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Elementary divisor rings with Dubrovin-Komarnytsky property*

Victor Bovdi and Bohdan Zabavsky

Abstract. Elementary divisor rings were first introduced by Kaplansky in his seminal work. The purpose of this research is to extend Kaplansky's study of commutative elementary divisor rings to certain classes of associative rings under weaker conditions than commutativity. We introduce two new classes of non-commutative rings: those with the DK-property (Dubrovin–Komarnytsky property) and those with the D-property (Dubrovin property), and investigate the structure of elementary divisor rings within these settings. Our main focus is on non-commutative rings of stable range 1. For such rings, we develop a theory of reduction matrices, which allows us to construct and analyze new families of non-commutative elementary divisor rings. In addition, we introduce the concept of an elementary element in a non-commutative ring. We prove that for Bézout domains of stable range 1 with the DK-property, a ring R is an elementary divisor ring if and only if every nonzero element of R is elementary.

Dedicated to Professor M. A. Salim on the occasion of his 70th birthday.

Contents

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Contact information:

V. Bovdi:

Affiliation: United Arab Emirates University, UAE.

Email: vbovdi@gmail.com

1 Introduction

Let R be an associative (but not necessarily commutative) ring with $1 \neq 0$, let $U(R)$ be the group of units of R, and let $R^{n\times m}$ be the vector space of $n \times m$ matrices over R with $n, m \geq 1$. Let $\mathrm{GL}_n(R)$ be the group of units of the matrix ring $R^{n\times n}$. The matrix $D := \mathrm{diag}(d_1, d_2, \dots, d_s) \in R^{n\times m}$ means a (possibly rectangular) matrix having $d_1, \dots, d_s$ (in which $s := \min(n, m)$) on the main diagonal and zeros elsewhere. By the main diagonal we mean the one beginning at the upper left corner. According to Kaplansky (see [18, p. 465]), a ring R is called an *elementary divisor ring* if for any matrix A over R there exist invertible matrices P and Q of suitable sizes such that

$$PAQ = D := \mathrm{diag}(d_1, \dots, d_k, 0, \dots, 0), \tag{1}$$

in which d_i is a total divisor of d_{i+1}, i.e., $Rd_{i+1}R \subseteq d_iR \cap Rd_i$ for each $i = 1, \dots, k-1$.

The class of elementary divisor rings is contained in the class of Bézout rings (for example, see [18,27,33]), i.e., rings with nonzero unit in which every finitely generated one-sided ideal is a principal one-sided ideal. Note that elementary divisor rings are Hermite rings. Hermite rings are rings in which each 1×2 and 2×1 matrix has a diagonal reduction, i.e., $(a, b)P = (c, 0)$ and $Q(a, b)^T = (d, 0)^T$, where $a, b, c, d \in R$ and $P, Q \in \mathrm{GL}_2(R)$ (see [18, 33]). Elementary divisor rings have been studied by many authors (for example, see [3, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 24, 31]) and an overview of this topic can be found in [12, 18, 26, 27, 33].

It is well known [28] that any finitely presented module over a valuation ring is isomorphic to a direct sum of cyclic modules. Thus, a natural question arises: whether there are other classes of rings that satisfy the above property. In the articles of Kaplansky [18] and Larsen, Lewis and Shores [22], it was shown that the above problem and the problem of reducing an arbitrary matrix to a diagonal form (1) are equivalent over commutative rings. Thus, such a reduction combines the theory of rings and the theory of modules.

In the article [24], the notion of stable range comes to the theory of diagonalizability of matrices from K-theory. Using this notation, one of the classes of rings was characterized in [24], namely the class of regular rings, which is important for solving the problem of reducing an arbitrary matrix to a diagonal form (1). During the last decade, algebraic K-theory has been actively used for the study of elementary divisor rings. The use of invariants as the stable range of rings is of particular importance (for some examples, see [2, 7, 8, 17, 23, 24, 25, 27, 30, 33, 34]). The aim of our study is an attempt to extend the results obtained for commutative rings to the case of non-commutative rings. Even the case of a 2×2 matrix ring over non-commutative rings shows that there are significant difficulties in realizing this project. We have succeeded in obtaining some results with natural restrictions on non-commutative rings. In the general case, as part of his study of elementary divisor rings and related classes of rings, Kaplansky proved that if R is an elementary divisor ring, then every finitely presented R-module is a direct sum of cyclic modules. The results of this article may therefore be of interest for the study of finitely presented modules over certain classes of noncommutative rings.

