi_3)$. We need to show that

$$\phi_1 \circ (\phi_2 \circ \phi_3) = (\phi_1 \circ \phi_2) \circ \phi_3.$$

We have proved above that these compositions are well-defined. The left side is

$$\phi_2 \circ \phi_3 = ((-1)^{|v_2|} v_2 u_3, v_2 v_3))$$
$$\phi_1 \circ (\phi_2 \circ \phi_3) = (u_1, v_1) \circ ((-1)^{|v_2|} v_2 u_3, v_2 v_3)) = ((-1)^{|v_1|+|v_2|} v_1 v_2 u_3, v_1 v_2 v_3).$$

While the right one is equal to it,

$$(\phi_1 \circ \phi_2) \circ \phi_3 = ((-1)^{|v_1|} v_1 u_2, v_1 v_2) \circ (u_3, v_3) = ((-1)^{|v_1|+|v_2|} v_1 v_2 u_3, v_1 v_2 v_3),$$

proving associativity.
Let us show that there is an element 1_g such that

$$s(1_g) = t(1_g) = g,$$
$$\phi \cdot 1_{s(\phi)} = \phi = 1_{t(\phi)} \cdot \phi, \ \forall \phi \in \mathrm{Hom}(\Gamma).$$

Take $1_g = (g, e)$, where e is the identity of the group $G:$

$$\begin{tikzcd} (-1)^{|e|} e^{-1} g = g = g e^{-1} \arrow[loop above, "{(g,e)}"] \end{tikzcd}$$

Indeed, choosing an arbitrary morphism (u, v), we have:

$$(uv^{-1}, e)(u, v) = (u, v);$$
$$(u, v)((-1)^{|v|} v^{-1} u, e) = ((-1)^{|v|} v (-1)^{|v|} v^{-1} u, e) = (u, v).$$

We have now shown that Γ is a category. It remains to check the invertibility of the arrows. We claim the inverse of a morphism of the form (u, v) is $((-1)^{|v|} v^{-1} u v^{-1}, v^{-1})$.

Indeed,

$$(u, v)((-1)^{|v|} v^{-1} u v^{-1}, v^{-1}) = (uv^{-1}, e) = (t(\phi), e),$$
$$((-1)^{|v|} v^{-1} u v^{-1}, v^{-1})(u, v) = ((-1)^{|v|} v^{-1} u, e) = (s(\phi), e).$$

$$
\begin{tikzcd}
(-1)^{|v|}v^{-1}u \arrow[rr, bend left, "{(u,v)}"] & & uv^{-1} \arrow[ll, bend left, "{((-1)^{|v|}v^{-1}uv^{-1},v^{-1})}"]
\end{tikzcd}
$$

This proves Proposition 2.3. $\square$

The following properties of the groupoid are easily verified by direct calculation.

Proposition 2.4. *1. The endomorphisms* $\mathrm{Hom}(a, a)$, $a \in \pm G$, *admit the following description:*

$$\mathrm{Hom}(a, a) = \{\chi \in \mathrm{Hom}(\Gamma) \mid s(\chi) = (-1)^{|t|}t^{-1}a,\ t(\chi) = at^{-1},\ t \in Z(a) \text{ and } t \in A_{2i}\}.$$

2. *For* $[a] := \{b \in \pm G \mid \exists t \in \pm G : b = (-1)^{|t|}t^{-1}at\}$, *denote by* $\Gamma_{[a]}$ *the subgroupoid such that* $\mathrm{Obj}(\Gamma_{[a]}) = [a]$. *The following properties hold:*

   (a) $\mathrm{Hom}(\Gamma_{[a]}) = \{\phi \in \mathrm{Hom}(\Gamma) \mid s(\phi) \in [a], t(\phi) \in [a]\}$.

   (b) $\Gamma = \coprod \Gamma_{[a]}$.

3. *The left action* $\mathrm{Hom}(a, a) \times \mathrm{Hom}(a, b) \to \mathrm{Hom}(a, b)$: $(\phi, \psi) \to \psi \circ \phi$ *is free and transitive.*

Let us show that, with the help of characters of Γ, it is possible to describe derivations satisfying the graded Leibniz rule.

