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Deformed homogeneous polynomials and the deformed q-exponential operator

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Abstract. In this paper, we introduce the deformed homogeneous polynomials $\mathrm{R}_n(x, y; u|q)$. These polynomials generalize some classical polynomials: the Rogers-Szegö polynomials $\mathrm{h}_n(x|q)$, the generalized Rogers-Szegö polynomials $\mathrm{r}_n(x, y)$, the Stieltjes-Wigert polynomials $\mathrm{S}_n(x; q)$, among others. Basic properties of the polynomial R_n are given, along with recurrence relations, its q-difference equation, and representations. Generating functions for the polynomials $\mathrm{R}_n(x, y; u|q)$ are given. These functions include generalizations of the Mehler and Rogers formulas. In addition, generalizations of the q-binomial formula and the Heine transformation formula are obtained. These results are obtained via the u-deformed q-exponential operator $\mathrm{E}(yD_q|u)$, defined here. From this operator, we obtain for free the operators $\mathrm{T}(yD_q)$ the Chen, $\mathrm{R}(yD_q)$ of Saad, $\mathcal{E}(yD_q)$ of Exton, and $\mathcal{R}(yD_q)$ of Rogers-Ramanujan when $u = 1, q, \sqrt{q}, q^2$, respectively. We introduce the deformed basic hypergeometric series ${}_r\Phi_s$, a generalization of the classical basic hypergeometric series. New transformation formulas for basic hypergeometric series are obtained.

Contents

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1 Introduction

We begin with some notation and terminology for basic hypergeometric series [1]. The q-shifted factorial is defined by

$$(a;q)_n=\begin{cases}1 & \text{if } n=0;\\ \prod_{k=0}^{n-1}(1-q^ka), & \text{if } n\neq 0,\end{cases}\qquad q\in\mathbb{C},$$

$$(a;q)_\infty=\lim_{n\to\infty}(a;q)_n=\prod_{k=0}^{\infty}(1-aq^k),\qquad |q|<1.$$

The multiple q-shifted factorials are defined by

$$(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,\quad q\in\mathbb{C},$$
$$(a_1,a_2,\ldots,a_m;q)_\infty=(a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty,\quad |q|<1.$$

Some useful identities for q-shifted factorial:

$$(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty},\ aq^n\neq q^{-k},\ k=1,2,3,\ldots,\tag{1}$$
$$(a;q)_{n+k}=(a;q)_n(aq^n;q)_k,\tag{2}$$
$$(a;q)_{2n}=(a;q^2)_n(aq;q^2)_n,\tag{3}$$
$$(a^2;q^2)_n=(a;q)_n(-a;q)_n.\tag{4}$$

In our work, we will use the identities for binomial coefficients:

$$\binom{n+k}{2} = \binom{n}{2} + \binom{k}{2} + nk,$$
$$\binom{n-k}{2} = \binom{n}{2} + \binom{k}{2} + k(1-n).$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

The q-binomial coefficient satisfies

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q, \tag{5}$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^{-n};q)_k}{(q;q)_k} (-q^n)^k q^{-\binom{k}{2}}.$$

The ${}_r\phi_s$ basic hypergeometric series is defined by

$${}_r\phi_s \left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q;q)_n (b_1, b_2, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n.$$

In this paper, we will frequently use the q-binomial theorem:

$${}_1\phi_0 \left(\begin{array}{c} a \\ - \end{array} ; q, -z \right) = \frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n. \tag{6}$$

The q-exponential $\mathrm{e}_q(z)$ is defined by

$$\mathrm{e}_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = {}_1\phi_0 \left(\begin{array}{c} 0 \\ - \end{array} ; q, z \right) = \frac{1}{(z;q)_\infty}.$$

Another q-analogue of the classical exponential function is

$$\mathrm{E}_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{(q;q)_n} = {}_0\phi_0 \left(\begin{array}{c} - \\ - \end{array} ; q, -z \right) = (-z;q)_\infty.$$

For all $u \in \mathbb{C}$, we define the following function, which we will call the deformed q-exponential function,

$$\mathrm{e}_q(z,u) = \begin{cases} \sum_{n=0}^{\infty} u^{\binom{n}{2}} \frac{z^n}{(q;q)_n} & \text{if } u \neq 0; \\ 1 + \frac{z}{1-q} & \text{if } u = 0. \end{cases}$$

Some deformed q-exponential functions are:

$$\begin{aligned}
\mathrm{e}_q(z,1)&=e_q(z)=\frac{1}{(z;q)_\infty},\ |z|<1,\\
\mathrm{e}_q(-z,q)&=\mathrm{E}_q(-z)=(z;q)_\infty,\ z\in\mathbb{C},\\
e_q(z,\sqrt{q})&=\mathcal{E}_q(z)=\sum_{n=0}^{\infty}q^{\frac{1}{2}\binom{n}{2}}\frac{z^n}{(q;q)_n}={}_1\phi_1\left(\begin{matrix}0\\-\sqrt{q}\end{matrix};\sqrt{q},-z\right),\ z\in\mathbb{C},\\
\mathrm{e}_q(qz,q^2)&=\mathcal{R}_q(z)=\sum_{n=0}^{\infty}q^{n^2}\frac{z^n}{(q;q)_n},\ z\in\mathbb{C},
\end{aligned}$$

where $\mathcal{E}_q(z)$ is the Exton q-exponential function and $\mathcal{R}_q(z)$ is the Rogers-Ramanujan function. The function $\mathrm{e}_q(z,u)$ is an u-deformed q-exponential function since when $u\mapsto 1$, then $\mathrm{e}_q(z,u)\mapsto\mathrm{e}_q(z)$ and since $\mathrm{e}_q(z)$ is a q-deformation of the exponential function e^z, then the function $\mathrm{e}_q(z,u)$ defines a double deformation of the exponential function.

Associated to the deformed q-exponential function $\mathrm{e}_q(z,u)$ we have the deformed homogeneous polynomials $\mathrm{R}_n(x,y;u|q)$ defined by

$$\mathrm{R}_n(x,y;u|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q u^{\binom{k}{2}}x^{n-k}y^k, \tag{7}$$

whose generating function is

$$\mathrm{e}_q(xt)\,\mathrm{e}_q(yt,u)=\sum_{n=0}^{\infty}\mathrm{R}_n(x,y;u|q)\frac{t^n}{(q;q)_n}. \tag{8}$$

The polynomials $\mathrm{R}_n(x,y;u|q)$ are a generalization of classical polynomials. For example:

- If $x=1$, $y=x$, and $u=1$, then

  $$\mathrm{R}_n(1,x;1|q)=\mathrm{h}_n(x|q), \tag{9}$$

  where $\mathrm{h}_n(x|q)$ is the classical Rogers-Szegö polynomial of degree n. The Rogers-Szegö polynomials play an important role in the theory of the orthogonal polynomials [8] and physics [3].

- If $u=1$, then

  $$\mathrm{R}_n(x,y;1|q)=\mathrm{r}_n(x,y), \tag{10}$$

  where $\mathrm{r}_n(x,y)$ is a generalization of the Rogers-Szegö polynomials given by Saad et al. [6].

- If $y\mapsto -y$ and $u=q$, then

  $$\mathrm{R}_n(x,-y;q|q)=\mathrm{P}_n(x,y), \tag{11}$$

  where $\mathrm{P}_n(x,y)$ are the Cauchy polynomials

  $$\mathrm{P}_n(x,y)=(x-y)(x-qy)(x-q^2y)\cdots(x-q^{n-1}y).$$

- If $x = 1$, $y = qx$, and $u = q^2$, then

$$\mathrm{R}_n(1, qx; q^2|q) = (q;q)_n \, \mathrm{S}_n(x;q) = {}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix} ; q, -q^{n+1}x \right), \tag{12}$$

where $\mathrm{S}_n(x;q)$ are the Stieltjes-Wigert polynomials.

In this paper, we give basic properties of the polynomials R_n, their q-difference equation, representations, and recurrence relations. Five types of generating functions for the polynomials $\mathrm{R}_n(x, y; u|q)$ are obtained

$$\sum_{n=0}^{\infty} v^{\binom{n}{2}} \, \mathrm{R}_n(x, y; u|q) \frac{t^n}{(q;q)_n}, \text{ (Deformed generating function)}$$

$$\sum_{n=0}^{\infty} \mathrm{R}_n(x, y; u|q) \frac{(a;q)_n}{(q;q)_n} t^n, \text{ (Srivastava-Agarwal type)}$$

$$\sum_{n=0}^{\infty} \mathrm{R}_n(x, y; u|q) \frac{(z;q)_n t^n}{(az;q)_n (q;q)_n},$$

$$\sum_{n=0}^{\infty} \mathrm{R}_n(x, y; u|q) \, \mathrm{R}_n(z, w; v|q) \frac{t^n}{(q;q)_n}, \text{ (Mehler type)}$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathrm{R}_{n+m}(x, y; u|q) \frac{v^{\binom{n}{2}} w^{\binom{m}{2}} t^n}{(q;q)_n (q;q)_m}, \text{ (Rogers type)}$$

and many special cases are given. These generating functions are expressed in terms of the deformed basic hypergeometric series ${}_r\Phi_s$ defined by

$$\begin{aligned} {}_r\Phi_s&\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, u, z \right) \\ &= \sum_{n=0}^{\infty} u^{\binom{n}{2}} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n. \end{aligned} \tag{13}$$

The paper is divided into the following sections: Section 2 discusses the basic identities of the q-differential operator. In Section 3, the deformed basic hypergeometric series ${}_r\Phi_s$ is introduced. As we will note later, these series are closely related to the polynomials $\mathrm{R}_n(x, y; u, v|q)$. In this section, we will give the equation in q-difference of the ${}_2\Phi_1$-series. Section 4 introduces the deformed homogeneous polynomials. Basic properties, q-difference equations, basic hypergeometric representations, and recurrence relations are given. Section 5 introduces the deformed q-exponential operator and provides some identities for this operator. In Section 6, generating functions for $\mathrm{R}_n(x, y; u|q)$ are given: generalized q-binomial theorem, generalization of Heine's transformation formula, Mehler's, Srivastava-Agarwal, and Rogers type formulas. New transformation formulas for basic hypergeometric series are given.