2 Main Results and relations between them

The *coboundary* of a one-sided ideal I of a ring R is a two-sided ideal which is equal to the intersection of all two-sided ideals which contain I. Note that this definition is left-right symmetric.

A nonzero element $a \in R$ is called a *right (left) duo* if aR (Ra) is a two-sided ideal. Moreover, if $aR = Ra$, then the element a is called *duo*.

A ring R has the *D-property (Dubrovin's property)* [33, p. 33] if for each $a \in R$ there exists $a_* \in R$ such that $RaR = a_*R = Ra_*$ (in other words, the coboundary of R is a principal ideal).

In the sequel, we consider only rings R with the D-property. Examples of such rings are simple rings [33, §4.2], quasi-duo elementary divisor rings [31, Theorem 1], and semi-local semi-prime elementary divisor rings [15, Theorem 1]. Several examples of such rings are given in [6].

Two matrices A and B over a ring R are called *equivalent* over the ring R if there exist invertible matrices P and Q over R of suitable sizes such that $A = PBQ$ and it will be denoted by $A \sim B$.

Let $A = (a_{ij}) \in R^{m\times n}$. The coboundary of the right ideal generated by all elements of the matrix $A = (a_{ij})$ is denoted by A_*, that is $A_* = \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} Ra_{ij}R$.

A ring R has *stable range* 1 if the property $aR + bR = R$ implies $(a + bt)R = R$ for some $t \in R$. Semi-local rings and unit-regular rings [33, p. 45] are examples of rings of stable range 1. Each commutative Bézout ring of stable range 1 is an elementary divisor ring. It is known [30, Theorem 2] that each non-commutative right Bézout ring of stable range 1 is a right Hermitian ring.

Our first result is of a technical nature, but it is actively used in what follows.

Theorem 2.1. *If R is a Bézout ring of stable range 1, then*

(i) for any $A \in R^{2\times 2}$ there exist $z, \gamma, d \in R$ such that $A \sim \begin{bmatrix} z & \gamma \\ d & 0 \end{bmatrix}$ and $RzR = A_*$;

(ii) for any $A \in R^{2\times 2}$ there exist $a, b, c \in R$ such that $A \sim \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ and $RaR = A_*$.

A ring R has the *K-property* (Komarnytsky's property) (see [21]) if each factor of a duo element is a duo element.

Duo-rings and simple rings are rings with K-property. However, a principal ideal domain $\mathbb{H}[x]$ over the classical quaternion divison ring $\mathbb{H}$ contains the element $1+x^2$ which is duo but not evert factor of $1 + x^2 = (1 + ix)(1 - ix)$ is duo.

Of course, in any ring the central elements and invertible elements are always duo. More examples and relations between properties of such elements can be found in [11, 15, 16, 19, 20, 31, 33].

We now introduce the following class of noncommutative rings. A ring is said to have the DK-property (Dubrovin–Komarnytsky property) if it satisfies both the D-property and the K-property.

Our second main result is the following.

Theorem 2.2. *Let R be an elementary divisor ring. If R has the DK-property, then for each matrix A over R there exists a matrix $C := \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_k, 0, \dots, 0)$ such that $A \sim C$, $\varepsilon_1, \dots, \varepsilon_{k-1}$ are duo elements and*

$$R\varepsilon_{i+1}R \subseteq R\varepsilon_i \cap \varepsilon_i R, \qquad (i = 1, \dots, k-1).$$

We present one more class of non-commutative rings associated with the DK-property.

A ring R has the *elementary DK-property* or the (EDK-property) if for each matrix A over R there exist invertible matrices P and Q of suitable sizes such that

$$PAQ = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_k, 0, \dots, 0),$$

in which $R\varepsilon_{i+1}R \subseteq R\varepsilon_i \cap \varepsilon_i R$ for all $i = 1, \dots, k-1$ and $\varepsilon_1, \dots, \varepsilon_{k-1}$ are duo elements of R.

Evidently, elementary divisor rings with the DK-property, simple elementary divisor rings and elementary divisor duo-rings are examples of rings with the EDK-property.

Note that to date we do not know examples of non-commutative rings that are EDK-rings but are not DK-rings. But for PI-rings holds the following.

Theorem 2.3. *A principal ideal domain R is an elementary divisor ring with the DK-property if and only if R is a ring with the EDK-property.*

Our next main result is the following.