Definition 2.5. The linear operator d: $\mathbb{C}[G] \to \mathbb{C}[G]$ satisfies the graded Leibniz rule if

$$d(uv) = d(u)v + (-1)^{|u|}ud(v),\ \forall\ u, v \in \cup A_i.$$

Definition 2.6. The map $\chi : \mathrm{Hom}(\Gamma) \to \mathbb{C}$ is a character if it satisfies the identity

$$
\chi(\psi \circ \phi) = \chi(\psi) + \chi(\phi) : \quad
\begin{tikzcd}
& \bullet \arrow[dr, "\chi(\psi)"] & \\
\bullet \arrow[ur, "\chi(\phi)"] \arrow[rr, "\chi(\psi\circ\phi)=\chi(\psi)+\chi(\phi)"'] & & \bullet
\end{tikzcd}
$$

Definition 2.7. A character χ is locally finite if for all $v \in \pm G$ we have $\chi(u, v) = 0$ for all except a finite number of u.

We will denote by $\chi(\Gamma)$ the space of locally finite characters. In Section 2.2 we will show that $\chi(\Gamma)$ has a natural Lie algebra structure.

The graded Leibniz rule in a group algebra can be expressed in terms of characters on the groupoid Γ as follows.

Proposition 2.8. *For any graded derivation d, there exists a locally finite character χ such that*

$$d(u) = \sum_{g \in G} \Big(\sum_{h \in \pm G} \chi(\phi(h,g)\lambda^h \Big) g, \ \forall u \in A_i. \tag{1}$$

Proof. It is enough to check the statement on the elements $g \in \pm G$, since they generate $\mathbb{C}[G]$ as a vector space.

For any element g of group G we have $d(g) = \sum_{h \in G} d_g^h h$.

Let us define $\chi(\phi(h,g)) := d_g^h$, $\chi(\phi(-h,g)) := -d_g^h$ and show that χ it is a character. We have

$$d(g_2 g_1) = d(g_2) g_1 + (-1)^{|g_2|} g_2 d(g_1),$$

or after a basis expansion

$$\sum_{h \in G} d_{g_2 g_1}^h h = \sum_{h \in G} d_{g_2}^h h g_1 + (-1)^{|g_2|} g_2 \sum_{h \in G} d_{g_1}^h h.$$

It is clear that summing over all h and all gh is equivalent, so we can convert the right-hand side to the following form:

$$\sum_{h \in G} d_{g_2 g_1}^h h = \sum_{h \in G} d_{g_2}^{h g_1^{-1}} h + \sum_{h \in G} d_{g_1}^{(-1)^{|g_2|} g_2^{-1} h} h \Rightarrow d_{g_2 g_1}^h = d_{g_2}^{h g_1^{-1}} + d_{g_1}^{(-1)^{|g_2|} g_2^{-1} h}. \tag{2}$$

Let's make a substitution:

$$h_1 = h g_1^{-1}, \quad h_2 = (-1)^{|g_2|} g_2^{-1} h.$$

Then the last equality (2) takes the form:

$$d_{g_2 g_1}^{(-1)^{|g_2|} g_2 h_2} = d_{g_2}^{h_1} + d_{g_1}^{h_2} \Leftrightarrow \chi(\phi((-1)^{|g_2|} g_2 h_2, g_2 g_1)) = \chi(\phi(h_1, g_2)) + \chi(\phi(h_2, g_1)).$$

It remains only to note that $((-1)^{|g_2|} g_2 h_2, g_2 g_1) = (h_1, g_2)(h_2, g_1)$. □

A corollary of the above statement is the following theorem, proved similarly to the non-graded case, see Theorem 1 of [2].

Theorem 2.9. *For any graded derivation d there exists a unique locally finite character χ_d satisfying*

$$d(a) = (-1)^{|a|} a \Big(\sum_{t \in G} \chi((-1)^{|a|} at, a) t \Big).$$

There is therefore an isomorphism $\mathrm{Der}(\mathbb{C}[G]) \cong \chi(\Gamma)$.