2 The q-differential operator

The q-differential operator D_q is defined by:

$$D_q f(x) = \frac{f(x) - f(qx)}{x} \tag{14}$$

and the Leibniz rule for D_q is

$$D_q^n\{f(x)g(x)\} = \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^k\{f(x)\} D_q^{n-k}\{g(q^k x)\}. \tag{15}$$

Then

$$D_q^n x^k = \frac{(q;q)_k}{(q;q)_{k-n}} x^{k-n}.$$

From Eq. (14), we have identities for the deformed q-exponential function:

$$\mathrm{e}_q(z,u) - \mathrm{e}_q(qz,u) - z\,\mathrm{e}_q(uz,u) = 0. \tag{16}$$

$$D_q^k\{\mathrm{e}_q(ax,u)\} = a^k u^{\binom{k}{2}}\,\mathrm{e}_q(au^k x,u). \tag{17}$$

According to the Leibniz formula and from Eqs. (15) and (16),

$$\begin{aligned} &D_q^n\{\mathrm{e}_q(ax,u)\,\mathrm{e}_q(bx,v)\} \\ &\qquad= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q u^{\binom{k}{2}} v^{\binom{n-k}{2}} a^k b^{n-k}\,\mathrm{e}_q\left(au^k x,u\right) \mathrm{e}_q\left(bq^k v^{n-k} x,v\right). \end{aligned} \tag{18}$$

If $u = v = q$ in Eq. (18), then

$$D_q^n\{(ax,bx;q)_\infty\} = q^{\binom{n}{2}}(ax,bq^n x;q)_\infty \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} \frac{a^k b^{n-k}}{(ax;q)_k}. \tag{19}$$

If $u = q$ and $v = 1$ in Eq. (18), then

$$D_q^n\left\{\frac{(-ax;q)_\infty}{(bx;q)_\infty}\right\} = \frac{(-ax;q)_\infty}{(bx;q)_\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} a^k b^{n-k} \frac{(bx;q)_k}{(-ax;q)_k}. \tag{20}$$

If $u = v = 1$ in Eq. (18), then

$$D_q^n\left\{\frac{1}{(ax,bx;q)_\infty}\right\} = \frac{1}{(ax,bx;q)_\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k b^{n-k} (bx;q)_k. \tag{21}$$

3 Deformed basic hypergeometric series

Definition 3.1. Set $0<|q|<1$ and take $u\in\mathbb{C}$. We define the u-deformed basic hypergeometric series ${}_r\Phi_s$ as

$$\begin{aligned}&{}_r\Phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,u,z\right)\\ &\qquad=\sum_{n=0}^{\infty}u^{\binom{n}{2}}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_s;q)_n}\left[(-1)^nq^{\binom{n}{2}}\right]^{1+s-r}z^n.\end{aligned}\tag{22}$$

By letting that u take values $1,q,q^2$ or $\sqrt{q}$, we obtain the following basic hypergeometric series:

- If $u=1$, then ${}_r\Phi_s={}_r\phi_s$.
- If $u=q$,
$$ {}_{r+1}\Phi_r\left(\begin{matrix}a_1,a_2,\ldots,a_r,0\\ b_1,\ldots,b_r\end{matrix};q,q,z\right)={}_r\phi_r\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_r\end{matrix};q,-z\right)\tag{23}$$
for all $z\in\mathbb{C}$.
- If $u=q^2$ and mapping $z\mapsto qz$,
$$ {}_{r+1}\Phi_r\left(\begin{matrix}a_1,\ldots,a_{r+1}\\ b_1,\ldots,b_r\end{matrix};q,q^2,qz\right)={}_{r+1}\phi_{r+2}\left(\begin{matrix}a_1,\ldots,a_{r+1}\\ b_1,\ldots,b_r,0,0\end{matrix};q,qz\right)\tag{24}$$
for all $z\in\mathbb{C}$.
- If $u=\sqrt{q}$,
$$\begin{aligned}&{}_{r+1}\Phi_r\left(\begin{matrix}a_1,\ldots,a_{r+1}\\ b_1,\ldots,b_r\end{matrix};q,\sqrt{q},z\right)\\ &\qquad={}_{2r+2}\phi_{2r+2}\left(\begin{matrix}\sqrt{a_1},-\sqrt{a_1}&,\ldots,&\sqrt{a_{r+1}},-\sqrt{a_{r+1}}\\ \sqrt{b_1},-\sqrt{b_1}&,\ldots,&\sqrt{b_r},-\sqrt{b_r},-\sqrt{q},0\end{matrix};\sqrt{q},-z\right)\end{aligned}\tag{25}$$
for all $z\in\mathbb{C}$.

Therefore, a deformed basic hypergeometric series generalizes the basic hypergeometric series. A series $\sum_{n=0}^{\infty}v_n$ is a deformed basic hypergeometric series if the quotient v_{n+1}/v_n is a rational function of q^n for a fixed base q and if it is proportional to u^n. The most general form of the quotient is

$$\frac{v_{n+1}}{v_n}=u^n\frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-q^{n+1})(1-b_1q^n)\cdots(1-b_sq^n)}(-q^n)^{1+s-r}z$$

normalizing $v_0=1$.

Theorem 3.2. *Some convergence conditions for the ${}_r\Phi_s$-series are*

- $1+s-r\neq 0$. *If* $0<|uq^{1+s-r}|<1$, *then* ${}_r\Phi_s$ *is an entire function. If* $|uq^{1+s-r}|=1$, *then* ${}_r\Phi_s$ *converges for* $|z|<1$. *If* $|uq^{1+s-r}|>1$, *then* ${}_r\Phi_s$ *is divergent.*
- $1+s-r=0$. *If* $0<|u|<1$, *then* ${}_r\Phi_s$ *is an entire function. If* $|u|=1$, *then* ${}_r\Phi_s$ *converges for* $|z|<1$. *If* $|u|>1$, *then* ${}_r\Phi_s$ *is divergent.*

The deformed q-exponential function has the following representation in u-deformed basic hypergeometric series

$$\mathrm{e}_q(z,u)={}_1\Phi_0\left(\begin{array}{c}0\\ -\end{array};q,u,z\right). \tag{26}$$

The deformed basic hypergeometric series ${}_2\Phi_1$ is

$${}_2\Phi_1\left(\begin{array}{c}a,b\\ c\end{array};q,u,z\right)=\sum_{n=0}^{\infty}u^{\binom{n}{2}}\frac{(a,b;q)_n}{(c;q)_n(q;q)_n}z^n. \tag{27}$$

The q-difference operator applied to the ${}_2\Phi_1$-series:

$$D_q^n{}_2\Phi_1\left(\begin{array}{c}a,b\\ c\end{array};q,u,z\right)=u^{\binom{n}{2}}\frac{(a,b;q)_n}{(c;q)_n}{}_2\Phi_1\left(\begin{array}{c}aq^n,bq^n\\ cq^n\end{array};q,u,u^nz\right) \tag{28}$$

which can be checked directly.

Theorem 3.3. *The deformed basic hypergeometric series ${}_2\Phi_1$, Eq. (27), satisfies the q-difference equation*

$$\begin{aligned}&czD_q^2f(z)-abqz^2D_q^2f(uz)+(1-c)D_qf(z)\\&\qquad+\big[(1-a)(1-b)-(1-abq)\big]zD_qf(uz)-(1-a)(1-b)f(uz)=0.\end{aligned} \tag{29}$$

Proof. Suppose that $f(x)=\sum_{n=0}^{\infty}v_nz^n$ is the solution of Eq. (29). Then

$$\begin{aligned}&czD_q^2f(z)-abqz^2D_q^2f(uz)+(1-c)D_qf(z)\\&\qquad+\big[(1-a)(1-b)-(1-abq)\big]zD_qf(uz)-(1-a)(1-b)f(uz)\\&=c\sum_{n=0}^{\infty}v_n(1-q^n)(1-q^{n-1})z^{n-1}-abq\sum_{n=0}^{\infty}v_n(1-q^n)(1-q^{n-1})u^nz^n\\&\qquad+(1-c)\sum_{n=0}^{\infty}v_n(1-q^n)z^{n-1}+\big[(1-a)(1-b)-(1-abq)\big]\sum_{n=0}^{\infty}v_n(1-q^n)u^nz^n\\&\qquad-(1-a)(1-b)\sum_{n=0}^{\infty}v_nu^nz^n=0,\end{aligned}$$

Therefore,

$$\begin{aligned}
&v_{n+1}\\
&=\frac{u^n[abq(1-q^n)(1-q^{n-1})-((1-a)(1-b)-(1-abq))(1-q^n)+(1-a)(1-b)]}{c(1-q^{n+1})(1-q^n)+(1-c)(1-q^{n+1})}v_n\\
&=u^n\frac{(1-aq^n)(1-bq^n)}{(1-q^{n+1})(1-cq^n)}v_n.
\end{aligned}$$

Then

$$v_n=\prod_{k=0}^{n-1}u^k\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}=u^{\binom{n}{2}}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n},$$

normalizing $v_0=1$. □

Replacing a,b,c with q^a, q^b, q^c in Eq. (29) and then dividing by $(1-q)^2$, and taking formal limits, shows that Eq. (29) tends to the functional-differential equation

$$zf''(z)-z^2f''(uz)+cf'(z)-(a+b+1)zf'(uz)-abf(uz)=0 \tag{30}$$

as $q\to 1$. Equation (30) has the solution

$${}_2\mathrm{F}_1\left(\begin{matrix}a,b\\c\end{matrix};u,z\right)=\sum_{n=0}^{\infty}u^{\binom{n}{2}}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}. \tag{31}$$

When $u=1$, we recover the classical hypergeometric function.

4 Deformed homogeneous polynomials

4.1 Definition and basic properties

Definition 4.1. We define the (u,v)-deformed homogeneous polynomial as

$$\mathrm{R}_n(x,y;u,v|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q u^{\binom{n-k}{2}}v^{\binom{k}{2}}x^{n-k}y^k. \tag{32}$$

Some specializations of $\mathrm{R}_n(x,y;u|q)$ are:

$$\mathrm{R}_n(x,y;1,1|q)=\mathrm{r}_n(x,y), \tag{33}$$

$$\mathrm{R}_n(1,x;q,q|q)=\mathrm{h}_n(x|q^{-1})=q^{\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-n)}x^k, \tag{34}$$

$$\mathrm{R}_n(1,-x;1,q|q)=(x;q)_n, \tag{35}$$

$$\mathrm{R}_n(1,qx;1,q^2|q)=\mathrm{S}_n^*(x;q)=(q;q)_n\mathrm{S}_n(x;q). \tag{36}$$

We define two new polynomials: The homogeneous Stieltjes-Wigert polynomials

$$\mathrm{S}_n(x,y;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{k^2}x^{n-k}y^k \tag{37}$$

and the Exton polynomials

$$\mathrm{E}_n(x,y;q)=\mathrm{R}_n(x,y;1,\sqrt{q}|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \sqrt{q}^{\binom{k}{2}}x^{n-k}y^k \tag{38}$$

$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{\sqrt{q}}\frac{(-\sqrt{q};\sqrt{q})_n}{(-\sqrt{q};\sqrt{q})_k(-\sqrt{q};\sqrt{q})_{n-k}}\sqrt{q}^{\binom{n}{2}}x^{n-k}y^k. \tag{39}$$

As we will see below, the Stieltjes-Wigert and Exton polynomials are related to the Rogers-Ramanujan and Exton q-exponential functions, respectively. As

$$\mathrm{R}_n(x,y;u,v|q)=u^{\binom{n}{2}}\,\mathrm{R}_n(x,u^{1-n}y;1,uv|q), \tag{40}$$

then we will deal with u-deformed homogeneous polynomials

$$\mathrm{R}_n(x,y;u|q)\equiv\mathrm{R}_n(x,y;1,u|q). \tag{41}$$

4.2 Recurrence relations

Theorem 4.2. *The polynomials* $\mathrm{R}_n(x,y;u|q)$ *satisfy the following recursion relations*

$$\mathrm{R}_{n+1}(x,y;u|q)=x\,\mathrm{R}_n(x,qy;u|q)+y\,\mathrm{R}_n(x,uy;u|q), \tag{42}$$

$$\mathrm{R}_{n+1}(x,y;u|q)=x\,\mathrm{R}_n(x,y;u|q)+y\,\mathrm{R}_n(qx,uy;u|q). \tag{43}$$

Proof. By Eq. (41),

$$\begin{aligned}
&x\,\mathrm{R}_n(x,qy;u|q)+y\,\mathrm{R}_n(x,uy;u|q)\\
&\quad=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^k u^{\binom{k}{2}}x^{n+1-k}y^k+\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q u^{\binom{k+1}{2}}x^{n-k}y^{k+1}\\
&\quad=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^k u^{\binom{k}{2}}x^{n+1-k}y^k+\sum_{k=1}^{n+1}\begin{bmatrix}n\\k-1\end{bmatrix}_q u^{\binom{k}{2}}x^{n+1-k}y^k\\
&\quad=x^{n+1}+u^{\binom{n+1}{2}}y^{n+1}+\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^k u^{\binom{k}{2}}x^{n+1-k}y^k+\sum_{k=1}^{n}\begin{bmatrix}n\\k-1\end{bmatrix}_q u^{\binom{k}{2}}x^{n+1-k}y^k.
\end{aligned}$$

Now, by using Eq. (5) we have,

$$\begin{aligned}
&x\,\mathrm{R}_n(x,qy;u|q)+y\,\mathrm{R}_n(x,uy;u|q)\\
&\quad= x^{n+1}+u^{\binom{n+1}{2}}y^{n+1}+\sum_{k=1}^{n}\left\{\begin{bmatrix}n\\k\end{bmatrix}_q q^k+\begin{bmatrix}n\\k-1\end{bmatrix}_q\right\}u^{\binom{k}{2}}x^{n+1-k}y^k\\
&\quad= x^{n+1}+u^{\binom{n+1}{2}}y^{n+1}+\sum_{k=1}^{n}\begin{bmatrix}n+1\\k\end{bmatrix}_q u^{\binom{k}{2}}x^{n+1-k}y^k\\
&\quad= \sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}_q u^{\binom{k}{2}}x^{n+1-k}y^k\\
&\quad= \mathrm{R}_{n+1}(x,y;u|q).
\end{aligned}$$

The proof is reached. □

By iterating the above theorem, we have the following result.