Theorem 2.4. *A Hermite ring R has the EDK-property if and only if each $A \in R^{2\times 2}$ is equivalent to $\operatorname{diag}(\varepsilon, a) \in R^{2\times 2}$, in which $a \in R$ and*

$$\text{either} \quad RaR \subseteq \varepsilon R = R\varepsilon \quad \text{or} \quad \varepsilon = 0.$$

The next result extends a well-known Kaplansky's criterion [18, Theorem 5.2, p. 472] for non-commutative Hermite rings.

Theorem 2.5. *A Hermite ring R with the DK-property is an elementary divisor ring if and only if for each $a, b, c \in R$ with the property*

$$RaR + RbR + RcR = R \tag{2}$$

there exist $p, q \in R$ such that $paR + (pb + qc)R = R$.

Let $a \in R$ such that $RaR = R$. There exist $n \in \mathbb{N}$ and $u_1, \dots, u_n, v_1, \dots, v_n \in R$ such that

$$u_1 a v_1 + u_2 a v_2 + \cdots + u_n a v_n = 1. \tag{3}$$

If $n \in \mathbb{N}$ is the minimal number satisfying (3), then $a \in R$ is called *n-simple*. We mostly concentrate on 2-simple elements of the ring R. Recall that an element $a \in R$ is called 2-*simple* if $u_1 a v_1 + u_2 a v_2 = 1$ for some $u_1, u_2, v_1, v_2 \in R$ and $n = 2$ is minimal.

The structure of elements of an elementary divisor domain with the DK-property is presented by the following.

Theorem 2.6. *Let R be an elementary divisor domain. If R has the DK-property, then every $a \in R \setminus \{0\}$ can be written in the form $a = \alpha b = c\alpha$, where $\alpha \in U(R)$ and $b, c \in R$ are 2-simple elements.*

In the case of a ring of stable range 1, we have the following.

Theorem 2.7. *Let R be a ring of stable range 1. If $a \in R$ is a 2-simple element, then*

$$\operatorname{diag}(a, a) \sim \operatorname{diag}(1, \Delta), \qquad (\Delta \in R).$$

Please note that the question still remains open whether a non-commutative ring of stable range 1 will be an elementary divisor ring.

Now we use the following definition. A ring R has the *L-property* if it follows from the condition $RaR = R$ that $a \in U(R)$.

Theorem 2.8. *Let R be a domain with the D-property. If R has an L-property, then R is a duo domain.*

Note that a ring is called a *quasi-duo ring* if every maximal one-sided ideal is a two-sided ideal. Every n-simple element is invertible in a quasi-duo domain; and in a quasi-duo elementary divisor domain the D-property is always satisfied [31, Theorem 1].

As a consequence of the previous theorem we have the following.

Corollary 2.9. *(see [31, Theorems 1 and 2]) A quasi-duo Bézout domain R is an elementary divisor domain if and only if R is a duo domain.*

Let R be a Bézout domain. An element $a \in R \setminus 0$ is called *finite* if each right and left ideal that contains a is principal. In a Bézout domain this condition is equivalent to the a.c.c. for principal right and left ideals which contain the element $a \in R$.

Corollary 2.10. *Let R be a Bézout domain of stable range 1 with the DK-property. If it follows from the condition $RaR = R$ that $a \in R$ is a finite element, then R is an elementary divisor ring.*

In a more general situation as in Theorem 2.6, we have the following.

Theorem 2.11. *Let R be a domain with the D-property. If it follows from the condition $RbR = R$ that $b \in R$ is a finite element, then any element $a \in R \setminus 0$ can be written as*

$$a = \alpha f = \varphi \alpha,$$

in which $\alpha \in R$ is a duo element and $f, \varphi \in R$ are finite elements.

Let R be a Bézout domain with the DK-property. An element $a \in R$ is called *elementary* if $RaR = R$ and for each $b, c \in R$ there exist $p, q \in R$ such that

$$paR + (pb + qc)R = R.$$

Note that invertible and finite elements are elementary elements by [4, Theorem 2].

Our last result is the following.

Theorem 2.12. *Let R be a Bézout domain of stable range 1 with the DK-property. The ring R is an elementary divisor ring if and only if each nonzero element of R is elementary.*

As a consequence we have the following.

Corollary 2.13. *Each quasi-duo elementary divisor domain of stable range 1 is a duo domain.*

3 Proofs

Let $A = (a_{ij}) \in R^{m\times n}$. The coboundary of the right ideal generated by all elements of the matrix $A = (a_{ij})$ is denoted by A_*, that is $A_* = \sum_{i=1}^{m}\sum_{j=1}^{n} Ra_{ij}R$.