Proof. To prove this, we need to apply formula (1) to $a \in \pm G$, write an arbitrary element $g \in G$ as $g = at$, and then drop the terms with a zero coefficient from the sum. See the proof of Theorem 1 in [2] for more details. □

2.2 Characters

Locally finite characters are isomorphic to derivations, which means that the commutator of derivations yields a binary operation on characters. Formula (1) allows us to define a commutator in character space. Let d_1, d_2 be the derivations defined by the characters χ_{d_1}, χ_{d_2}. Denote the character that corresponding to the commutator $[d_1, d_2]$ by $\chi_{[d_1,d_2]}$. This defines an operation on the space of characters.

$$\{\chi_{d_1}, \chi_{d_2}\} := \chi_{[d_1,d_2]}. \tag{3}$$

For this operation, we have the following equality:

Proposition 2.10. *The operation on characters above can be expressed as*

$$\{\chi_{d_1}, \chi_{d_2}\}(a, g) = \sum_{h\in G} \Big(\chi_{d_1}(a, h)\chi_{d_2}(h, g) - \chi_{d_2}(a, h)\chi_{d_1}(h, g)\Big). \tag{4}$$

Proof. Let $g \in G$. Then we have

$$\begin{aligned} d_1(g) &= \sum_{h\in G} \chi_{d_1}(h, g)h, \\ d_2(g) &= \sum_{h\in G} \chi_{d_2}(h, g)h, \\ \{\chi_{d_1}, \chi_{d_2}\}(g) &= \sum_{h\in G} \{\chi_{d_1}, \chi_{d_2}\}(a, g)a. \end{aligned}$$

Write down the equation for the commutator

$$\begin{aligned} [d_1, d_2] &= d_1d_2 - d_2d_1, \\ d_1d_2(g) &= \sum_{h\in G} \chi_{d_2}(h, g)\Big(\sum_{a\in\pm G} \chi_{d_1}(a, h)a\Big), \\ d_2d_1(g) &= \sum_{h\in G} \chi_{d_1}(h, g)\Big(\sum_{a\in\pm G} \chi_{d_2}(a, h)a\Big). \end{aligned}$$

Thanks to local finiteness, we can swap the sums in the last equation and get

$$[d_1, d_2](h) = \sum_{a\in G} \Big(\sum_{h\in G} \chi_{d_2}(h, g)\chi_{d_1}(a, h) - \chi_{d_1}(h, g)\chi_{d_2}(a, h)\Big)a.$$

The value of $\{\chi_{d_1}, \chi_{d_2}\}(a, g)$ is the coefficient of a, so

$$\{\chi_{d_1}, \chi_{d_2}\}(a, g) = \sum_{h\in G} \chi_{d_1}(a, h)\chi_{d_2}(h, g) - \chi_{d_2}(a, h)\chi_{d_1}(h, g).$$

□

It follows that $(\chi_\Gamma, \{\cdot, \cdot\})$ is a Lie algebra, isomorphic to the algebra of all graded derivations by Proposition 2.8.

3 Examples of graded derivations and their description in terms of characters

We will illustrate the correspondence between characters on a groupoid and graded derivations using examples of inner derivations, central derivations, and quasi-inner derivations. They will serve as easy examples of such operators.

3.1 Inner derivations

Consider an element $a \in \pm G$. Define the mapping $\chi^a : \mathrm{Hom}(\Gamma) \to \mathbb{C}$ as follows (assuming $a \neq b$):

$$\chi^a(\phi) := \begin{cases} 1, & \phi \in \mathrm{Hom}(b, a) \\ -1, & \phi \in \mathrm{Hom}(a, b) \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Clearly χ^a is trivial on endomorphisms.

Proposition 3.1. *The map χ^a defines an inner derivations for $a \in \pm G$ by the formula*

$$d_a : x \mapsto [a, x]_{grad} = ax - (-1)^{|x|}xa. \tag{6}$$

Proof. Let's check that d_a is a graded derivation (that is, satisfies the graded Leibniz rule):

$$d_a(x)y + (-1)^{|x|}xd_a(y) = (ax - (-1)^{|x|}xa)y + (-1)^{|x|}x(ay - (-1)^{|y|}ya) =$$
$$axy - (-1)^{|xy|}xya = d_a(xy).$$

Let's check that it really satisfies the formula (6). Due to linearity, it is enough to check the statement for the element $g \in \pm G$. According to the previous corollary, we have an equality:

$$d_a(g) = \sum_{h \in G} \chi(h, g)h.$$

We have that $\chi(h, g) \neq 0 \Leftrightarrow t((h, g)) = a$ or $s((h, g)) = a$. Let's consider these two cases:

1. $s((h, g)) = a \Leftrightarrow (-1)^{|g|}g^{-1}h = a \Leftrightarrow h = (-1)^{|g|}ga.$
2. $t((h, g)) = a \Leftrightarrow hg^{-1} = a \Leftrightarrow h = ag.$

Therefore, we have: $d_a(g) = ag - (-1)^{|g|}ga.$ ☐

3.2 Central derivations

We will now give an example of a derivation that is not inner. We will show that it admits a simple explicit form and is not trivial on endomorphisms. That entails that it cannot be expressed as a (perhaps formal) sum of inner derivations.