Theorem 4.3. *For $m \geq 0$,*

$$\mathrm{R}_{n+m}(x,y;u|q)=\sum_{k=0}^{m}\begin{bmatrix}m\\k\end{bmatrix}_q u^{\binom{k}{2}}x^{m-k}y^k\,\mathrm{R}_n(x,q^{m-k}u^ky;u|q). \tag{44}$$

4.3 Deformed q-exponential function as limit of deformed homogeneous polynomials

Theorem 4.4. *If $0<|q|<1$, then*

$$\lim_{n\to\infty}\mathrm{R}_n(1,x;u|q)=\mathrm{e}_q(x,u). \tag{45}$$

Proof. As

$$\lim_{n\to\infty}\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{1}{(q;q)_k},$$

then

$$\lim_{n\to\infty}\mathrm{R}_n(1,x;u|q)=\lim_{n\to\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q u^{\binom{k}{2}}x^k=\sum_{k=0}^{\infty}u^{\binom{k}{2}}\frac{x^k}{(q;q)_k}=\mathrm{e}_q(x,u).$$

□

4.4 Deformed basic hypergeometric representation

Theorem 4.5. *For all $n\geq 0$ and $x\neq 0$,*

$$\mathrm{R}_n(x,y;u|q)=x^n{}_2\Phi_0\left(\begin{array}{c}q^{-n},0\\-\end{array};q,u,q^ny/x\right). \tag{46}$$

Proof.

$$\begin{aligned}x^n{}_2\Phi_0\left(\begin{matrix}q^{-n},0\\ -\end{matrix};q,u,q^ny/x\right)&=x^n\sum_{k=0}^{\infty}u^{\binom{k}{2}}\frac{(q^{-n};q)_k}{(q;q)_k}(-1)^kq^{-\binom{k}{2}}(q^ny/x)^k\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_qu^{\binom{k}{2}}x^{n-k}y^k.\end{aligned}$$

□

The basic hypergeometric representation of polynomials Eqs. (33)-(38) are

$$\begin{aligned}\mathrm{r}_n(x,y)&=x^n{}_2\Phi_0\left(\begin{matrix}q^{-n},0\\ -\end{matrix};q,1,q^ny/x\right)=x^n{}_2\phi_0\left(\begin{matrix}q^{-n},0\\ -\end{matrix};q,q^ny/x\right).\\ \mathrm{h}_n(x|q^{-1})&=q^{\binom{n}{2}}{}_2\Phi_0\left(\begin{matrix}q^{-n},0\\ -\end{matrix};q,q^2,qy\right)=q^{\binom{n}{2}}{}_1\phi_1\left(\begin{matrix}q^{-n},0\\ -\end{matrix};q,qy\right).\\ (x;q)_n&={}_2\Phi_0\left(\begin{matrix}q^{-n},0\\ -\end{matrix};q,q,-q^nx\right)={}_2\phi_1\left(\begin{matrix}q^{-n},0\\ 0\end{matrix};q,q^nx\right).\\ \mathrm{S}_n(x,y;q)&=x^n{}_2\Phi_0\left(\begin{matrix}q^{-n},0\\ -\end{matrix};q,q^2,q^{n+1}y/x\right)=x^n{}_1\phi_1\left(\begin{matrix}q^{-n}\\ 0\end{matrix};q,-q^{n+1}y/x\right).\\ \mathrm{E}_n(x,y;q)&=x^n{}_2\Phi_0\left(\begin{matrix}q^{-n},0\\ -\end{matrix};q,\sqrt{q},q^ny/x\right)\\&=x^n{}_4\phi_2\left(\begin{matrix}q^{-n/2},-q^{-n/2},q,0\\ \sqrt{q},-\sqrt{q}\end{matrix};\sqrt{q},q^ny/x\right).\end{aligned}$$

4.5 The q-difference equation

Theorem 4.6. *The polynomials $y(x)=\mathrm{R}_n(1,x;u|q)$ are solutions of the proportional functional difference equation*

$$(D_qy)(u^{-1}x)+q^{n-1}x(D_qy)(q^{-1}x)=(1-q^n)y(x),\tag{47}$$

with initial value $y(0)=1$.

Proof. As

$$D_q\,\mathrm{R}_n(1,x;u|q)=(1-q^n)\,\mathrm{R}_{n-1}(1,ux;u|q),$$

then the left side of Eq. (47) is equal to

$$\begin{aligned}&(D_qy)(u^{-1}x)+q^{n-1}x(D_qy)(q^{-1}x)\\&\quad=(1-q^n)\,\mathrm{R}_{n-1}(1,x;u|q)+q^{n-1}(1-q^n)x\,\mathrm{R}_{n-1}(1,uq^{-1}x;u|q)\\&\quad=(1-q^n)\left(\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_qu^{\binom{k}{2}}x^k+\sum_{k=1}^{n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_qu^{\binom{k}{2}}q^{n-k}x^k\right)\\&\quad=(1-q^n)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_qu^{\binom{k}{2}}x^k=(1-q^n)\,\mathrm{R}_n(1,x;u|q).\end{aligned}$$

□

Eq. (47) is equivalent to

$$f(u^{-1}x) - f(qu^{-1}x) + q^n u^{-1} x f(q^{-1}x) - u^{-1} x f(x) = 0. \tag{48}$$

5 The deformed q-exponential operator

Definition 5.1. Set $u \in \mathbb{C}$. Define the u-deformed q-exponential operator $\mathrm{E}(yD_q|u)$ by letting

$$\mathrm{E}(yD_q|u) = \sum_{n=0}^{\infty} u^{\binom{n}{2}} \frac{(yD_q)^n}{(q;q)_n}. \tag{49}$$

The u-deformed q-exponential operator $\mathrm{E}(yD_q|u)$ generalizes the operators given by Chen [9]:

$$\mathrm{T}(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n} = \mathrm{E}(bD_q|1), \tag{50}$$

and by Saad et al. [7]:

$$\mathrm{R}(bD_q) = \sum_{n=0}^{\infty} (-1)^n \frac{(bD_q)^n q^{\binom{n}{2}}}{(q;q)_n} = \mathrm{E}(-bD_q|q). \tag{51}$$

When $u = \sqrt{q}$, we define the Exton operator $\mathcal{E}(yD_q)$ as

$$\mathcal{E}(yD_q) = \mathrm{E}(yD_q|\sqrt{q}) = \sum_{n=0}^{\infty} \sqrt{q}^{\binom{n}{2}} \frac{(yD_q)^n}{(q;q)_n} = {}_1\phi_1 \left(\begin{matrix} 0 \\ -\sqrt{q} \end{matrix} ; \sqrt{q}, -yD_q \right). \tag{52}$$

When $u = q^2$ and by the mapping $y \mapsto qy$, we define the Rogers-Ramanujan operator $\mathcal{R}(yD_q)$ as

$$\mathcal{R}(yD_q) = \mathrm{E}(qyD_q|q^2) = \sum_{n=0}^{\infty} q^{n^2} \frac{(yD_q)^n}{(q;q)_n}. \tag{53}$$

The deformed homogeneous polynomials $\mathrm{R}_n(x, y; u|q)$ can be represented by the deformed q-exponential operator $\mathrm{E}(yD_q|u)$ as follows:

Proposition 5.2. *For all $u, v \in \mathbb{C}$*

$$\mathrm{E}(yD_q|u)\{x^n\} = \mathrm{R}_n(x, y; u|q). \tag{54}$$

Proof. By applying the deformed q-exponential operator, we get

$$\begin{aligned} \mathrm{E}(yD_q|u)\{x^n\} &= \sum_{k=0}^{\infty} u^{\binom{k}{2}} \frac{y^k}{(q;q)_k} D_q^k\{x^n\} \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q u^{\binom{k}{2}} x^{n-k} y^k \\ &= \mathrm{R}_n(x, y; u|q), \end{aligned}$$

which implies the statement. ∎

Below, we have some q-operator identities.

Theorem 5.3. *If $u, v \in \mathbb{C}$, $u \neq 0$, then*

$$\mathrm{E}(yD_q|u)\big\{\, \mathrm{e}_q(ax,v)\big\} = \sum_{k=0}^{\infty} (uv)^{\binom{k}{2}} \frac{(ay)^k}{(q;q)_k} \,\mathrm{e}_q\left(av^k x, v\right). \tag{55}$$

Proof. From Eq. (16),

$$\begin{aligned}\mathrm{E}(yD_q|u)\big\{\, \mathrm{e}_q(ax,v)\big\} &= \sum_{k=0}^{\infty} \frac{u^{\binom{k}{2}} y^k}{(q;q)_k} D_q^k \{\mathrm{e}_q(ax,v)\} \\ &= \sum_{k=0}^{\infty} u^{\binom{k}{2}} \frac{y^k}{(q;q)_k} a^k v^{\binom{k}{2}} \,\mathrm{e}_q\left(av^k x, v\right).\end{aligned}$$

The proof is completed. $\square$

If $v = 1$ in Eq. (55), then

$$\mathrm{E}(yD_q|u)\left\{\frac{1}{(ax;q)_\infty}\right\} = \frac{\mathrm{e}_q(ay,u)}{(ax;q)_\infty}, \tag{56}$$

for $|ax| < 1$. If $v = q$ in Eq. (55), then

$$\mathrm{E}(yD_q|u)\{(ax;q)_\infty\} = (ax;q)_\infty \cdot {}_2\Phi_1\left(\begin{array}{c} 0,0 \\ ax \end{array}; q, qu, ay\right), \tag{57}$$

for $|ax| < 1$.