We start with the following well-known result.

Lemma 3.1. *If A and B are equivalent matrices over R then $A_* = B_*$.*

Proof. If $A = (a_{ij}) = PBQ$, where $B = (b_{ij})$, then

$$a_{ij} \in \sum_k \sum_s Rb_{ks}R \quad \text{and} \quad b_{ij} \in \sum_i \sum_j Ra_{ij}R.$$

It follows that $\sum_i \sum_j Ra_{ij}R = \sum_k \sum_s Rb_{ks}R$, so $A_* = B_*$. □

We shall freely use the following well-known results.

Lemma 3.2. *The following statements hold:*

(i) each Bézout ring of stable range 1 is Hermite (see [30, Theorem 2]);

(ii) if R is a right Bézout ring of stable range 1, then for any $a, b \in R$ there exist $x, d \in R$ such that $a + bx = d$ and $aR + bR = dR$ ($a + xb = d$ and $Ra + Rb = Rd$, respectively) (see [32, Proposition 6]);

(iii) if $a \in R$ is right duo, then every factor of a is a left factor. Moreover, if a is duo, then every proper factor of $a \in R$ is a proper left factor (see [4, Proposition 2]).

Remark 3.3. Let R be a left Bézout ring of stable range 1. For each $a, b \in R$ there exist $x, y, d \in R$ such that $xa + yb = d$, so $Ra + Rb = Rd$.

Proof of Theorem 2.1. (i) Since R is a Hermite ring (see Lemma 3.2(i)), up to the equivalence of matrices, we can assume that $A = \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix}$, so $x\alpha + \beta = d$ and $R\alpha + R\beta = Rd$ for some $x, d \in R$ by Lemma 3.2(ii) and Remark 3.3. Thus

$$\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix} = \begin{bmatrix} x\alpha + \beta & \gamma \\ \alpha & 0 \end{bmatrix}$$

in which $\alpha = \alpha_0 d$ for some $\alpha_0 \in R$.

Let $\gamma R + dR = zR$ and $\gamma y + d = z$ for some $y \in R$. Evidently,

$$\begin{bmatrix} d & \gamma \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} = \begin{bmatrix} d + \gamma y & \gamma \\ \alpha & 0 \end{bmatrix} = \begin{bmatrix} z & \gamma \\ \alpha & 0 \end{bmatrix},$$

where $d = zt$ and $\gamma = z\gamma_0$ for some $t, \gamma_0 \in R$. It means that $A \sim \begin{bmatrix} z & \gamma \\ d & 0 \end{bmatrix}$, where $d = zt$ and $\gamma = z\gamma_0$. This yields that $RdR \subseteq RzR$ and $R\gamma R \subseteq RzR$, i.e.,

$$RzR + R\gamma R + RdR = RzR.$$

Finally, $A_* = RzR + R\gamma R + RdR = RzR$ by Lemma 3.1.

(ii) For each matrix $A \in R^{2\times 2}$ there exists an equivalent matrix $B = \begin{bmatrix} z & \gamma \\ \delta & 0 \end{bmatrix}$ such that $RzR = A_*$ by Theorem 2.1. If $t \in R$ such that $zR + \gamma R = (z + \gamma t)R$ (see Lemma 3.2(ii)), then

$$\begin{bmatrix} z & \gamma \\ \delta & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} z + \gamma t & \gamma \\ \delta & 0 \end{bmatrix}.$$

Let $z + \gamma t = a$ and $\gamma = as$ for some $s \in R$. Obviously,

$$\begin{bmatrix} z + \gamma t & \gamma \\ \delta & 0 \end{bmatrix} = \begin{bmatrix} a & as \\ \delta & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} a & as \\ \delta & 0 \end{bmatrix} \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ \delta & -\delta s \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}.$$

Hence $RzR = RaR$ because $\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \in GL_2(R)$. Also $RbR \subseteq RaR$, and $RcR \subseteq RaR$. □

Lemma 3.4. *Let R be a right Hermite ring. For each $a, b \in R$ there exist $a_0, b_0, d \in R$ such that $a = da_0$, $b = db_0$ and $a_0R + b_0R = R$.*