Definition 3.2. That the map $\tau : G \to \mathbb{C}$ is a graded group character if it satisfies the following property:

$$\tau(ab) = \tau(a) + (-1)^{|a|}\tau(b),\ \forall a, b \in G.$$

Let's fix a graded group character τ, and let $z \in Z(G)$ be a central element of the group.

Definition 3.3. Define the map $d_z^\tau : \mathbb{C}[G] \to \mathbb{C}[G]$ on the generators $g \in \pm G$ as

$$d_z^\tau : g \to \tau(g)gz$$

and extend it by linearity to the whole algebra $\mathbb{C}[G]$. We will call it a central graded derivation.

Proposition 3.4. *The map d_z^τ is a graded derivation.*

Proof. It is enough to check the graded Leibniz rule on arbitrary generators $u, v \in G$:

$$d_z^\tau(uv) = \tau(uv)uvz = \tau(u)uzv + (-1)^{|u|}u\tau(v)vz = d_z^\tau(u)v + (-1)^{|u|}ud_z^\tau(v).$$

On arbitrary elements of a group algebra, the graded Leibniz rule is satisfied due to the linearity of d_z^τ. □

Proposition 3.5. *Nontrivial central derivations are not inner.*

Proof. Let $\chi_{\tau,z}$ be a character that corresponds to the derivation d_z^τ. By Theorem 2.9 we have the equality

$$d_z^\tau(g) = \tau(g)gz = g\Big(\sum_{t\in G}\chi((-1)^{|g|}gt, g)(-1)^{|g|}t\Big).$$

Therefore, this character is not equal to 0 only in the case $t = z$:

$$\chi((-1)^{|g|}gz, g) = (-1)^{|g|}\tau(g).$$

Define the morphism $\phi = ((-1)^{|g|}gz, g)$ and calculate its beginning and

$$s(\phi) = z, \quad t(\phi) = (-1)^{|g|}z.$$

Then in the case when $g \in A_{2n}$, we have: $s(\phi) = t(\phi) \Rightarrow \chi_{\tau,z}$ is non–trivial on a loop ϕ, thus d_z^τ it is not an inner derivation. □

A direct check shows that inner and central derivations constitute subalgebras. For more details, see [2].

3.3 Quasi-inner derivations

As seen from the formula (5), inner derivations are trivial on endomorphisms; however, the converse is not true. We will call such derivations quasi-inner.

Definition 3.6. Define $\mathrm{QInnDer}(\mathbb{C}[G])$ as follows:

$$\mathrm{QInnDer}(\mathbb{C}[G]) = \{d \in \mathrm{Der} \mid \forall a \in \mathrm{Obj}(\Gamma), \forall \phi \in \mathrm{Hom}(a,a) : \chi_d(\phi) = 0\}. \tag{7}$$

Quasi-inner derivations are clearly a subspace, but the more significant statement is that they constitute an ideal in the algebra of all derivations.

Proposition 3.7. $\mathrm{QInnDer}(\mathbb{C}[G]) \triangleleft \mathrm{Der}(\mathbb{C}[G])$ *is an ideal.*

Proof. Let us prove that $\mathrm{QInnDer}(\mathbb{C}[G])$ is a subalgebra, i.e.

$$d_1,\, d_2 \in \mathrm{QInnDer}(\mathbb{C}[G]) \Rightarrow [d_1, d_2] \in \mathrm{QInnDer}(\mathbb{C}[G]).$$

The character χ_{d_1} can be represented as

$$\chi_{d_1} = \sum_{a\in G} \lambda^a \chi^a,\ \lambda^a \in \mathbb{C},$$

where χ_{d_1} defined by formula (5), since χ^a is trivial on endomorphisms. Similarly for the character χ_{d_2}:

$$\chi_{d_2} = \sum_{b\in G} \lambda^b \chi^b,\ \lambda^b \in \mathbb{C}.$$

Because of the bilinearity of the commutator,

$$\{\chi_{d_1}, \chi_{d_2}\} = \sum_{a\in G}\sum_{b\in G} \lambda^a \mu^b \{\chi^a, \chi^b\}.$$