Theorem 5.4. *If $u, v, w \in \mathbb{C}$, $u \neq 0$, then*

$$\begin{aligned}&\mathrm{E}(yD_q|u)\{\mathrm{e}_q(ax,v)\,\mathrm{e}_q(bx,w)\} \\ &\qquad = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (uv)^{\binom{k}{2}} (uw)^{\binom{n}{2}} \frac{(ay)^k}{(q;q)_k} \frac{(u^k by)^n}{(q;q)_n} \,\mathrm{e}_q(av^k x, v)\,\mathrm{e}_q(bq^k w^n x, w).\end{aligned} \tag{58}$$

Proof.

$$\begin{aligned}&\mathrm{E}(yD_q|u)\{\mathrm{e}_q(ax,v)\,\mathrm{e}_q(bx,w)\} \\ &\quad = \sum_{n=0}^{\infty} u^{\binom{n}{2}} \frac{y^n}{(q;q)_n} D_q^n\{\mathrm{e}_q(ax,v)\,\mathrm{e}_q(bx,w)\} \\ &\quad = \sum_{n=0}^{\infty} u^{\binom{n}{2}} \frac{y^n}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k v^{\binom{k}{2}} b^{n-k} w^{\binom{n-k}{2}} \,\mathrm{e}_q(av^k x, v)\,\mathrm{e}_q(bq^k w^{n-k} x, w) \\ &\quad = \sum_{k=0}^{\infty} (uv)^{\binom{k}{2}} \frac{(ay)^k}{(q;q)_k} \,\mathrm{e}_q(av^k x, v) \sum_{n=0}^{\infty} \frac{(uw)^{\binom{n}{2}} (u^k by)^n}{(q;q)_n} \,\mathrm{e}_q(bq^k w^n x, w) \\ &\quad = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (uv)^{\binom{k}{2}} (uw)^{\binom{n}{2}} \frac{(ay)^k}{(q;q)_k} \frac{(u^k by)^n}{(q;q)_n} \,\mathrm{e}_q(av^k x, v)\,\mathrm{e}_q(bq^k w^n x, w).\end{aligned}$$

$\square$

We have some specializations of the above theorem:

Corollary 5.5. *If $v = w = 1$, then*

$$\mathrm{E}(yD_q|u)\left\{\frac{1}{(ax,bx;q)_\infty}\right\} = \frac{1}{(ax,bx;q)_\infty}\sum_{k=0}^{\infty} u^{\binom{k}{2}}\frac{(bx;q)_k(ay)^k}{(q;q)_k}\,\mathrm{e}_q(u^k by,u). \tag{59}$$

Corollary 5.6. *If $v = 1$, $w = q$ and setting $b \mapsto -b$, then*

$$\mathrm{E}(yD_q|u)\left\{\frac{(bx;q)_\infty}{(ax;q)_\infty}\right\} = \frac{(bx;q)_\infty}{(ax;q)_\infty}\sum_{k=0}^{\infty}\frac{u^{\binom{k}{2}}(ay)^k}{(q;q)_k(bx;q)_k}{}_1\Phi_1\left(\begin{matrix}0\\ bxq^k\end{matrix};q,u,u^k by\right). \tag{60}$$

Corollary 5.7. *If $v = q$, $w = 1$ and setting $a \mapsto -a$, then*

$$\mathrm{E}(yD_q|u)\left\{\frac{(ax;q)_\infty}{(bx;q)_\infty}\right\} = \frac{(ax;q)_\infty}{(bx;q)_\infty}\sum_{k=0}^{\infty}\frac{(uq)^{\binom{k}{2}}(-ay)^k}{(q;q)_k(ax;q)_k}\,\mathrm{e}_q(u^k by,u). \tag{61}$$

Theorem 5.8.

$$\mathrm{E}(yD_q|u)\left\{\frac{(ax;q)_\infty}{(bx;q)_\infty}\right\} = \frac{(ax;q)_\infty}{(bx;q)_\infty}{}_2\Phi_1\left(\begin{matrix}a/b,0\\ ax\end{matrix};q,u,by\right). \tag{62}$$

Proof.

$$\begin{aligned}
\mathrm{E}(yD_q|u)\left\{\frac{(ax;q)_\infty}{(bx;q)_\infty}\right\} &= \sum_{n=0}^{\infty}u^{\binom{n}{2}}\frac{y^n}{(q;q)_n}D_q^n\left\{\frac{(ax;q)_\infty}{(bx;q)_\infty}\right\}\\
&= \sum_{n=0}^{\infty}u^{\binom{n}{2}}\frac{y^n}{(q;q)_n}D_q^n\left\{{}_1\phi_0\left(\begin{matrix}a/b\\ -\end{matrix};q,bx\right)\right\}\\
&= \sum_{n=0}^{\infty}u^{\binom{n}{2}}\frac{(by)^n}{(q;q)_n}(a/b;q)_n\cdot{}_1\phi_0\left(\begin{matrix}aq^n/b\\ -\end{matrix};q,bx\right)\\
&= \sum_{n=0}^{\infty}u^{\binom{n}{2}}\frac{(a/b;q)_n}{(q;q)_n}(by)^n\frac{(aq^nx;q)_\infty}{(bx;q)_\infty}\\
&= \frac{(ax;q)_\infty}{(bx;q)_\infty}{}_2\Phi_1\left(\begin{matrix}a/b,0\\ ax\end{matrix};q,u,by\right).
\end{aligned}$$

$\square$

Corollary 5.9. *If $v = w = q$ and setting $a \mapsto -a$ and $b \mapsto -b$, then*

$$\begin{aligned}
&\mathrm{E}(yD_q|u)\{(ax,bx;q)_\infty\}\\
&\quad= (ax,bx;q)_\infty\sum_{k=0}^{\infty}\frac{(qu)^{\binom{k}{2}}(-ay)^k}{(ax;q)_k(bx;q)_k(q;q)_k}{}_1\Phi_1\left(\begin{matrix}0\\ bxq^k\end{matrix};q,u,u^k by\right).
\end{aligned} \tag{63}$$

6 Generating functions for $\mathrm{R}_n(x, y; u|q)$

6.1 Generalized q-binomial theorem

Theorem 6.1.

$$\sum_{n=0}^{\infty} v^{\binom{n}{2}} \mathrm{R}_n(x, y; u|q) \frac{z^n}{(q; q)_n} = \sum_{k=0}^{\infty} (uv)^{\binom{k}{2}} \frac{(yz)^k}{(q; q)_k} \mathrm{e}_q\left(v^k xz, v\right). \tag{64}$$

Proof. From Theorem 5.3

$$\begin{aligned}\sum_{n=0}^{\infty} v^{\binom{n}{2}} \mathrm{R}_n(x, y; u|q) \frac{z^n}{(q; q)_n} &= \sum_{n=0}^{\infty} v^{\binom{n}{2}} \mathrm{E}(yD_q|u) \left\{\frac{(xz)^n}{(q; q)_n}\right\} \\ &= \mathrm{E}(yD_q|u) \left\{\sum_{n=0}^{\infty} v^{\binom{n}{2}} \frac{(xz)^n}{(q; q)_n}\right\} \\ &= \mathrm{E}(yD_q|u) \left\{\mathrm{e}_q(xz, v)\right\} \\ &= \sum_{k=0}^{\infty} (uv)^{\binom{k}{2}} \frac{(yz)^k}{(q; q)_k} \mathrm{e}_q\left(v^k xz, v\right).\end{aligned}$$

□

Corollary 6.2. *Set $|yz| < 1$ and $|v| < 1$. Then*

$$\sum_{n=0}^{\infty} v^{\binom{n}{2}} \mathrm{R}_n(x, y; v^{-1}|q) \frac{z^n}{(q; q)_n} = \sum_{n=0}^{\infty} v^{\binom{n}{2}} \frac{(xz)^n}{(q; q)_n} \frac{1}{(v^n yz; q)_\infty}. \tag{65}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} v^{\binom{n}{2}} \mathrm{R}_n(x, y; v^{-1}|q) \frac{z^n}{(q; q)_n} &= \sum_{k=0}^{\infty} \frac{(yz)^k}{(q; q)_k} \mathrm{e}_q\left(v^k xz, v\right) \\ &= \sum_{k=0}^{\infty} \frac{(yz)^k}{(q; q)_k} \sum_{n=0}^{\infty} v^{\binom{n}{2}} \frac{(v^k xz)^n}{(q; q)_n} \\ &= \sum_{n=0}^{\infty} v^{\binom{n}{2}} \frac{(xz)^n}{(q; q)_n} \sum_{k=0}^{\infty} \frac{(v^n yz)^k}{(q; q)_k} \\ &= \sum_{n=0}^{\infty} v^{\binom{n}{2}} \frac{(xz)^n}{(q; q)_n} \frac{1}{(v^n yz; q)_\infty}.\end{aligned}$$

□

From Corollary 6.2 we have the following results:

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}} \mathrm{R}_n(x, y; q^{-1}|q) \frac{z^n}{(q; q)_n} = \frac{1}{(yz; q)_\infty} {}_1\phi_1\left(\begin{matrix} yz \\ 0 \end{matrix}; q, -xz\right). \tag{66}$$

Corollary 6.3. *Set $v = q^2$, $u = q^{-2}$, and $z \mapsto qz$ in Theorem 6.1. Then*

$$\sum_{n=0}^{\infty} q^{n^2}\,\mathrm{R}_n(x,y;q^{-2}|q)\frac{z^n}{(q;q)_n} = \frac{1}{(yz;q)_\infty}\,{}_4\phi_5\left(\begin{matrix}\sqrt{yz}, -\sqrt{yz}, \sqrt{yzq}, -\sqrt{yzq}\\ 0,0,0,0,0\end{matrix};q,q^2xz\right). \quad (67)$$

Proof. Set $v = q^2$, $u = q^{-2}$, and by mapping $z \mapsto qz$, then

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{n^2}\,\mathrm{R}_n(x,y;q^{-2}|q)\frac{z^n}{(q;q)_n}\\
&= \sum_{m=0}^{\infty} q^{m^2}\frac{(xz)^m}{(q;q)_m}\frac{1}{(q^{2m}yz;q)_\infty}\\
&= \frac{1}{(yz;q)_\infty}\sum_{m=0}^{\infty} q^{m^2}\frac{(xz)^m(yz;q)_{2m}}{(q;q)_m}\\
&= \frac{1}{(yz;q)_\infty}\sum_{m=0}^{\infty} q^{m^2}\frac{(xz)^m(yz;q^2)_m(yzq;q^2)_m}{(q;q)_m}\\
&= \frac{1}{(yz;q)_\infty}\sum_{m=0}^{\infty} q^{m^2}\frac{(xz)^m(\sqrt{yz};q)_m(-\sqrt{yz};q)_m(\sqrt{yzq};q)_m(-\sqrt{yzq};q)_m}{(q;q)_m}\\
&= \frac{1}{(yz;q)_\infty}\,{}_4\phi_5\left(\begin{matrix}\sqrt{yz}, -\sqrt{yz}, \sqrt{yzq}, -\sqrt{yzq}\\ 0,0,0,0,0\end{matrix};q,qxz\right).
\end{aligned}$$

□

Corollary 6.4. *Set $v = \sqrt{q}$ and $u = \sqrt{q^{-1}}$ in Theorem 6.1. Then*

$$\begin{aligned}
\sum_{n=0}^{\infty} \sqrt{q}^{\binom{n}{2}}\,\mathrm{R}_n(x,y;\sqrt{q^{-1}}|q)\frac{z^n}{(q;q)_n}
&= \frac{1}{(yz;q)_\infty}\,{}_3\phi_4\left(\begin{matrix}0,0,0\\ \sqrt{q}, -\sqrt{q}, -q, yz\end{matrix};q,\sqrt{q}x^2z^2\right)\\
&\quad + \frac{xz}{(1-q)(\sqrt{q}yz;q)_\infty}\,{}_3\phi_4\left(\begin{matrix}0,0,0\\ q\sqrt{q}, -q\sqrt{q}, -q, \sqrt{q}yz\end{matrix};q,q\sqrt{q}x^2z^2\right). \quad (68)
\end{aligned}$$

Proof. Set $v=\sqrt{q}$, $u=\sqrt{q^{-1}}$. Then

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sqrt{q}^{\binom{n}{2}}\,\mathrm{R}_n(x,y;\sqrt{q^{-1}}|q)\frac{z^n}{(q;q)_n}\\
&=\sum_{m=0}^{\infty}\sqrt{q}^{\binom{m}{2}}\frac{(xz)^m}{(q;q)_m}\frac{1}{(q^{m/2}yz;q)_\infty}\\
&=\sum_{m=0}^{\infty}\sqrt{q}^{\binom{2m}{2}}\frac{(xz)^{2m}}{(q;q)_{2m}}\frac{1}{(q^{m}yz;q)_\infty}+\sum_{m=0}^{\infty}\sqrt{q}^{\binom{2m+1}{2}}\frac{(xz)^{2m+1}}{(q;q)_{2m+1}}\frac{1}{(q^{m+1/2}yz;q)_\infty}\\
&=\frac{1}{(yz;q)_\infty}\sum_{m=0}^{\infty}\sqrt{q}^{\binom{2m}{2}}\frac{(xz)^{2m}}{(q;q)_{2m}}(yz;q)_m\\
&\qquad\qquad+\frac{1}{(\sqrt{q}yz;q)_\infty}\sum_{m=0}^{\infty}\sqrt{q}^{\binom{2m+1}{2}}\frac{(xz)^{2m+1}}{(q;q)_{2m+1}}(\sqrt{q}yz;q)_m.
\end{aligned}$$

