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Communications in Mathematics **34** (2026), no. 1, Paper no. 15
DOI: https://doi.org/10.46298/cm.17612

Integro-derivation Dzhumadildaev algebras: from the algebra of polynomials

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Abstract. This paper introduces and investigates some properties of algebras constructed from the algebra of polynomials via derivation and integration operators using a process presented by Dzhumadildaev in an earlier work. In particular, we discover new classes of infinite-dimensional simple conservative algebras and describe the derivations of these algebras for ranks 1 and 2.

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MSC 2020: 17A30 (primary); 17A36 (secondary).

Keywords: derivation, Rota-Baxter operator, conservative algebra, algebra of polynomials.

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Editorial information:
Received: March 2, 2026
Accepted: March 6, 2026
Communicated by: Pasha Zusmanovich

Acknowledgments: The work is supported by FCT 2023.08031.CEECIND and UID/00212/2025.

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1 Introduction

The idea of obtaining new objects from old ones by using derivative operations has long been known in algebra [2]. In its most general form, the idea was realized by Malcev [21]. Let $\mathrm{M_n}$ be the algebra of matrices of order n over a field. Assume that some finite collection $\Lambda = \big(a_{ij}, b_{ij}, c_{ij}\big)$ of matrices in $\mathrm{M_n}$ is given. Denote by $\mathrm{M_n^{(\Lambda)}}$ an algebra defined on a space of matrices in $\mathrm{M_n}$ with respect to new multiplication

$$x \cdot_{\Lambda} y = \sum_{i,j} a_{ij} x b_{ij} y c_{ij}.$$

It was proved that every n-dimensional algebra is isomorphic to a subalgebra of $\mathrm{M_n^{(\Lambda)}}$ [21]. Other interesting ways to derive the initial multiplication are isotopes, homotopes, and mutations [3,22,25]. The concept of an isotope was introduced by Albert [2]. Let algebras A and $\mathrm{A_0}$ have a common linear space on which right multiplication operators R_x and $R_x^{(0)}$ are defined (for A and $\mathrm{A_0}$, resp.). We say that A and $\mathrm{A_0}$ are isotopic if there exist invertible linear operators ϕ, ψ, ξ such that $R_x^{(0)} = \phi R_{x\psi} \xi$. We call $\mathrm{A_0}$ an isotope of A. Another similar construction appears in the study of Novikov algebras and their generalizations. Namely, let A be an associative commutative algebra with a derivation d, then a new multiplication $x * y = xd(y)$ gives a structure of a Novikov algebra [11]. The present construction was generalized to the case of noncommutative Novikov algebras [24] and δ-Novikov algebras [16]. On the other hand, an associative commutative algebra with a Rota-Baxter operator R gives a structure of a Zinbiel algebra under the multiplication $x * y = xR(y)$ [11]. Recently, in a paper of Dzhumadildaev [11], an idea of mixing the two above-presented constructions was introduced. The present paper is dedicated to the study of non-associative algebras, which we call *integro-derivation Dzhumadildaev* algebras (see, Definition 1), obtained analogously by the construction of Dzhumadildaev given in [11].

The algebra of restricted polynomials (also known as null-filiform associative algebra) is still a subject of interest [1, 10, 15, 19]. The present paper is dedicated to the study of some algebras obtained from the n-dimensional algebra of restricted polynomials (and their unital versions) under a specific multiplication given by derivation-integration operators. Namely, we give the definition of algebras under our consideration in Proposition 1 and define the exact table of multiplications of algebras under our consideration in Definition 5. Proposition 7 gives a characterization of nontrivial algebras under our consideration, and the next subsection 2.2 characterizes simple, perfect, and nilpotent algebras. Subsection 2.3 provides some new examples of simple conservative algebras. The second part of the paper is dedicated to the study of derivations of integro-derivation Dzhumadildaev algebras of ranks 1 and 2. The description of algebras of derivations and their generalizations of associative and non-associative algebras is a classical, but still current problem [4–6, 8, 13, 20].

Notations We do not provide some well-known definitions (such as definitions of Lie algebras, Leibniz algebras, nilpotent algebras, solvable algebras, etc.), and refer the readers to consult previously published papers. For a set of vectors S, we denote by $\langle S\rangle$ the vector space generated by S. For an algebra A and an element $a \in$ A, we denote the right multiplication on a as R_a. In general, we are working in the complex field, but some results are applicable to other fields as well. We also always assume algebras under our consideration are nontrivial, i.e., they have nonzero multiplications.