Proof. Since R is Hermite, $(a, b)P = (d, 0) \in R^{2\times 2}$ for an invertible $P = \begin{bmatrix} u & * \\ v & * \end{bmatrix}$. If $P^{-1} := \begin{bmatrix} a_0 & b_0 \\ * & * \end{bmatrix}$, then

$$(d, 0)P^{-1} = (d, 0) \begin{bmatrix} a_0 & b_0 \\ * & * \end{bmatrix} = (a, b),$$

so $da_0 = a$, $db_0 = b$ and $a_0u + b_0v = 1$. □

Lemma 3.5. *Each Hermite ring R with the D-property is an elementary divisor ring if and only if every matrix $A = (a_{ij})$ over R with the property $\sum_i \sum_j Ra_{ij}R = R$ has a form (1).*

Proof. Since the proof of the "if" part is obvious, we start with the proof of the "only if" part. Let $Ra_{ij}R = a^*_{ij}R = Ra^*_{ij}R$ for each i, j and for some duo-element $\alpha \in R$ we have

$$\sum_i \sum_j Ra_{ij}R = \sum_i \sum_j a^*_{ij}R = \sum_i \sum_j Ra^*_{ij} = \alpha R = R\alpha.$$

That yields $a_{ij} = \alpha a^0_{ij}$ and $A = \mathrm{diag}(\alpha, \dots, \alpha)A_0$, where $A_0 = (a^0_{ij})$. Since R is a Hermite ring, $\sum_i \sum_j Ra^0_{ij}R = R$ by Lemma 3.4. ◻

Proof of Theorem 2.2. Since R is an elementary divisor ring, we assume that

$$A = \mathrm{diag}(d_1, \dots, d_k, 0, \dots, 0)$$

in which $Rd_{i+1}R \subseteq Rd_i \cap d_iR$ for $i = 1, \dots, k-1$. The ring R has the D-property, so

$$Rd_{i+1}R = d^*_{i+1}R = Rd^*_{i+1},$$

i.e., d^*_{i+1} is a duo element. From $d^*_{i+1}R = Rd^*_{i+1} \subseteq d_iR \cap Rd_i$ and the definition of the K-property we obtain that each d_i is a duo element. ◻

Proof of Theorem 2.3. Since a principal ideal domain is an elementary divisor ring with the D-property [33, p. 5], the proof of the "if" part follows from Theorem 2.2.

Let $a, z \in R$ such that a is duo and z is a divisor of a. Since a is duo, z is a left and right divisor of a by Lemma 3.2(iii). Consequently, $RaR \subseteq zR \cap Rz$ and $A = \begin{bmatrix} z & 0 \\ 0 & a \end{bmatrix}$ has a normal canonical form (1). Since R is an elementary divisor ring with the DK-property, $A \sim \begin{bmatrix} x & 0 \\ 0 & b \end{bmatrix}$ in which $RbR \subset xR = Rx$. Moreover, $R/zR \cong R/xR$ as a right R-module (see [1, Theorem 2.8]) and x is a duo element. Consequently, z is a duo element by [12, Exercise 4.2.16]. ◻

Proof of Theorem 2.4. Taking into account that the elements $\varepsilon_1, \dots, \varepsilon_{k-1}$ are duo in the definition of the EDK-property, the proof is the same as the proof of [18, Theorem 51]. ◻

Proof of Theorem 2.5. Let (2) hold for some $a, b, c \in R$. For the matrix $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ there exist invertible matrices $P := \begin{bmatrix} p & q \\ * & * \end{bmatrix}$ and $Q := \begin{bmatrix} x & * \\ y & * \end{bmatrix}$ such that

$$PAQ = \mathrm{diag}(z, d) \qquad \text{and} \qquad RdR \subseteq zR = Rz.$$

Evidently, $RzR + RdR = RzR = RaR + RbR + RcR = R$. Since $zR = Rz$, then

$$RzR = zR = Rz \quad \text{and} \quad zR = Rz = R,$$

i.e., $z \in U(R)$. On the other hand, we have that $paxz^{-1} + (pb + qc)yz^{-1} = 1$ which we intended to prove in the first place.

The ring R is Hermite, so according to Lemma 3.5 and Theorem 2.4 for the proof of the "only if" part it is enough to prove for the matrices $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, where $a, b, c \in R$ which satisfies (2).