Now by using the identities Eq. (3) and Eq. (4),

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sqrt{q}^{\binom{n}{2}}\,\mathrm{R}_n(x,y;\sqrt{q^{-1}}|q)\frac{z^n}{(q;q)_n}\\
&=\frac{1}{(yz;q)_\infty}\sum_{m=0}^{\infty}\sqrt{q}^{\binom{2m}{2}}\frac{(yz;q)_m(xz)^{2m}}{(\sqrt{q};q)_m(-\sqrt{q};q)_m(-q;q)_m(q;q)_m}\\
&\quad+\frac{1}{(1-q)(\sqrt{q}yz;q)_\infty}\sum_{m=0}^{\infty}\sqrt{q}^{\binom{2m+1}{2}}\frac{(\sqrt{q}yz;q)_m(xz)^{2m+1}}{(-q;q)_m(q;q)_m(-q\sqrt{q};q)_m(q\sqrt{q};q)_m}\\
&=\frac{1}{(yz;q)_\infty}{}_2\phi_3\left(\begin{matrix}yz,0\\ \sqrt{q},-\sqrt{q},-q\end{matrix};q,\sqrt{q}x^2z^2\right)\\
&\qquad\qquad+\frac{xz}{(1-q)(\sqrt{q}yz;q)_\infty}{}_2\phi_3\left(\begin{matrix}\sqrt{q}yz,0\\ q\sqrt{q},-q\sqrt{q},-q\end{matrix};q,q\sqrt{q}x^2z^2\right).
\end{aligned}$$

□

Corollary 6.5 (Generalized q-binomial theorem)**.** *If $v=1$ in Theorem 6.1, then*

$$\sum_{n=0}^{\infty}\mathrm{R}_n(x,y;u|q)\frac{t^n}{(q;q)_n}=\frac{\mathrm{e}_q(yt,u)}{(xt;q)_\infty}. \tag{69}$$

Eq. (69) is a generalization of the q-binomial theorem, and Eq. (6) and Theorem 6.1 provide a very straightforward proof for this. On the other hand,

- If $x=1$, $y=x$, and $u=1$ in Eq. (69), then

$$\sum_{n=0}^{\infty}\mathrm{h}_n(x|q)\frac{t^n}{(q;q)_n}=\frac{1}{(t,xt;q)_\infty},\ \max\{|t|,|xt|\}<1.$$

- If $y \mapsto -y$ and $u = q$ in Eq. (69), then

$$\sum_{n=0}^{\infty} \mathrm{P}_n(x,y)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \ |xt|<1.$$

- If $u = q^2$, $y \mapsto qy$, and $y \neq 0$ in Eq. (69), and if $yz = 1$, then

$$\sum_{n=0}^{\infty} \mathrm{S}_n(x,y^{-1};q)\frac{y^n}{(q;q)_n} = \frac{\mathcal{R}_q(1)}{(xy^{-1};q)_\infty} = \frac{1}{(xy^{-1};q)_\infty(q;q^5)_\infty(q^4;q^5)_\infty}. \tag{70}$$

If $yz = q$, then

$$\sum_{n=0}^{\infty} \mathrm{S}_n(x,qy^{-1};q)\frac{y^n}{(q;q)_n} = \frac{\mathcal{R}_q(q)}{(xy^{-1};q)_\infty} = \frac{1}{(xy^{-1};q)_\infty(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{71}$$

$\mathcal{R}_q(1)$ and $\mathcal{R}_q(q)$ are the Rogers-Ramanujan functions [4, 5]

$$\mathcal{R}_q(1) = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \text{ and } \mathcal{R}_q(q) = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{72}$$

- If $u = \sqrt{q}$ in Eq. (69). Then

$$\sum_{n=0}^{\infty} \mathrm{E}_n(x,y;q)\frac{t^n}{(q;q)_n} = \frac{\mathcal{E}_q(yt)}{(xt;q)_\infty}. \tag{73}$$

Corollary 6.6. *If $v = q$ in Theorem 6.1, then*

$$\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\,\mathrm{R}_n(x,y;u|q)\frac{z^n}{(q;q)_n} = (xz;q)_\infty {}_1\Phi_1\left(\begin{matrix}0\\ xz\end{matrix};q,u,yz\right). \tag{74}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\,\mathrm{R}_n(x,y;u|q)\frac{z^n}{(q;q)_n} &= \sum_{k=0}^{\infty}(uq)^{\binom{k}{2}}\frac{(-yz)^k}{(q;q)_k}\,\mathrm{e}_q\left(-q^k xz,q\right)\\ &= (xz;q)_\infty\sum_{k=0}^{\infty}(uq)^{\binom{k}{2}}\frac{(-yz)^k}{(xz;q)_k(q;q)_k}\\ &= (xz;q)_\infty\cdot {}_1\Phi_1\left(\begin{matrix}0\\ xz\end{matrix};q,u,yz\right).\end{aligned}$$

□

Some specializations of Corollary 6.6 are

- If $u=1$, then

$$\sum_{n=0}^{\infty}(-1)^{n}q^{\binom{n}{2}}\,\mathrm{R}_{n}(x,y)\frac{z^{n}}{(q;q)_{n}}=(xz;q)_{\infty}\cdot{}_{1}\phi_{1}\left(\begin{matrix}0\\ xz\end{matrix};q,yz\right). \tag{75}$$

- If $u=q$, then

$$\sum_{n=0}^{\infty}(-1)^{n}q^{\binom{n}{2}}\,\mathrm{P}_{n}(x,y)\frac{z^{n}}{(q;q)_{n}}=(xz;q)_{\infty}\cdot{}_{0}\phi_{1}\left(\begin{matrix}-\\ xz\end{matrix};q,-yz\right). \tag{76}$$

- If $u=q^{2}$, then

$$\sum_{n=0}^{\infty}(-1)^{n}q^{\binom{n}{2}}\,\mathrm{S}_{n}(x,y;q)\frac{z^{n}}{(q;q)_{n}}=(xz;q)_{\infty}\cdot{}_{0}\phi_{2}\left(\begin{matrix}0\\ xz,0\end{matrix};q,yz\right). \tag{77}$$

- If $u=\sqrt{q}$, then

$$\sum_{n=0}^{\infty}(-1)^{n}q^{\binom{n}{2}}\,\mathrm{E}_{n}(x,y;q)\frac{z^{n}}{(q;q)_{n}}=(xz;q)_{\infty}\cdot{}_{1}\phi_{3}\left(\begin{matrix}0\\ \sqrt{xz},-\sqrt{xz},-\sqrt{q}\end{matrix};\sqrt{q},yz\right). \tag{78}$$

6.2 Generalization of Heine's transformation formula

Heine's transformation formula is

$${}_{2}\phi_{1}\left(\begin{matrix}a,b\\ c\end{matrix};q,z\right)=\frac{(b,az;q)_{\infty}}{(c,z;q)_{\infty}}{}_{2}\phi_{1}\left(\begin{matrix}c/b,z\\ az\end{matrix};q,b\right) \tag{79}$$

where $|z|<1$ and $|b|<1$. In this section, we investigate a generalization of Heine's transformation formula via the generating function

$$\sum_{n=0}^{\infty}\mathrm{R}_{n}(x,y;u|q)\frac{(z;q)_{n}}{(az;q)_{n}(q;q)_{n}}t^{n}.$$

Definition 6.7. We define the following representation for the deformed homogeneous polynomials $\mathrm{R}_{n}(x,y;u|q)$ based on the basic hypergeometric series

$$\begin{aligned}&{}_{r}\mathcal{R}_{s}\left(x,y;u;\begin{matrix}a_{1},\ldots,a_{r}\\ b_{1},\ldots,b_{s}\end{matrix};q,t\right)\\&\qquad=\sum_{n=0}^{\infty}\mathrm{R}_{n}(x,y;u|q)\frac{(a_{1},\ldots,a_{r};q)_{n}}{(q;q)_{n}(b_{1},\ldots,b_{s};q)_{n}}\left[(-1)^{n}q^{\binom{n}{2}}\right]^{1+s-r}t^{n}.\end{aligned} \tag{80}$$

For $u = 1, q, q^2, \sqrt{q}$, we have, respectively:

$$
{}_r\mathrm{R}_s\left(x, y;\ \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\ ; q, t\right) = \sum_{n=0}^{\infty} \mathrm{r}_n(x, y) \frac{(a_1, \ldots, a_r; q)_n}{(q; q)_n (b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} t^n. \tag{81}
$$

$$
{}_r\mathrm{P}_s\left(x, y;\ \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\ ; q, t\right) = \sum_{n=0}^{\infty} \mathrm{P}_n(x, y) \frac{(a_1, \ldots, a_r; q)_n}{(q; q)_n (b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} t^n. \tag{82}
$$

$$
{}_r\mathrm{S}_s\left(x, y;\ \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\ ; q, t\right) = \sum_{n=0}^{\infty} \mathrm{S}_n(x, y; q) \frac{(a_1, \ldots, a_r; q)_n}{(q; q)_n (b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} t^n. \tag{83}
$$

$$
{}_r\mathrm{E}_s\left(x, y;\ \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\ ; q, t\right) = \sum_{n=0}^{\infty} \mathrm{E}_n(x, y; q) \frac{(a_1, \ldots, a_r; q)_n}{(q; q)_n (b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} t^n. \tag{84}
$$

From Eqs. (66),

$$
\begin{aligned}
{}_0\mathcal{R}_0\left(x, y; q^{-1};\ \begin{matrix} - \\ - \end{matrix}\ ; q, -z\right) &= \frac{1}{(yz; q)_\infty}\, {}_1\phi_1\left(\begin{matrix} yz \\ 0 \end{matrix}\ ; q, -xz\right). \\
{}_1\mathcal{R}_0\left(x, y; u;\ \begin{matrix} 0 \\ - \end{matrix}\ ; q, z\right) &= \frac{\mathrm{e}_q(yt, u)}{(xt; q)_\infty}. \\
{}_0\mathcal{R}_0\left(x, y; u;\ \begin{matrix} - \\ - \end{matrix}\ ; q, z\right) &= (xz; q)_\infty\, {}_1\Phi_1\left(\begin{matrix} 0 \\ xz \end{matrix}\ ; q, u, yz\right).
\end{aligned}
$$

Theorem 6.8. *We get the following representation for the deformed homogeneous polynomials* $\mathrm{R}_n(x, y; u|q)$

$$
{}_1\mathcal{R}_1\left(x, y; u;\ \begin{matrix} z \\ az \end{matrix}\ ; q, b\right) = \frac{(z; q)_\infty}{(xb, az; q)_\infty} \sum_{n=0}^{\infty} \mathrm{e}_q(ybq^n, u) \frac{(a; q)_n (xb; q)_n}{(q; q)_n} z^n, \tag{85}
$$

where $|b| < 1$ *and* $|z| < 1$.