2 Preliminaries

Definition 1. Let $\mathrm{K}[x]$ be a polynomial algebra (with a unit x^0), generated by x, and let S be a set of elements $\{x^j\}_{j\in J}$, where $J \subseteq \mathbb{N}\cup\{0\}$. Then $(\mathrm{S},\diamond)$ is an integro-derivation Dzhumadildaev algebra of type (n,m)[*], where $n,m\in\mathbb{Z}$, if

$$x^i\diamond x^j=\begin{cases} T^n(x^i)T^m(x^j), & \text{if } T^n(x^i)T^m(x^j) \text{ is linearly dependent with one } x^l, l\in J;\\ 0, & \text{otherwise,}\end{cases}$$

where, for any positive $k>0$, we define as follows

T^k is the kth partial derivation "∂^k", where $\partial(x^i) = ix^{i-1}$;
$T^0 = \mathrm{Id}$, and
T^{-k} is the kth usual integral "$\int\ldots\int$", where $\int(x^i) = \frac{1}{i+1}x^{i+1}$.

Rank of $\mathrm{IDD}(\mathrm{S},n,m)$ is equal to $|n|+|m|$ and denoted as $\mathrm{Rank}\big(\mathrm{IDD}(\mathrm{S},n,m)\big)$.
Level of $\mathrm{IDD}(\mathrm{S},n,m)$ is equal to $n+m$ and denoted as $\mathrm{lev}\big(\mathrm{IDD}(\mathrm{S},n,m)\big)$.

Proposition 2. $\mathrm{IDD}(\mathrm{S},\mathrm{n},\mathrm{m})$ *is opposite to* $\mathrm{IDD}(\mathrm{S},\mathrm{m},\mathrm{n})$. *In particular,* $\mathrm{IDD}(\mathrm{S},\mathrm{n},\mathrm{n})$ *is commutative.*

Proof. It follows from Definition 1 and commutativity of $\mathrm{K}[x]$. □

Below, due to Proposition 2, we will consider algebras $\mathrm{IDD}(\mathrm{S},\mathrm{n},\mathrm{m})$ only with $n\geq m$.

Definition 3. Let $B=\{b_i\}_{i\in I}$ be a basis of an algebra A. It is said that B is a multiplicative basis[†] if for each e_i and e_j from B, we have $e_ie_j\in\langle e_k\rangle$ for some k. A multiplicative basis $B=\{b_i\}_{i\in I}$ of an algebra A is called strong multiplicative if for each e_i and e_j from B, we have $e_ie_j\neq 0$.

[*] We will denote it as IDD(S, n, m).

[†] About algebras with a multiplication basis see [17] and references therein.

Proposition 4. $\mathrm{IDD}(\mathrm{S},\mathrm{n},\mathrm{m})$ *has a multiplicative basis.*

Proof. It follows from Definitions 1 and 3. □

2.1 The multiplication table of IDD algebras from polynomials

Below, we will consider IDD algebras, constructed from the following sets

$$\begin{array}{lcl} K_n^k & := & \{x^k = e_k,\ x^{k+1} = e_{k+1},\ \ldots,\ x^n = e_n\}; \\ K_\infty^k & := & \{x^k = e_k,\ x^{k+1} = e_{k+1},\ \ldots\ \}. \end{array}$$

Let us remember that $\mathrm{IDD}(K_n^1, m_1, m_2)$ $\big($resp., $\mathrm{IDD}(K_n^0, m_1, m_2)\big)$ algebras of rank 0 are $\mathrm{IDD}(K_n^1, 0, 0)$ $\big($resp., $\mathrm{IDD}(K_n^0, 0, 0)\big)$ algebras, which are the well-known algebras of restricted polynomials[‡] $\big($resp., restricted polynomials with unit$\big)$.
Let $\mathfrak{n} \in \mathbb{N} \cup \{\infty\}$. The multiplication tables of algebras of rank 1 are:

$\mathrm{IDD}(K_\mathfrak{n}^0,\ 1,\ 0)$	:	$e_i \diamond e_j$	$=$	$ie_{i+j-1},$	$1 \leq i+j \leq \mathfrak{n}+1$
$\mathrm{IDD}(K_\mathfrak{n}^1,\ 1,\ 0)$	:	$e_i \diamond e_j$	$=$	$ie_{i+j-1},$	$2 \leq i+j \leq \mathfrak{n}+1$
$\mathrm{IDD}(K_\mathfrak{n}^0,\ 0,\ -1)$	:	$e_i \diamond e_j$	$=$	$\frac{1}{j+1}e_{i+j+1},$	$0 \leq i+j \leq \mathfrak{n}-1$
$\mathrm{IDD}(K_\mathfrak{n}^1,\ 0,\ -1)$	:	$e_i \diamond e_j$	$=$	$\frac{1}{j+1}e_{i+j+1},$	$2 \leq i+j \leq \mathfrak{n}-1$

The multiplication tables of algebras of rank 2 are:

$\mathrm{IDD}(K_\mathfrak{n}^0,\ 2,\ 0)$	:	$e_i \diamond e_j$	$=$	$i(i-1)e_{i+j-2},$	$2 \leq i+j \leq \mathfrak{n}+2$
$\mathrm{IDD}(K_\mathfrak{n}^1,\ 2,\ 0)$	:	$e_i \diamond e_j$	$=$	$i(i-1)e_{i+j-2},$	$3 \leq i+j \leq \mathfrak{n}+2$
$\mathrm{IDD}(K_\mathfrak{n}^0,\ 1,\ 1)$	:	$e_i \diamond e_j$	$=$	$ije_{i+j-2},$	$2 \leq i+j \leq \mathfrak{n}+2$
$\mathrm{IDD}(K_\mathfrak{n}^1,\ 1,\ 1)$	:	$e_i \diamond e_j$	$=$	$ije_{i+j-2},$	$3 \leq i+j \leq \mathfrak{n}+2$
$\mathrm{IDD}(K_\mathfrak{n}^0,\ 1,\ -1)$	:	$e_i \diamond e_j$	$=$	$\frac{i}{j+1}e_{i+j},$	$1 \leq i+j \leq \mathfrak{n}$
$\mathrm{IDD}(K_\mathfrak{n}^1,\ 1,\ -1)$	:	$e_i \diamond e_j$	$=$	$\frac{i}{j+1}e_{i+j},$	$2 \leq i+j \leq \mathfrak{n}$
$\mathrm{IDD}(K_\mathfrak{n}^0,\ -1,\ -1)$	:	$e_i \diamond e_j$	$=$	$\frac{1}{(i+1)(j+1)}e_{i+j+2},$	$0 \leq i+j \leq \mathfrak{n}-2$
$\mathrm{IDD}(K_\mathfrak{n}^1,\ -1,\ -1)$	:	$e_i \diamond e_j$	$=$	$\frac{1}{(i+1)(j+1)}e_{i+j+2},$	$2 \leq i+j \leq \mathfrak{n}-2$
$\mathrm{IDD}(K_\mathfrak{n}^0,\ 0,\ -2)$	:	$e_i \diamond e_j$	$=$	$\frac{1}{(j+1)(j+2)}e_{i+j+2},$	$0 \leq i+j \leq \mathfrak{n}-2$
$\mathrm{IDD}(K_\mathfrak{n}^1,\ 0,\ -2)$	:	$e_i \diamond e_j$	$=$	$\frac{1}{(j+1)(j+2)}e_{i+j+2},$	$2 \leq i+j \leq \mathfrak{n}-2$

In the general case, we have the following multiplication table.

Proposition 5. *Let* $l := m_1 + m_2$*, then the multiplication table of* $\mathrm{IDD}(K_\mathfrak{n}^k, m_1, m_2)$ *is given below:*

[‡] Also known as null-filiform associative algebras.

$$\boxed{\begin{array}{rcr} \mathrm{IDD}(K^k_{\mathfrak{n}}, m_1, m_2) & : & e_i \diamond e_j \; = \; \frac{i!}{(i-m_1)!}\frac{j!}{(j-m_2)!}e_{i+j-l}, \\ & & \textit{where } \max\{k, m_1\} \leq i \leq \mathfrak{n}, \\ & & \max\{k, m_2\} \leq j \leq \mathfrak{n}, \\ & & \textit{and } \max\{k, k+l\} \leq i+j \leq \mathfrak{n}+l. \end{array}}$$

Proposition 6. *If $k \geq m_1 \geq m_2$, then $\mathrm{IDD}(K^k_\infty, m_1, m_2)$ has a strong multiplicative basis.*

Proof. It follows from Definitions 3 and Proposition 5. □

It is obvious that each algebra $\mathrm{IDD}(K^k_\infty, m_1, m_2)$ is nontrivial. The finite-dimensional case is more complicated, and our first aim is to identify all nontrivial algebras $\mathrm{IDD}(K^k_n, m_1, m_2)$ for $m_1 \geq m_2$.