The commutator of the d^a and d^b can be represented as follows:

$$[d^a, d^b] = d^{ab} - d^{ba}.$$

Then define the character $\{\chi^a, \chi^b\}$ by the following formula

$$\{\chi^a, \chi^b\} = \chi^{ab} - \chi^{ba}.$$

Claim:

$$\{\chi_{d_1}, \chi_{d_2}\} = \sum_{a\in G}\sum_{b\in G} \lambda^a \mu^b \chi^{ab} - \sum_{a\in G}\sum_{b\in G} \lambda^a \mu^b \chi^{ba},$$

thus $\{\chi_{d_1}, \chi_{d_2}\} \in \mathrm{QInnDer}(\mathbb{C}[G])$. To prove that, consider the value of the character on the loop (uz, z), $z \in Z_G(u)$:

$$\{\chi_{d_1}, \chi_{d_2}\}(uz, z) = \sum_{ab=zuz^{-1}} \lambda^a \mu^b - \sum_{ab=u} \lambda^a \mu^b - \sum_{ba=u} \lambda^a \mu^b + \sum_{ba=zuz^{-1}} \lambda^a \mu^b = 0.$$

We will now turn to the proof of the theorem. Let's represent χ_{d_0} as

$$\chi_{d_0} = \sum_{a\in G} \lambda^a \chi^a.$$

It is sufficient to prove the theorem for χ^a and extend the result to the χ_{d_0} using bilinearity of the commutator. Consider the character $\{\chi_d, \chi^a\}$. We need to prove that it is trivial on endomorphisms, i.e., that for all $b \in G$ and for $z \in Z_G(b)$ it satisfies $\{\chi_d, \chi^a\}(bz, z) = 0$. By Proposition 2.3:

$$\{\chi_d, \chi^a\}(bz, z) = \sum_{h \in G} \chi_d(bz, h)\chi^a(h, z) - \chi^a(bz, h)\chi_d(h, z).$$

Here $\chi^a(h, z) \neq 0$ only in two cases: when $h = za$ and when $h = az$; and $\chi^a(bz, h) \neq 0$ only in two cases: when $h = bza^{-1}$ and when $h = a^{-1}bz$. This means that

$$\{\chi_d, \chi^a\}(bz, z) = \chi_d(bz, za) - \chi_d(bz, az) + \chi_d(a^{-1}bz, z) - \chi_d(bza^{-1}, z).$$

But also

$$(a^{-1}bz, z) \circ (bz, za) = (bz, az) \circ (bza^{-1}, z).$$

Hence

$$\begin{gathered} \chi_d(bz, za) + \chi_d(a^{-1}bz, z) = \chi_d(bz, az) + \chi_d(bza^{-1}, z), \\ \{\chi_d, \chi^a\}(bz, z) = 0. \end{gathered}$$

□

It is easy to see that central derivations are not quasi-inner by repeating the reasoning from Proposition 3.5.

4 Description of the structure of a DG-algebra in terms of characters on a groupoid

Our next goal is to describe the possible DG-algebra structures up to isomorphism. We follow the definitions of [5].

Definition 4.1. Let $A \cong \bigoplus A_i$ be a $\mathbb{Z}$–graded algebra equipped with a derivation

$$d : A \to A$$

of degree 1 (the case of a cochain complex) or -1 (the case of a chain complex) and satisfying the following conditions:

1. $d \circ d = 0$,
2. $d(uv) = d(u)v + (-1)^{|u|}ud(v)$.

We say that (A, d) is a differential graded algebra or DG-algebra.

We will give the necessary and sufficient conditions from the point of view of groupoid characters for a derivation to define a DG-algebra structure on a graded group algebra. We will describe the cochain complex case, since the reasoning in the chain complex is similar.

Theorem 4.2. *A character χ on the groupoid Γ defines a DG-algebra structure if and only if the following conditions are satisfied:*

1. *The convolution is trivial:*
$$\sum_{h\in G}\chi(h,g)\chi(x,h)=0,\quad \forall g,x\in G. \tag{8}$$

2. *For all $(h,g)\in Hom(\Gamma_{A_i})$, $i\neq 1$, we have $\chi(h,g)=0$.*

The main idea is to prove that the first condition is equivalent to $d^2(g)=0$ and the second one is equivalent to $d(A^i)\subseteq A^{i+1}$.

Proof. Using Proposition 2.8, we rewrite the defining properties of a DG-algebra in terms of characters.