Proof. From generalized q-binomial theorem

$$\frac{\mathrm{e}_q(ybq^n,u)}{(xbq^n;q)_\infty}=\sum_{m=0}^{\infty}\mathrm{R}_m(x,y;u|q)\frac{(bq^n)^m}{(q;q)_m}. \tag{86}$$

Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathrm{e}_q(ybq^n,u)\frac{(a;q)_n(xb;q)_n}{(q;q)_n}z^n &= (xb;q)_\infty\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\frac{\mathrm{e}_q(ybq^n,u)}{(xbq^n;q)_\infty}z^n\\
&= (xb;q)_\infty\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\sum_{m=0}^{\infty}\mathrm{R}_m(x,y;u|q)\frac{(bq^n)^m}{(q;q)_m}z^n\\
&= (xb;q)_\infty\sum_{m=0}^{\infty}\mathrm{R}_m(x,y;u|q)\frac{b^m}{(q;q)_m}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(zq^m)^n\\
&= (xb;q)_\infty\sum_{m=0}^{\infty}\mathrm{R}_m(x,y;u|q)\frac{b^m}{(q;q)_m}\frac{(azq^m;q)_\infty}{(zq^m;q)_\infty}\\
&= \frac{(az,xb;q)_\infty}{(z;q)_\infty}\sum_{m=0}^{\infty}\mathrm{R}_m(x,y;u|q)\frac{(z;q)_m b^m}{(az;q)_m(q;q)_m}
\end{aligned}$$

□

Corollary 6.9. *From Theorem 6.8 we have the following representation for the generalized Rogers-Szegö polynomials*

$${}_2\mathrm{R}_1\left(x,y;\begin{array}{c}z,0\\az\end{array};q,b\right)=\frac{(z;q)_\infty}{(az,bx,by;q)_\infty}{}_3\phi_2\left(\begin{array}{c}a,bx,by\\0,0\end{array};q,z\right). \tag{87}$$

Proof. Set $u=1$ in Theorem 6.8. Then

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathrm{r}_n(x,y)\frac{(z;q)_n}{(az;q)_n(q;q)_n}b^n &= \frac{(z;q)_\infty}{(bx,by,az;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(bx;q)_n(by;q)_n}{(q;q)_n}z^n\\
&= \frac{(z;q)_\infty}{(az,bx,by;q)_\infty}{}_3\phi_2\left(\begin{array}{c}a,bx,by\\0,0\end{array};q,z\right).
\end{aligned}$$

□

Corollary 6.10. *From Theorem 6.8, we have the following representation for the Cauchy polynomials*

$${}_2\mathrm{P}_1\left(x,y;\begin{array}{c}z,0\\az\end{array};q,b\right)=\frac{(z,by;q)_\infty}{(az,bx;q)_\infty}{}_2\phi_1\left(\begin{array}{c}a,bx\\by\end{array};q,z\right). \tag{88}$$

Proof. Set $u=q$ and $y\mapsto -y$ in Theorem 6.8. Then

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{P}_n(x,y)\frac{(z;q)_n}{(az;q)_n(q;q)_n}b^n &= \frac{(z;q)_\infty}{(bx,az;q)_\infty}\sum_{n=0}^{\infty}(byq^n;q)_\infty\frac{(a;q)_n(bx;q)_n}{(q;q)_n}z^n\\ &= \frac{(z,by;q)_\infty}{(bx,az;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(bx;q)_n}{(by;q)_n(q;q)_n}z^n\\ &= \frac{(z,by;q)_\infty}{(bx,az;q)_\infty}{}_2\phi_1\left(\begin{matrix}a,bx\\ by\end{matrix};q,z\right).\end{aligned}$$

□

Corollary 6.11. *From Theorem 6.8 we have the following representation for the homogeneous Stieltjes-Wigert polynomials*

$$ {}_2\mathrm{S}_1\left(x,y;\begin{matrix}z,0\\ az\end{matrix};q,b\right)=\frac{(z;q)_\infty}{(xb,az;q)_\infty}\sum_{n=0}^{\infty}\mathcal{R}_q(ybq^n)\frac{(a;q)_n(xb;q)_n}{(q;q)_n}z^n. \tag{89}$$

Corollary 6.12. *From Theorem 6.8, we have the following representation for the Exton polynomials*

$$ {}_2\mathrm{E}_1\left(x,y;\begin{matrix}z,0\\ az\end{matrix};q,b\right)=\frac{(z;q)_\infty}{(xb,az;q)_\infty}\sum_{n=0}^{\infty}\mathcal{E}_q(ybq^n)\frac{(a;q)_n(xb;q)_n}{(q;q)_n}z^n, \tag{90}$$

where $|bx|<1$ *and* $|az|<1$.

Theorem 6.13 (Generalized Heine's transformation formula)**.**

$$ {}_1\mathcal{R}_1\left(x,y;u;\begin{matrix}z\\ az\end{matrix};q,b\right)=\frac{(a,bxz;q)_\infty}{(az,bx;q)_\infty}{}_1\mathcal{R}_1\left(a/b,y;u;\begin{matrix}z\\ bxz\end{matrix};q,b\right), \tag{91}$$

where $0<|b|<1$.

Proof. Repeat the proof of the above theorem with

$$\frac{\mathrm{e}_q(ybq^n,u)}{(aq^n;q)_\infty}=\sum_{m=0}^{\infty}\mathrm{R}_m(a/b,y;u|q)\frac{(bq^n)^m}{(q;q)_m}. \tag{92}$$

□

Some specialization of the Theorem 6.13 are

- We get Heine's transformation formula Eq. (79) if we set $u=q$, $x=1$, $y=-c/b$, $z=b$, and $a=b$ in Theorem 6.13.

- $$ {}_1\mathrm{R}_1\left(x,y;\begin{array}{c}z\\az\end{array};q,b\right)=\frac{(a,bxz;q)_\infty}{(az,bx;q)_\infty}{}_1\mathrm{R}_1\left(a/b,y;\begin{array}{c}z\\bxz\end{array};q,b\right) $$
- $$ {}_1\mathrm{P}_1\left(x,y;\begin{array}{c}z\\az\end{array};q,b\right)=\frac{(a,bxz;q)_\infty}{(az,bx;q)_\infty}{}_1\mathrm{P}_1\left(a/b,y;\begin{array}{c}z\\bxz\end{array};q,b\right) $$
- $$ {}_1\mathrm{S}_1\left(x,y;\begin{array}{c}z\\az\end{array};q,b\right)=\frac{(a,bxz;q)_\infty}{(az,bx;q)_\infty}{}_1\mathrm{S}_1\left(a/b,y;\begin{array}{c}z\\bxz\end{array};q,b\right) $$
- $$ {}_1\mathrm{E}_1\left(x,y;\begin{array}{c}z\\az\end{array};q,b\right)=\frac{(a,bxz;q)_\infty}{(az,bx;q)_\infty}{}_1\mathrm{E}_1\left(a/b,y;\begin{array}{c}z\\bxz\end{array};q,b\right) $$

6.3 Mehler's formula for $\mathrm{R}_n(x,y;u|q)$

Theorem 6.14 (Mehler's formula)**.**

$$ \begin{aligned}&\sum_{n=0}^\infty \mathrm{R}_n(x,y;u|q)\,\mathrm{R}_n(z,w;v|q)\frac{t^n}{(q;q)_n}\\&\qquad=\frac{1}{(tzx;q)_\infty}\sum_{k=0}^\infty (uv)^{\binom{k}{2}}\frac{(twy)^k(tzx;q)_k}{(q;q)_k}\,\mathrm{e}_q(twxv^k,v)\,\mathrm{e}_q(tyzu^k,u). \end{aligned} \tag{93} $$

Proof. By Eq. (69),

$$ \begin{aligned}\sum_{n=0}^\infty \mathrm{R}_n(x,y;u|q)\,\mathrm{R}_n(z,w;v|q)\frac{t^n}{(q;q)_n}&=\sum_{n=0}^\infty \mathrm{E}(yD_q|u)\{x^n\}\,\mathrm{R}_n(z,w;v|q)\frac{t^n}{(q;q)_n}\\&=\mathrm{E}(yD_q|u)\left\{\sum_{n=0}^\infty \mathrm{R}_n(z,w;v|q)\frac{(xt)^n}{(q;q)_n}\right\}\\&=\mathrm{E}(yD_q|u)\left\{\frac{\mathrm{e}_q(wxt,v)}{(zxt;q)_\infty}\right\}\\&=\sum_{n=0}^\infty u^{\binom{n}{2}}\frac{y^n}{(q;q)_n}D_q^n\left\{\frac{\mathrm{e}_q(wxt,v)}{(zxt;q)_\infty}\right\}\end{aligned} $$

Now, by applying Leibniz's formula Eq. (15)

$$ \begin{aligned}&\sum_{n=0}^\infty \mathrm{R}_n(x,y;u|q)\,\mathrm{R}_n(z,w;v|q)\frac{t^n}{(q;q)_n}\\&\quad=\frac{1}{(zxt;q)_\infty}\sum_{n=0}^\infty u^{\binom{n}{2}}\frac{(yt)^n}{(q;q)_n}\sum_{k=0}^n \begin{bmatrix}n\\k\end{bmatrix}_q v^{\binom{k}{2}}w^kz^{n-k}(zxt;q)_k\,\mathrm{e}_q(wtxv^k,v)\\&\quad=\frac{1}{(zxt;q)_\infty}\sum_{k=0}^\infty (uv)^{\binom{k}{2}}\frac{(zxt;q)_k(wyt)^k}{(q;q)_k}\,\mathrm{e}_q(wtxv^k,v)\sum_{n=0}^\infty u^{\binom{n}{2}}\frac{(u^kytz)^n}{(q;q)_n}.\end{aligned} $$

This completes the proof. $\square$

- If $u=v=1$, $x=z=1$, $y=x$, $w=y$ in the Theorem 6.14, we obtain the following identity for the Rogers-Szegö polynomials

$$\sum_{n=0}^{\infty}\mathrm{h}_n(x|q)\,\mathrm{h}_n(y|q)\frac{t^n}{(q;q)_n}=\frac{(xyt^2;q)_\infty}{(t,xt,yt,xyt;q)_\infty},$$

where $\max\{|t|,|xt|,|yt|,|xyt|\}<1$.

- If $u=v=1$, we obtain the following identity [6] for generalized the Rogers-Szegö polynomials

$$\sum_{n=0}^{\infty}\mathrm{r}_n(x,y)\,\mathrm{r}_n(z,w)\frac{t^n}{(q;q)_n}=\frac{(xyzwt^2;q)_\infty}{(txz,txw,tyw,tyz;q)_\infty},$$

where $\max\{|txz|,|xwt|,|tyw|,|tyw|\}<1$

- If $u=v=q$, $y\mapsto -y$, and $w\mapsto -w$, we obtain the following identity

$$\sum_{n=0}^{\infty}\mathrm{P}_n(x,y)\,\mathrm{P}_n(z,w)\frac{t^n}{(q;q)_n}=\frac{(twx,tyz;q)_\infty}{(txz;q)_\infty}{}_1\phi_2\left(\begin{matrix}txz\\ txw,tyz\end{matrix};q,tyw\right). \tag{94}$$

- If $u=q^2$, $v=q^2$, and with the maps $y\mapsto qy$ and $w\mapsto qw$, then

$$\sum_{n=0}^{\infty}\mathrm{S}_n(x,y;q)\,\mathrm{S}_n(z,w;q)\frac{t^n}{(q;q)_n}$$
$$=\frac{1}{(tzx;q)_\infty}\sum_{k=0}^{\infty}q^{4k^2}\frac{(tzx;q)_k}{(q;q)_k}(txy)^k\mathcal{R}_q(tyzq^{2k})\mathcal{R}_q(txwq^{2k}). \tag{95}$$