According to the condition of our theorem, there exist $p, q \in R$ such that

$$paR + (pb + qc)R = R.$$

Let $pax + (pb + qc)y = 1$. Since $pR + qR = R$ and $Rx + Ry = R$, then there exist matrices $P := \begin{bmatrix} p & q \\ * & * \end{bmatrix} \in \mathrm{GL}_2(R)$ and $Q := \begin{bmatrix} x & * \\ y & * \end{bmatrix} \in \mathrm{GL}_2(R)$ such that

$$PAQ = \begin{bmatrix} 1 & * \\ * & * \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & \Delta \end{bmatrix}, \qquad (\Delta \in R).$$

□

Proof of Theorem 2.6. For every $a \in R \setminus \{0\}$, we have $RaR = \alpha R = R\alpha$ because R is a ring with the DK-property. This yields $a = \alpha b = c\alpha$ for some $b, c \in R$.

Since R is a domain, $RbR = R$ and $RcR = R$. Using the fact that R is an elementary divisor domain with the DK-property, we have

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} P = Q \begin{bmatrix} z & 0 \\ 0 & d \end{bmatrix} \tag{4}$$

in which $P = (p_{ij}) \in \mathrm{GL}_2(R)$, $Q = (q_{ij}) \in \mathrm{GL}_2(R)$ and $RdR \subseteq zR = Rz$.

Let us show that $d \neq 0$. Indeed, if $d = 0$, then $ap_{12} = 0$ and $ap_{22} = 0$ by (4). Since $a \neq 0$ and R is a domain, $p_{12} = p_{22} = 0$ which is impossible because $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ is invertible. Hence $d \neq 0$ and z is a duo element. Moreover, $z \in U(R)$, because $R = RaR = RzR$. Without loss of generality, we can assume that $z = 1$. From (4), we obtain that

$$ap_{11} = q_{11} \qquad \text{and} \qquad ap_{12} = q_{21}. \tag{5}$$

Taking into account that $Q \in \mathrm{GL}_2(R)$, we deduce that $uq_{11} + vq_{12} = 1$ for some $u, v \in R$, so $uap_{11} + vap_{12} = 1$ by (5). Consequently, a is a 2-simple element. □

Proof of Theorem 2.7. Since the element a is 2-simple, there exist $u_1, u_2, v_1, v_2 \in R$ such that $u_1av_1 + u_2av_2 = 1$. Hence $u_1aR + u_2aR = R$, because R is a ring of stable range 1. It follows that

$$u_1a + u_2at = w_1 \in U(R) \quad (\text{for some}\, t \in R)$$

and $Ra+Rt=R$. Obviously, $xa+t=w_2\in U(R)$ for some $x\in R$. Since $t=w_2-xa$ and $u_1a+u_2at=w_1$, we have that

$$\begin{aligned}u_1a+u_2a(w_2-xa)&=u_1a+u_2aw_2-u_2axa\\&=(u_1-u_2ax)a+u_2aw_2=w_1.\end{aligned}$$

This yields $Ra+Raw_2=R$ and $sa+aw_2=w_3\in U(R)$ for some $s\in R$. Consequently,

$$\begin{aligned}&\begin{bmatrix} s & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} a & 0\\ 0 & a\end{bmatrix}\begin{bmatrix} 1 & 0\\ 0 & w_2\end{bmatrix}\begin{bmatrix} 1 & 0\\ 1 & 1\end{bmatrix}=\\&\qquad=\begin{bmatrix} sa+aw_2 & aw_2\\ a & 0\end{bmatrix}=\begin{bmatrix} w_3 & aw_2\\ a & 0\end{bmatrix}\end{aligned}$$

and $\begin{bmatrix} w_3 & aw_2\\ a & 0\end{bmatrix}\sim \operatorname{diag}(1,\Delta)$ for some $\Delta\in R$. ☐

Now we consider rings of stable range 1 with the D-property. According to Lemma 3.5, all possible diagonal reductions to the form (1) for rings R are conditional on elements $a\in R$ such that $RaR=R$.

Lemma 3.6. *Let R be a domain with the D-property. Each $a\in R\setminus\{0\}$ can be written in the form*

$$a=\alpha b=c\alpha,$$

where α is a duo element and b,c are n-simple elements for some $n\in\mathbb{N}$.