1. By direct calculation we can see that
$$d^2(g)=d\Big(\sum_{h\in G}d_g^h h\Big)=\sum_{h\in G}d_g^h\Big(\sum_{x\in G}d_h^x x\Big)=$$
$$\sum_{h\in G}(h,g)\Big(\sum_{x\in G}(x,h)x\Big)=\sum_{x\in G}\Big(\sum_{h\in G}(\chi(h,g)\chi(x,h))\Big)x.$$

2. Let $g\in A^i$, then $d(g)=\sum\limits_{h\in G}\chi(h,g)h\in A^{i+1}\Leftrightarrow\chi(h,g)=0$ when $h\notin A^{i+1}$.
Note that if $u\in A^{i+1}, v\in A^i$, then for $\phi=(u,v)$ we have: $s(\phi)=(-1)^{|v|}v^{-1}u\in A^1$, $t(\phi)=uv^{-1}\in A^1$, which means that in the case of a DG-algebra, the characters may not be equal to 0 only on morphisms between elements from A^1.

□

Now, let us recall the definition of an isomorphism of DG-algebras.

Definition 4.3. An isomorphism $f\colon\ A_1\to A_2$ between two DG-algebras (A_1,d_1) and (A_2,d_2) is an algebra isomorphism such that following properties hold:

1. f respects the grading, $f(A_1^i)\subset A_2^i$;

2. for all $a\in A_1$, $f(d_1(a))=d_2(f(a))$.

Now we describe the structures of DG-algebras up to isomorphism per Definition 4.3.

Let $X_i, i=1,2$ be two matrices (infinity ones, if G is an infinite group) in following sense:
$$X_i: G\times G\to\mathbb{C},$$
$$(x,h)\mapsto\chi_i(x,h).$$

Theorem 4.4. *Two DG-algebras* $(\mathbb{C}[G], d_1)$ *and* $(\mathbb{C}[G], d_2)$ *are isomorphic iff there is a matrix* C *such that*

1. *matrices* X_1 *and* X_2 *are conjugate,* $CX_2 = X_1C$;
2. *the mapping* f *is in* $\mathrm{Aut}(\mathbb{C}[G])$.

In other words, that means that sets of constants c_h^g, gives $f(g) = \sum\limits_{h\in G} c_h^g h \in \mathrm{Aut}(\mathbb{C}[G])$ and

$$\sum_{h\in G} c_h^g \chi_2(x,h) = \sum_{h\in G} \chi_1(h,g) c_x^h, \forall g, x \in G, \tag{9}$$

where χ_1 and χ_2 denotes the characters that define the derivations d_1 and d_2.

Proof. Suppose that we have an isomorphism of DG-algebras $f : (\mathbb{C}[G], d_1) \to (\mathbb{C}[G], d_2)$, given on generators by the formula $f(g) = \sum\limits_{h\in G} c_h^g h$. By definition, we have

$$f(d_1(g)) = d_2(f(g)).$$

Then

$$f(d_1(g)) = f\Big(\sum_{h\in G} \chi_1(h,g)h\Big) = \sum_{h\in G} \chi_1(h,g) f(h) = \sum_{h\in G} \chi_1(h,g)\Big(\sum_{x\in G} c_x^h x\Big) = \Big(\sum_{h\in G} \chi_1(h,g) c_x^h\Big)x;$$

$$d_2(f(g)) = d_2\Big(\sum_{h\in G} c_h^g h\Big) = \sum_{h\in G} c_h^g d_2(h) = \sum_{h\in G} c_h^g\big(\sum_{x\in G} \chi_2(x,h)x\big) = \Big(\sum_{h\in G} c_h^g \chi_2(x,h)\Big)x.$$

Whence we get our equality (9). To prove the converse, just define $f(g) = \sum\limits_{h\in G} c_h^g h$ and for the same reasons, get the necessary equalities. ☐

Corollary 4.5. *In the case when* $f \in \mathrm{Aut(G)}$ *the equality* (9) *takes the form*

$$\chi_1(h,g) = \chi_2(f(h), f(g)).$$

That is, all characters giving isomorphic structures on a group algebra differ by the action of the automorphism f on the groupoid Γ.

Example 4.6. In a group algebra $\mathbb{C}[G] \cong \bigoplus A^i$ such that $A^1 \neq 0$, and given $a \in Z(G) \cap A^1$, the inner derivation generated by the element a according to the formula from Proposition 3.1 defines a DG-algebra structure on $\mathbb{C}[G]$. Note that the identity element of the group lies in A^0.