Proposition 7. *$\mathrm{IDD}(K^k_n, m_1, m_2)$ is trivial if and only if one of the following conditions is true*

(1) $m_1 > n$;

(2) $n \geq m_1$ *and* $l > 2n - k$;

(3) $n \geq m_1$ *and* $l < 2k - n$.

Proof. Firstly, we consider the case $m_1 > n$. Then $m_1 > 0$ and for each pair (i, j), such that $k \leq i, j \leq n$, we have $T^{m_1}(e_i) = 0$ and $e_i \diamond e_j = T^{m_1}(e_i)T^{m_2}(e_j) = 0$, i.e., $\mathrm{IDD}(K^k_n, m_1, m_2)$ is trivial.

Secondly, if $n \geq m_1$, then for each pair i and j, such that $k \leq i, j \leq n$, we have

$$T^{m_1}(e_i) \in \langle e_{i-m_1} \rangle \text{ and } T^{m_2}(e_j) \in \langle e_{j-m_2} \rangle, \text{ i.e., } T^{m_1}(e_i)T^{m_2}(e_j) \in \langle e_{i+j-l} \rangle.$$

(A) If $l > 2n - k$, then $i + j - l < i + j - 2n + k < k$, i.e., $e_i \diamond e_j = 0$ for $k \leq i, j \leq n$ and $\mathrm{IDD}(K^k_n, m_1, m_2)$ is trivial.

(B) If $l < 2k - n$, then $i + j - l > i + j - 2k + n > n$, i.e., $e_i \diamond e_j = 0$ for $k \leq i, j \leq n$ and $\mathrm{IDD}(K^k_n, m_1, m_2)$ is trivial.

On the other hand, if $2k - n \leq l \leq 2n - k$, we have the following two cases.

(A) If $l = 2n - t$, where $k \leq t \leq n$, then $0 \neq e_n \diamond e_n \in \langle e_t \rangle$, i.e., $\mathrm{IDD}(K^k_n, m_1, m_2)$ is nontrivial.

(B) If $2k - n \leq l < n$, then taking $i = \left[\frac{l+n}{2}\right]$ we have that $0 \neq e_i \diamond e_i \in \langle e_{n-1}, e_n \rangle$, i.e., $\mathrm{IDD}(K^k_n, m_1, m_2)$ is nontrivial. □

From now on, we are only interested in nontrivial IDD algebras, i.e., $\mathrm{IDD}(K^k_{\mathfrak{n}}, m_1, m_2)$, such that

$$\boxed{\infty > \mathfrak{n} \geq m_1 \text{ and } 2k - \mathfrak{n} \leq m_1 + m_2 \leq 2\mathfrak{n} - k, \text{ or } \mathfrak{n} = \infty.}$$

2.2 Simple, perfect and nilpotent IDD algebras

Proposition 8. *If* $n < \mathrm{lev}\big(\mathrm{IDD}(K_n^k, m_1, m_2)\big) \leq 2n-k$, *then* $\mathrm{IDD}(K_n^k, m_1, m_2)$ *is nilpotent.*

Proof. Let us denote $\mathrm{A} := \mathrm{IDD}(K_n^k, m_1, m_2)$ and $l := \mathrm{lev}\big(\mathrm{IDD}(K_n^k, m_1, m_2)\big)$, then

$$\mathrm{A} \diamond \mathrm{A} \subseteq \big\langle e_k, \ldots, e_{n+(n-l)} \big\rangle \subset \mathrm{A};$$
$$\mathrm{A} \diamond \big(\mathrm{A} \diamond \mathrm{A}\big) + \big(\mathrm{A} \diamond \mathrm{A}\big) \diamond \mathrm{A} \subseteq \big\langle e_k, \ldots, e_{n+2(n-l)} \big\rangle \subset \mathrm{A} \diamond \mathrm{A}.$$