Proof. Let $a\in R\setminus\{0\}$ and $RaR=\alpha R=R\alpha$. It follows that $a=\alpha b=c\alpha$ for some $b,c\in R$, so $\sum_{i=1}^n u_i\alpha b v_i=\alpha$ because $\sum_{i=1}^n u_i a v_i=\alpha$ for some $u_1,\dots,u_n,v_1,\dots,v_n\in R$. Since α is a duo element, $\alpha\sum_{i=1}^n u_i' b v_i=\alpha$, i.e., $\sum_{i=1}^n u_i' b v_i=1$ for some $u_1',\dots,u_n'\in R$. Thus $RbR=R$. The proof of $RcR=R$ is similar. ☐

Proof of Theorem 2.8. Follows from Lemma 3.6. ☐

Proof of Corollary 2.9. Each quasi-duo ring is a ring in which every maximal one-sided ideal is a two-sided ideal. Since any n-simple element is invertible, then the D-property is always satisfied [31, Theorem 1] in a quasi-duo domain and in a quasi-duo elementary divisor domain. ☐

Proof of Corollary 2.10. The ring R is Hermite by [30, Theorem 2]. According to Theorems 2.1 and 2.4, it is sufficient to show our statement for a matrix A of the form $\begin{bmatrix} a & 0\\ b & c\end{bmatrix}$ in which $RaR=R$. Since a is a finite element, it is evident that $A\sim\begin{bmatrix} f & 0\\ 0 & d\end{bmatrix}$ in which $RdR\subseteq fR\cap Rf$ by [29, Theorem 6]. Consequently, R is an elementary divisor ring. ☐

Proof of Theorem 2.11. Follows from Lemma 3.6. ☐

Proof of Theorem 2.12. Let $a, b, c \in R$ such that $a \neq 0$ and $RaR = R$. Since R is an elementary divisor ring with the DK-property, for the matrix $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ there exist invertible matrices $P := \begin{bmatrix} p & q \\ * & * \end{bmatrix}$ and $Q := \begin{bmatrix} u & * \\ v & * \end{bmatrix}$ such that $PAQ = \begin{bmatrix} z & 0 \\ 0 & d \end{bmatrix}$ where $d = 0$ or $RdR \subseteq zR = Rz$ for some $z, d \in R$. Since $a \neq 0$ and R is a domain, the case $d = 0$ is impossible.

Evidently, $RdR \subseteq RzR$ and

$$RzR + RdR = RaR + RbR + RcR = R,$$

so $RzR = R$. Since z is a duo element, we have $z \in U(R)$. Clearly, $z = pau + (pb + qc)v$ and

$$pauz^{-1} + (pb + qc)vz^{-1} = 1,$$

i.e., $paR + (pb + qc)R = R$.

Proof of the "only if" part. Since each Bézout domain is a Hermite ring, it is sufficient to prove our statement for the matrices of the form $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ in which $a \neq 0$ and $RaR = R$ by Lemma 3.6 and Theorem 2.4. According to assumptions, there exist $p, q \in R$ such that

$$paR + (pb + qc)R = R,$$

that is, $pau + (pb + qc)v = 1$, where $u, v \in R$. Since R is Hermite, there exist matrices $P := \begin{bmatrix} p & q \\ * & * \end{bmatrix} \in \mathrm{GL}_2(R)$ and $Q := \begin{bmatrix} u & * \\ v & * \end{bmatrix} \in \mathrm{GL}_2(R)$ such that

$$mPAQ = \begin{bmatrix} 1 & * \\ * & * \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}, \qquad (t \in R).$$

□

References

[1] S. Amitsur. Remarks on principal ideal rings. *Osaka Math. J.*, 15:59–69, 1963.

[2] D. D. Anderson and J. R. Juett. Stable range and almost stable range. *J. Pure Appl. Algebra*, 216(10):2094–2097, 2012.

[3] P. Ara, K. R. Goodearl, K. C. O'Meara, and E. Pardo. Diagonalization of matrices over regular rings. *Linear Algebra Appl.*, 265:147–163, 1997.

[4] R. A. Beauregard. Left Ore principal right ideal domains. *Proc. Amer. Math. Soc.*, 102(3):459–462, 1988.

[5] V. Bovdi and B. Zabavsky. Reduction of matrices over simple Ore domains. *Linear Multilinear Algebra*, 70(4):642–649, 2022.

[6] V. Bovdi and B. Zabavsky. Rings of the right (left) almost stable range 1. *Carpathian Math. Publ.*, 17(2):461–471, 2025.

[7] V. A. Bovdi and V. P. Shchedryk. Commutative Bezout domains of stable range 1.5. *Linear Algebra Appl.*, 568:127–134, 2019.

[8] V. A. Bovdi and V. P. Shchedryk. Adequacy of nonsingular matrices over commutative principal ideal domains. *arXiv:2209.01408, Carpathian Math. Publ.*, accepted for publication:1–13, 2025.