Proof. We have to show that $d^2 = 0$ for such a derivation. We know from Theorem 4.2 that this condition is equivalent to

$$\sum_{h\in G} \chi(h,g)\chi(x,h) = 0; \forall g, x \in G.$$

Consider the product $\chi(h,g)\chi(x,h)$. By the definition of an inner derivation, this product is not equal to 0 if and only if (h,g) and $(x,h) \in Hom(a,-a)$ or $Hom(-a,a)$.

Let $s(h,g) = s(x,h) = a$ and $t(h,g) = t(x,h) = -a$, then $(-1)^{|g|}g^{-1}h = (-1)^{|h|}h^{-1}x$ and $hg^{-1} = xh^{-1}$; but this is impossible because $a \in A^1 \Rightarrow |h| \neq |g|$.

Let $s(h,g) = t(x,h) = a$ and $t(h,g) = s(x,h) = -a$, then $(-1)^{|g|}g^{-1}h = xh^{-1}$ and $hg^{-1} = (-1)^{|h|}h^{-1}x$; but this is similarly impossible because $|h| \neq |g|$.

The remaining two cases are similar. $\square$

Proposition 4.7. *Let $d_z^\tau(g) = \tau(g)gz$ be the central derivation given by the element $z \in A^1$. It defines a DG-algebra structure if and only if $\tau(g)\tau(gz) = 0\ \forall g \in G$.*

Proof. The fact that $d_z^\tau(g) : A^i \to A^{i+1}$ follows from the fact that $z \in A^1$.
It remains to check under which conditions $d_z^\tau(g)^2 = 0$. To do this, we simply write by definition:

$$(d_z^\tau)^2(g) = d_z^\tau(\tau(g)gz) = \tau(g)\tau(gz)gz^2 \Rightarrow (d_z^\tau)^2(g) = 0 \Leftrightarrow \tau(g)\tau(gz) = 0.$$

$\square$

The example below illustrates a non-trivial condition that is implied by the previous Proposition.

Example 4.8. Let $N \triangleleft G$ be a normal subgroup in G and $\exists f : {}^{G}\!/_{N} \to \mathbb{Z}$ an epimorphism. Then $\mathbb{C}[G] \cong \bigoplus_{k\in\mathbb{Z}} \mathbb{C}[\langle w_z^k \rangle N]$, where $z \in Z\left({}^{G}\!/_{N}\right)$ is such that $f(z) = 1$, and $\langle w_g^k \rangle$ is the set of words in which the total degree of z is k. In addition, the central derivation d_z^τ, where

$$\tau(z^k g) = \begin{cases} 1, & k = 2n+1, \\ 0, & k = 2n, \end{cases}$$

defines a DG-algebra structure on $\mathbb{C}[G]$.

Here $Z\left({}^{G}\!/_{N}\right)$ denotes the center of the group ${}^{G}\!/_{N}$.

Proof. It is only necessary to check that the graded Leibniz rule is satisfied. To do this, we can make sure that the given τ is indeed a group character and then take advantage of Proposition 4.7. It is not difficult to do this by simply going through all the possibilities. $\square$

Corollary 4.9. *In the graded group algebra $\mathbb{C}[\mathbb{Z}] \cong \bigoplus \mathbb{C}[x^k]$, where $\mathbb{Z} = \langle x \rangle$, it is possible to define a DG-algebra structure using the central derivation d_x^τ, where*

$$\tau(x^k) = \begin{cases} 1, & k = 2n+1, \\ 0, & k = 2n. \end{cases}$$

Corollary 4.10. *Consider the Heisenberg integer group of 3×3 upper-triangular matrices*

$$H_3 = \langle x, y, z \mid z = xyx^{-1}y^{-1}, xz = zx, yz = zy \rangle.$$

It is well known that any element of this group can be represented as $y^b z^c x^a$. *Then* $f(y^b z^c x^a) = c \in \mathbb{Z}$ *is an epimorphism into a group of integers. From here we get the grading* $\mathbb{C}[H_3] \cong \bigoplus_{k\in\mathbb{Z}} \mathbb{C}[z^k \langle x, y\rangle]$. *And the central derivation* d_z^τ, *constructed similarly to Example 4.8, gives a DG-algebra structure on* $\mathbb{C}[H_3]$.

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