Applying mathematical induction, we find that

$$\mathrm{A}^{\diamond t} := \sum_{i+j=t} \mathrm{A}^{\diamond i} \diamond \mathrm{A}^{\diamond j} \subset \big\langle e_k, \ldots, e_{n+(t-1)(n-l)} \big\rangle.$$

Due to $n < l$, there exists n_0, such that $\mathrm{A}^{\diamond n_0} = 0$, i.e., A is nilpotent. □

Proposition 9. *If* $\mathrm{lev}\big(\mathrm{IDD}(K_n^k, m_1, m_2)\big) = n$, *then* $\mathrm{IDD}(K_n^k, m_1, m_2)$ *is non-simple perfect and it has exactly* $n-k$ *proper ideals.*

Proof. Let us denote $\mathrm{A} := \mathrm{IDD}(K_n^k, m_1, m_2)$, then

$$\mathrm{A} \diamond \mathrm{A} \supseteq \big\langle e_k, \ldots, e_n \big\rangle = \mathrm{A}, \text{ i.e., A is perfect.}$$

Let I be an ideal of A and $n_0 \in \mathbb{N}$, such that there exists $\mathfrak{i}_0 = \sum\limits_{i=k}^{n_0} \alpha_i e_i \in I$, where $\alpha_{n_0} \neq 0$, and there are no elements $\mathfrak{i} = \sum\limits_{i=k}^{t} \alpha_i e_i \in I$, where $\alpha_t \neq 0$ and $t > n_0$. It is easy to see that $I = \big\langle e_k, \ldots, e_{n_0} \big\rangle$, i.e., there exists a 1-1 correspondence between the set of ideals of A and the set of numbers $\{k, \ldots, n\}$. Considering that the ideal $I = \big\langle e_k, \ldots, e_n \big\rangle$ is not proper, we have our statement. □

Proposition 10. *If* $0 \leq k < \mathrm{lev}\big(\mathrm{IDD}(K_{\mathfrak{n}}^k, m_1, m_2)\big) < \mathfrak{n}$, *then* $\mathrm{IDD}(K_{\mathfrak{n}}^k, m_1, m_2)$ *is simple.*

Proof. Let I be a nonzero ideal of $\mathrm{A} := \mathrm{IDD}(K_{\mathfrak{n}}^k, m_1, m_2)$ and $l := \mathrm{lev}\big(\mathrm{IDD}(K_{\mathfrak{n}}^k, m_1, m_2)\big)$. Then there exists an element $\mathfrak{i} \in I$, such that $\mathfrak{i} = \sum\limits_{i=k}^{k_0} \alpha_i e_i$, where $k_0 \geq k$ and $\alpha_{k_0} \neq 0$. Hence, $0 \neq \mathfrak{i}\mathrm{R}_{e_{l-1}}^{k_0-k} \in \big\langle e_k \big\rangle$, i.e., $e_k \in I$. The last gives that $\mathrm{A} \diamond \big\langle e_k \big\rangle \subseteq I$, i.e., $\{e_k, \ldots, e_{\mathfrak{n}-l+k}\} \subseteq I$. If $\mathfrak{n} = \infty$, we have our statement. On the other hand, if $\mathfrak{n} < \infty$, we have to mention that $e_{k+1} \in I$ and $0 \neq e_{\mathfrak{n}} \diamond e_{k+1} \in \big\langle e_{\mathfrak{n}-l+k+1} \big\rangle$, i.e., $e_{k+2} \in I$. That gives $0 \neq e_{\mathfrak{n}} \diamond e_{k+2} \in \big\langle e_{\mathfrak{n}-l+k+2} \big\rangle$, i.e., $e_{k+3} \in I$ and so on. At the end, we have that $\mathrm{A} = I$. The statement is proved. □

Proposition 11. *If* $\operatorname{lev}\big(\mathrm{IDD}(K^k_{\mathfrak{n}}, m_1, m_2)\big) = k$, *then* $\mathrm{IDD}(K^k_{\mathfrak{n}}, m_1, m_2)$ *is non-simple perfect and it has exactly* $\mathfrak{n} - k$ *proper ideals if* $\mathfrak{n} < \infty$ *and it has infinitely many proper ideals if* $\mathfrak{n} = \infty$.