[9] G. Calugareanu, H. F. Pop, and A. Vasiu. Matrix invertible extensions over commutative rings. Part I: General theory. *J. Pure Appl. Algebra*, 229(1):Paper No. 107852, 17, 2025.

[10] H. Chen and M. S. Abdolyousefi. Elementary matrix reduction over Bézout domains. *J. Algebra Appl.*, 18(8):1950141, 12, 2019.

[11] J. S. Cheon, T. K. Kwak, Y. Lee, Z. Piao, and S. J. Yun. Rings with a right duo factor ring by an ideal contained in the center. *Bull. Korean Math. Soc.*, 59(3):529–545, 2022.

[12] P. M. Cohn. *Free rings and their relations.* London Mathematical Society Monographs, No. 2. Academic Press, London-New York, 1971.

[13] F. Dopico, V. Noferini, and I. Zaballa. Rosenbrock's theorem on system matrices over elementary divisor domains. *Linear Algebra Appl.*, 710:10–49, 2025.

[14] N. I. Dubrovin. The projective limit of rings with elementary divisors. *Mat. Sb. (N.S.)*, 119(161)(1):89–95, 1982.

[15] N. I. Dubrovin. On rings with elementary divisors. *Russian Mathematics*, (11):14–20, 85, 1986.

[16] A. I. Gatalevich. On adequate and generalized adequate duo rings, and duo rings of elementary divisors. *Mat. Stud.*, 9(2):115–119, 223, 1998.

[17] J. R. Juett and J. L. Williams. Strongly stable rank and applications to matrix completion. *Comm. Algebra*, 45(9):3967–3985, 2017.

[18] I. Kaplansky. Elementary divisors and modules. *Trans. Amer. Math. Soc.*, 66:464–491, 1949.

[19] N. K. Kim, T. K. Kwak, and Y. Lee. On a generalization of right duo rings. *Bull. Korean Math. Soc.*, 53(3):925–942, 2016.

[20] M. T. Koşan, T.-K. Lee, and Y. Zhou. An intermediate ring between a polynomial ring and a power series ring. *Colloq. Math.*, 130(1):1–17, 2013.

[21] M. Y. Komarnitskii. On rings with almost invariant elementary divisors.(Ukrainian). *Visnyk. LNU.*, 49:21–29, 1998.

[22] M. D. Larsen, W. J. Lewis, and T. S. Shores. Elementary divisor rings and finitely presented modules. *Trans. Amer. Math. Soc.*, 187:231–248, 1974.

[23] W. McGovern. Bézout rings with almost stable range 1. *J. Pure Appl. Algebra*, 212(2):340–348, 2008.

[24] P. Menal and J. Moncasi. On regular rings with stable range 2. *J. Pure Appl. Algebra*, 24(1):25–40, 1982.

[25] R. Mortini and R. Rupp. The Bézout properties for some classical function algebras. *Indag. Math. (N.S.)*, 24(1):229–253, 2013.

[26] V. P. Shchedryk. Commutative domains of elementary divisors and some properties of their elements. *Ukrainian Math. J.*, 64(1):140–155, 2012.

[27] V. P. Shchedryk. *Arithmetic of matrices over rings.* Akademperiodyka, Kiyiv, 2021.

[28] R. B. Warfield, Jr. Decomposability of finitely presented modules. *Proc. Amer. Math. Soc.*, 25:167–172, 1970.

[29] B. V. Zabavskii. Noncommutative rings of elementary divisors. *Ukrainian Math. J.*, 39(4):440–444, 541, 1987.

[30] B. V. Zabavskii. Reduction of matrices over Bezout rings of stable rank at most 2. *Ukrainian Math. J.*, 55(4):550–554, 2003.

[31] B. V. Zabavskii and N. Y. Komarnitskii. Distributive domains with elementary divisors. *Ukrainian Math. J.*, 42(7):1000–1004, 1990.

[32] B. Zabavsky. Diagonalizability theorems for matrices over rings with finite stable range. *Algebra Discrete Math.*, (1):151–165, 2005.

[33] B. Zabavsky. *Diagonal reduction of matrices over rings*, volume 16 of *Mathematical Studies Monograph Series.* VNTL Publishers, Lviv, 2012.

[34] B. V. Zabavsky and A. Gatalevych. Diagonal reduction of matrices over commutative semi-hereditary Bezout rings. *Comm. Algebra*, 47(4):1785–1795, 2019.

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