- If $u=v=\sqrt{q}$, then

$$\sum_{n=0}^{\infty}\mathrm{E}_n(x,y;q)\,\mathrm{E}_n(z,w;q)\frac{t^n}{(q;q)_n}$$
$$=\frac{1}{(txz;q)_\infty}\sum_{k=0}^{\infty}q^{\binom{k}{2}}\frac{(txz;q)_k}{(q;q)_k}(twy)^k{}_1\phi_1\left(\begin{matrix}0\\ -\sqrt{q}\end{matrix};q,-txw\sqrt{q}^k.\right)$$
$$\times\,{}_1\phi_1\left(\begin{matrix}0\\ -\sqrt{q}\end{matrix};q,-tyz\sqrt{q}^k\right). \tag{96}$$

Other identities that arise as special cases of Theorem 6.14 are

- If $u=1$, $v=q$, and $w\mapsto -w$, then

$$\sum_{n=0}^{\infty}\mathrm{r}_n(x,y)\,\mathrm{P}_n(z,w)\frac{t^n}{(q;q)_n}=\frac{(twx;q)_\infty}{(tzx,tzy;q)_\infty}{}_1\phi_1\left(\begin{array}{c}tzx\\twx\end{array};q,twy\right). \tag{97}$$

- If $u=1$, $v=q^2$, and $w\mapsto qw$, then

$$\begin{aligned}\sum_{n=0}^{\infty}&\mathrm{r}_n(x,y)\,\mathrm{S}_n(z,w;q)\frac{t^n}{(q;q)_n}\\&=\frac{1}{(tyz,tzw;q)_\infty}\sum_{k=0}^{\infty}q^{k^2}\frac{(tzx;q)_k}{(q;q)_k}(twy)^k\mathcal{R}_q(twxq^{2k}) \qquad (98)\\&=\frac{1}{(tyz,tzw;q)_\infty}\sum_{n=0}^{\infty}q^{n^2}\frac{(twx)^n}{(q;q)_n}{}_1\phi_2\left(\begin{array}{c}tzx\\0,0\end{array};q,twyq^{2n-1}\right). \qquad (99)\end{aligned}$$

- If $u=1$ and $v=\sqrt{q}$, then

$$\begin{aligned}\sum_{n=0}^{\infty}&\mathrm{r}_n(x,y)\,\mathrm{E}_n(z,w;q)\frac{t^n}{(q;q)_n}\\&=\frac{1}{(tyz,tzx;q)_\infty}\sum_{k=0}^{\infty}\sqrt{q}^{\binom{k}{2}}\frac{(tzx;q)_k}{(q;q)_k}(twy)^k{}_1\phi_1\left(\begin{array}{c}0\\-\sqrt{q}\end{array};\sqrt{q},-twxq^{k/2}.\right). \qquad (100)\end{aligned}$$

- If $u=q^2$, $v=q$, and $y\mapsto qy$ and $w\mapsto -w$, then

$$\begin{aligned}\sum_{n=0}^{\infty}&\mathrm{S}_n(x,y;q)\,\mathrm{P}_n(z,w)\frac{t^n}{(q;q)_n}\\&=\frac{(twx;q)_\infty}{(tzx;q)_\infty}\sum_{k=0}^{\infty}(-1)^kq^{\frac{3k^2-k}{2}}\frac{(tzx;q)_k}{(twx;q)_k(q;q)_k}(twy)^k\mathcal{R}(tyzq^{2k}). \qquad (101)\end{aligned}$$

- If $u=q^2$, $v=\sqrt{q}$, and with the map $y\mapsto qy$, then

$$\begin{aligned}\sum_{n=0}^{\infty}&\mathrm{S}_n(x,y;q)\,\mathrm{E}_n(z,w;q)\frac{t^n}{(q;q)_n}\\&=\frac{1}{(tzx;q)_\infty}\sum_{k=0}^{\infty}q^{\frac{5k^2-k}{2}}\frac{(twy)^k(tzx;q)_k}{(q;q)_k}\mathcal{E}_q(twxq^{k/2})\mathcal{R}_q(tyzq^{2k}). \qquad (102)\end{aligned}$$

- If $u=\sqrt{q}$, $v=q$, and $w\mapsto -w$, then

$$\begin{aligned}\sum_{n=0}^{\infty}&\mathrm{E}_n(x,y;q)\,\mathrm{P}_n(z,w)\frac{t^n}{(q;q)_n}\\&=\frac{(twx;q)_\infty}{(tzx;q)_\infty}\sum_{k=0}^{\infty}q^{\frac{3}{2}\binom{k}{2}}\frac{(tzx;q)_k}{(twx;q)_k(q;q)_k}(twy)^k\mathcal{E}_q(tyzq^{k/2}). \qquad (103)\end{aligned}$$

6.4 Srivastava-Agarwal type formulas

Srivastava and Agarwal [2] derived a large number of generating functions of the form

$$\sum_{n=0}^{\infty} Q_n(x;q)\frac{(\lambda;q)_n}{(q;q)_n}t^n, \tag{104}$$

where $Q_n(x;q)$ is a q-polynomial. In this section, we will use Mehler's formula, Theorem 6.14, to derive the Srivastava-Agarwal generating function of the polynomials $\mathrm{R}_n(x,y;u|q)$.

Corollary 6.15. *The representation type Srivastava-Agarwal, Eq. (104), of* $\mathrm{R}_n(x,y;u|q)$ *is*

$$\sum_{n=0}^{\infty} \mathrm{R}_n(x,y;u|q)\frac{(a;q)_n}{(q;q)_n}t^n = \frac{(atx;q)_\infty}{(tx;q)_\infty}\sum_{k=0}^{\infty}(uq)^{\binom{k}{2}}\frac{(-aty)^k(tx;q)_k}{(atx;q)_k(q;q)_k}\,\mathrm{e}_q(tyu^k,u). \tag{105}$$

Proof. Set $v=q$, $z=1$, $w=-a$ in Theorem 6.14. Then

$$\begin{aligned}\sum_{n=0}^{\infty} &\mathrm{R}_n(x,y;u|q)\frac{(a;q)_n}{(q;q)_n}t^n\\ &= \frac{1}{(tzx;q)_\infty}\sum_{k=0}^{\infty}(uq)^{\binom{k}{2}}\frac{(-aty)^k(tzx;q)_k}{(q;q)_k}(atxq^k;q)_\infty\,\mathrm{e}_q(tyzu^k,u)\\ &= \frac{(atx;q)_\infty}{(tzx;q)_\infty}\sum_{k=0}^{\infty}(uq)^{\binom{k}{2}}\frac{(-aty)^k(tzx;q)_k}{(atx;q)_k(q;q)_k}\,\mathrm{e}_q(tyzu^k,u).\end{aligned}$$

□

- If $u=1$, $v=q$, $x=z=1$, $y=x$, $w=-y$ in Corollary 6.15, then we obtain the following result of Srivastava and Agarwal [2]

$$\sum_{n=0}^{\infty} \mathrm{r}_n(x,y)\frac{(a;q)_n}{(q;q)_n}t^n = \frac{(atx;q)_\infty}{(tx,ty;q)_\infty}\,{}_1\phi_1\left(\begin{matrix} tx\\ atx\end{matrix};q,aty\right).$$

- If $u=q$, $v=q^2$, $w\mapsto -w$, $z=1$, and $y\mapsto qy$ in Corollary 6.15, then the representation type Srivastava-Agarwal of $\mathrm{S}_n(x,y;q)$ is

$$\sum_{n=0}^{\infty} \mathrm{S}_n(x,y;q)\frac{(a;q)_n}{(q;q)_n}t^n = \frac{(atx;q)_\infty}{(tx;q)_\infty}\sum_{k=0}^{\infty}q^{\frac{3k^2-k}{2}}\frac{(tx;q)_k(-aty)^k}{(q;q)_k(atx;q)_k}\mathcal{R}_q(tyq^{2k}). \tag{106}$$

- The representation type Srivastava-Agarwal of $\mathrm{E}_n(x,y;q)$ is

$$\begin{aligned}\sum_{n=0}^{\infty} &\mathrm{E}_n(x,y;q)\frac{(a;q)_n}{(q;q)_n}t^n\\ &= \frac{(ty;q)_\infty}{(tx;q)_\infty}\sum_{k=0}^{\infty}(q\sqrt{q})^{\binom{k}{2}}\frac{(tx;q)_k}{(q;q)_k(ty;q)_k}(tay)^k\,{}_1\phi_1\left(\begin{matrix} 0\\ -\sqrt{q}\end{matrix};\sqrt{q},taxq^{k/2}\right).\end{aligned} \tag{107}$$

6.5 Transformation formulas for ${}_2\Phi_1$

Setting $a \mapsto b$ and $b \mapsto a$ in corollaries 5.6 and 5.7, then matching with Theorem 5.8, we obtain, respectively

$$
{}_2\Phi_1\left(\begin{matrix} a/b, 0 \\ ax \end{matrix}; q, u, by\right) = \sum_{k=0}^{\infty} \frac{u^{\binom{k}{2}}(by)^k}{(q;q)_k(ax;q)_k} {}_1\Phi_1\left(\begin{matrix} 0 \\ axq^k \end{matrix}; q, u, u^k ay\right). \tag{108}
$$

$$
= \sum_{k=0}^{\infty} \frac{(uq)^{\binom{k}{2}}(-ay)^k}{(q;q)_k(ax;q)_k} \mathrm{e}_q(u^k by, u). \tag{109}
$$

Corollary 6.16. *From Eq. (108) with $u = 1$, we obtain*

$$
{}_2\phi_1\left(\begin{matrix} a/b, 0 \\ ax \end{matrix}; q, ay\right) = \sum_{k=0}^{\infty} \frac{(by)^k}{(q;q)_k(ax;q)_k} {}_1\phi_1\left(\begin{matrix} 0 \\ axq^k \end{matrix}; q, ay\right). \tag{110}
$$

Corollary 6.17 (Transformation formula for ${}_2\phi_1$)**.** *From Eq. (109) with $u = 1$, we obtain the important transformation formula from ${}_2\phi_1$ sum to ${}_1\phi_1$*

$$
{}_2\phi_1\left(\begin{matrix} a/b, 0 \\ ax \end{matrix}; q, by\right) = \frac{1}{(by;q)_\infty} {}_1\phi_1\left(\begin{matrix} 0 \\ ax \end{matrix}; q, ay\right). \tag{111}
$$

Corollary 6.18. *If we set $u = q$ in Eq. (108), we obtain the identity*

$$
{}_1\phi_1\left(\begin{matrix} a/b \\ ax \end{matrix}; q, -by\right) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}(by)^k}{(q;q)_k(ax;q)_k} {}_0\phi_1\left(\begin{matrix} - \\ axq^k \end{matrix}; q, -q^k ay\right). \tag{112}
$$

Corollary 6.19 (Transformation formula for ${}_1\phi_1$)**.** *From Eq. (109) with $u = q$, we obtain the important transformation formula from ${}_1\phi_1$ sum to ${}_1\phi_2$*

$$
{}_1\phi_1\left(\begin{matrix} a/b \\ ax \end{matrix}; q, -by\right) = (by;q)_\infty {}_1\phi_2\left(\begin{matrix} 0 \\ ax, by \end{matrix}; q, ay\right). \tag{113}
$$

Corollary 6.20. *From Eqs. (108) and (109), with $u = q^2$, and $y \mapsto qy$, we obtain the identities*

$$
{}_1\phi_2\left(\begin{matrix} a/b \\ ax, 0 \end{matrix}; q, qby\right) = \sum_{k=0}^{\infty} \frac{q^{k^2}(by)^k}{(q;q)_k(ax;q)_k} {}_0\phi_2\left(\begin{matrix} - \\ axq^k, 0 \end{matrix}; q, q^{2k} ay\right)
$$

$$
= \sum_{k=0}^{\infty} \frac{q^{\frac{3k^2-k}{2}}(-ay)^k}{(q;q)_k(ax;q)_k} \mathcal{R}_q(q^{2k} by).
$$

Corollary 6.21. *From Eq. (108) with $u = \sqrt{q}$ we obtain*

$$
{}_3\phi_3\left(\begin{matrix} \sqrt{a/b}, -\sqrt{a/b}, 0 \\ \sqrt{ax}, -\sqrt{ax}, -\sqrt{q} \end{matrix}; \sqrt{q}, -by\right)
$$

$$
= \sum_{k=0}^{\infty} \frac{q^{\frac{1}{2}\binom{k}{2}}(by)^k}{(q;q)_k(ax;q)_k} {}_1\phi_3\left(\begin{matrix} 0 \\ \sqrt{axq^k}, -\sqrt{axq^k}, -\sqrt{q} \end{matrix}; \sqrt{q}, q^{k/2} ay\right). \tag{114}
$$

Finally, set $x = z = 1$, $y = x$, and $w = y$ in Eq. (94). Then we have the corollary.

Corollary 6.22 (Transformation formula for ${}_1\phi_2$)**.** *For* $|t| < 1$, *then*

$$ {}_1\phi_2\left(\begin{matrix} t \\ tx, ty \end{matrix}; q, txy\right) = \frac{(t;q)_\infty}{(tx, ty;q)_\infty} {}_2\phi_1\left(\begin{matrix} x, y \\ 0 \end{matrix}; q, t\right). \tag{115} $$

6.6 Rogers type formulas

Theorem 6.23 (Rogers formula)**.**

$$ \sum_{n=0}^\infty \sum_{m=0}^\infty \mathrm{R}_{n+m}(x, y; u|q) \frac{v^{\binom{n}{2}} w^{\binom{m}{2}} t^n s^m}{(q;q)_n (q;q)_m} = \sum_{k=0}^\infty \sum_{n=0}^\infty (uv)^{\binom{k}{2}} (uw)^{\binom{n}{2}} \frac{(ty)^k}{(q;q)_k} \frac{(u^k sy)^n}{(q;q)_n} \mathrm{e}_q(tv^k x, v)\, \mathrm{e}_q(sq^k w^n x, w). \tag{116} $$

Proof.

$$ \begin{aligned} \sum_{n=0}^\infty \sum_{m=0}^\infty \mathrm{R}_{n+m}(x, y; u|q) \frac{v^{\binom{n}{2}} w^{\binom{m}{2}} t^n s^m}{(q;q)_n (q;q)_m} &= \sum_{n=0}^\infty \sum_{m=0}^\infty \mathrm{E}(yD_q|u)\{x^{n+m}\} \frac{v^{\binom{n}{2}} w^{\binom{m}{2}} t^n s^m}{(q;q)_n (q;q)_m} \\ &= \mathrm{E}(yD_q|u) \left\{ \sum_{n=0}^\infty v^{\binom{n}{2}} \frac{(tx)^n}{(q;q)_n} \sum_{m=0}^\infty w^{\binom{m}{2}} \frac{(sx)^m}{(q;q)_m} \right\} \\ &= \sum_{k=0}^\infty \sum_{n=0}^\infty (uv)^{\binom{k}{2}} (uw)^{\binom{n}{2}} \frac{(ty)^k}{(q;q)_k} \frac{(u^k sy)^n}{(q;q)_n} \mathrm{e}_q(tv^k x, v)\, \mathrm{e}_q(sq^k w^n x, w). \end{aligned} $$

□

Corollary 6.24. *If* $v = w = 1$ *in Theorem 6.23, then*

$$ \sum_{n=0}^\infty \sum_{m=0}^\infty \mathrm{R}_{n+m}(x, y; u|q) \frac{t^n s^m}{(q;q)_n (q;q)_m} = \frac{1}{(tx, sx;q)_\infty} \sum_{k=0}^\infty u^{\binom{k}{2}} \frac{(sx;q)_k}{(q;q)_k} (ty)^k\, \mathrm{e}_q(u^k sy, u). \tag{117} $$

- The Rogers formula for $\mathrm{h}_n(x|q)$ is:

$$ \sum_{n=0}^\infty \sum_{m=0}^\infty \mathrm{h}_{n+m}(x|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} = \frac{(xst;q)_\infty}{(t, xt, s, xs;q)_\infty}, \quad \max\{|s|, |t|, |xs|, |xt|\} < 1. $$

- If $u=1$, we obtain the following results in [6]

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathrm{r}_{n+m}(x,y)\frac{t^ns^m}{(q;q)_n(q;q)_m}=\frac{(stxy;q)_\infty}{(tx,sx,sy,ty;q)_\infty}.$$

- If $u=q$, then we get the following result in [7],

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathrm{P}_{n+m}(x,y)\frac{t^ns^m}{(q;q)_n(q;q)_m}=\frac{(sy;q)_\infty}{(tx,sx;q)_\infty}{}_1\phi_1\left(\begin{matrix}sx\\ sy\end{matrix};q,ty\right)$$

where $|tx|<1$, $|sx|<1$.

- If $u=q^2$ and mapping $y\mapsto qy$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathrm{S}_{n+m}(x,y;q)\frac{t^ns^m}{(q;q)_n(q;q)_m}$$
$$=\frac{1}{(tx,sx;q)_\infty}\sum_{k=0}^{\infty}q^{k^2}\frac{(sx;q)_k}{(q;q)_k}(ty)^k\mathcal{R}_q(q^{2k}sy). \tag{118}$$

- If $u=\sqrt{q}$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathrm{E}_{n+m}(x,y;q)\frac{t^ns^m}{(q;q)_n(q;q)_m}$$
$$=\frac{1}{(tx,sx;q)_\infty}\sum_{k=0}^{\infty}\sqrt{q}^{\binom{k}{2}}\frac{(sx;q)_k}{(q;q)_k}(ty)^k\mathcal{E}_q(q^{k/2}sy). \tag{119}$$

Corollary 6.25. *If $v=1,w=q$ in Theorem 6.23, then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\,\mathrm{R}_{n+m}(x,y;u|q)\frac{q^{\binom{m}{2}}t^ns^m}{(q;q)_n(q;q)_m}$$
$$=\frac{(sx;q)_\infty}{(tx;q)_\infty}\sum_{k=0}^{\infty}u^{\binom{k}{2}}\frac{(ty)^k}{(sx;q)_k(q;q)_k}{}_1\Phi_1\left(\begin{matrix}0\\ sxq^k\end{matrix};q,u,u^ksy\right). \tag{120}$$

- If $u=1$ in Corollary 6.25, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\,\mathrm{r}_{n+m}(x,y)\frac{q^{\binom{m}{2}}t^ns^m}{(q;q)_n(q;q)_m}$$
$$=\frac{(sx;q)_\infty}{(tx;q)_\infty}\sum_{k=0}^{\infty}\frac{(ty)^k}{(sx;q)_k(q;q)_k}{}_1\phi_1\left(\begin{matrix}0\\ sxq^k\end{matrix};q,sy\right). \tag{121}$$

- If $u=q$ in Corollary 6.25, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m \mathrm{P}_{n+m}(x,y)\frac{q^{\binom{m}{2}}t^n s^m}{(q;q)_n(q;q)_m}$$
$$=\frac{(sx;q)_\infty}{(tx;q)_\infty}\sum_{k=0}^{\infty}q^{\binom{k}{2}}\frac{(ty)^k}{(sx;q)_k(q;q)_k}{}_0\phi_1\left(\begin{matrix}-\\ sxq^k\end{matrix};q,-q^k sy\right). \tag{122}$$

- If $u=q^2$ and mapping $y\mapsto qy$ in Corollary 6.25, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m \mathrm{S}_{n+m}(x,y;q)\frac{q^{\binom{m}{2}}t^n s^m}{(q;q)_n(q;q)_m}$$
$$=\frac{(sx;q)_\infty}{(tx;q)_\infty}\sum_{k=0}^{\infty}q^{k^2}\frac{(ty)^k}{(sx;q)_k(q;q)_k}{}_0\phi_2\left(\begin{matrix}-\\ sxq^k,0\end{matrix};q,q^{2k+1}sy\right). \tag{123}$$

- If $u=\sqrt{q}$ in Corollary 6.25, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m \mathrm{E}_{n+m}(x,y;q)\frac{q^{\binom{m}{2}}t^n s^m}{(q;q)_n(q;q)_m}$$
$$=\frac{(sx;q)_\infty}{(tx;q)_\infty}\sum_{k=0}^{\infty}\sqrt{q}^{\binom{k}{2}}\frac{(ty)^k}{(sx;q)_k(q;q)_k}{}_1\phi_3\left(\begin{matrix}0\\ \sqrt{sxq^k},-\sqrt{sxq^k},-\sqrt{q}\end{matrix};\sqrt{q},-q^{k/2}sy\right). \tag{124}$$

Corollary 6.26. *If $v=w=q$ in Theorem 6.23, then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m} \mathrm{R}_{n+m}(x,y;u|q)\frac{q^{\binom{n}{2}+\binom{m}{2}}t^n s^m}{(q;q)_n(q;q)_m}$$
$$=(tx,sx;q)_\infty\sum_{k=0}^{\infty}(qu)^{\binom{k}{2}}\frac{(-ty)^k}{(tx;q)_k(sx;q)_k(q;q)_k}{}_1\Phi_1\left(\begin{matrix}0\\ sxq^k\end{matrix};q,u,u^k sy\right). \tag{125}$$

- If $u=1$ in Corollary 6.26, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m} \mathrm{r}_{n+m}(x,y)\frac{q^{\binom{n}{2}+\binom{m}{2}}t^n s^m}{(q;q)_n(q;q)_m}$$
$$=(tx,sx;q)_\infty\sum_{k=0}^{\infty}q^{\binom{k}{2}}\frac{(-ty)^k}{(tx;q)_k(sx;q)_k(q;q)_k}{}_1\phi_1\left(\begin{matrix}0\\ sxq^k\end{matrix};q,sy\right). \tag{126}$$

- If $u=q$ in Corollary 6.26, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\,\mathrm{P}_{n+m}(x,y)\frac{q^{\binom{n}{2}+\binom{m}{2}}t^ns^m}{(q;q)_n(q;q)_m}$$
$$=(tx,sx;q)_\infty\sum_{k=0}^{\infty}q^{k^2}\frac{(-ty)^k}{(tx;q)_k(sx;q)_k(q;q)_k}{}_0\phi_1\left(\begin{matrix}-\\ sxq^k\end{matrix};q,-q^ksy\right). \tag{127}$$

- If $u=q^2$ and mapping $y\mapsto qy$ in Corollary 6.26, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\,\mathrm{S}_{n+m}(x,y;q)\frac{q^{\binom{n}{2}+\binom{m}{2}}t^ns^m}{(q;q)_n(q;q)_m}$$
$$=(tx,sx;q)_\infty\sum_{k=0}^{\infty}q^{\frac{3k^2-k}{2}}\frac{(-ty)^k}{(tx;q)_k(sx;q)_k(q;q)_k}{}_0\phi_2\left(\begin{matrix}-\\ sxq^k,0\end{matrix};q,q^{2k+1}sy\right). \tag{128}$$

- If $u=\sqrt{q}$ in Corollary 6.26, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\,\mathrm{E}_{n+m}(x,y;q)\frac{q^{\binom{n}{2}+\binom{m}{2}}t^ns^m}{(q;q)_n(q;q)_m}$$
$$=(tx,sx;q)_\infty\sum_{k=0}^{\infty}(q\sqrt{q})^{\binom{k}{2}}\frac{(-ty)^k}{(tx;q)_k(sx;q)_k(q;q)_k}$$
$$\times{}_1\phi_3\left(\begin{matrix}0\\ \sqrt{sxq^k},-\sqrt{sxq^k},-\sqrt{q}\end{matrix};\sqrt{q},q^{k/2}sy\right). \tag{129}$$

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