Proof. Let us denote $\mathrm{A} := \mathrm{IDD}(K^k_{\mathfrak{n}}, m_1, m_2)$. Firstly, we consider the finite-dimensional case $\mathfrak{n} = n \in \mathbb{N}$, then

$$\mathrm{A} \diamond \mathrm{A} \supseteq \big\langle e_k, \ldots, e_n \big\rangle = \mathrm{A}, \text{ i.e., A is perfect.}$$

Let I be an ideal of A and $n_0 \in \mathbb{N}$, such that there exists $\mathfrak{i}_0 = \sum\limits_{i=n_0}^{n} \alpha_i e_i \in I$, where $\alpha_{n_0} \neq 0$, and there are no elements $\mathfrak{i} = \sum\limits_{i=t}^{n} \alpha_i e_i \in I$, where $\alpha_t \neq 0$ and $t < n_0$. It is easy to see that $I = \big\langle e_{n_0}, \ldots, e_n \big\rangle$, i.e., there exists a 1-1 correspondence between the set of ideals of A and the set of numbers $\{k, \ldots, n\}$. Considering that the ideal $I = \big\langle e_k, \ldots, e_n \big\rangle$ is not proper, we have our statement.

Secondly, we consider the infinite-dimensional case $\mathfrak{n} = \infty$, then

$$\mathrm{A} \diamond \mathrm{A} \supseteq \big\langle e_k, \ldots, e_n, \ldots \big\rangle = \mathrm{A}, \text{ i.e., A is perfect.}$$

Each element e_{n_0} for $n_0 \geq k$ generates an ideal I_{n_0} of A. Namely, ideals $I_{n_0} = \langle e_{n_0}, .., e_n, .. \rangle$ are different and proper, hence A has infinitely many proper ideals. ☐

Proposition 12. *If* $2k - n \leq \operatorname{lev}\big(\mathrm{IDD}(K^k_n, m_1, m_2)\big) < k$, *then* $\mathrm{IDD}(K^k_n, m_1, m_2)$ *is nilpotent.*

Proof. Let us denote $\mathrm{A} := \mathrm{IDD}(K^k_n, m_1, m_2)$ and $l := \operatorname{lev}\big(\mathrm{IDD}(K^k_n, m_1, m_2)\big)$, then

$$\begin{array}{rcccl} \mathrm{A} \diamond \mathrm{A} & \subseteq & \big\langle e_{k+(k-l)}, \ldots, e_n \big\rangle & \subset & \mathrm{A}; \\ \mathrm{A} \diamond \big(\mathrm{A} \diamond \mathrm{A}\big) + \big(\mathrm{A} \diamond \mathrm{A}\big) \diamond \mathrm{A} & \subseteq & \big\langle e_{k+2(k-l)}, \ldots, e_n \big\rangle & \subset & \mathrm{A} \diamond \mathrm{A}. \end{array}$$

Applying the mathematical induction, we find that

$$\mathrm{A}^{\diamond t} := \sum_{i+j=t} \mathrm{A}^{\diamond i} \diamond \mathrm{A}^{\diamond j} \subset \big\langle e_{k+(t-1)(k-l)}, \ldots, e_n \big\rangle.$$

Due to $l < k$, there exists n_0, such that $\mathrm{A}^{\diamond n_0} = 0$, i.e., A is nilpotent. ☐

Definition 13 (see [7]). An infinite-dimensional algebra A is called pro-nilpotent if

$$\bigcap\nolimits_{n=1}^{\infty} \mathrm{A}^n = \big\{0\big\}, \text{ where } \mathrm{A}^n := \sum_{i+j=n} \mathrm{A}^i \mathrm{A}^j.$$

Proposition 14. *If* $\operatorname{lev}\big(\mathrm{IDD}(K^k_\infty, m_1, m_2)\big) < k$, *then* $\mathrm{IDD}(K^k_\infty, m_1, m_2)$ *is pro-nilpotent.*

Proof. Let us denote $\mathrm{A} := \mathrm{IDD}(K^k_\infty, m_1, m_2)$ and $l := \operatorname{lev}\big(\mathrm{IDD}(K^k_\infty, m_1, m_2)\big)$, then

$$\begin{array}{rcccl} \mathrm{A} \diamond \mathrm{A} & = & \big\langle e_{k+(k-l)}, \ldots, e_n, \ldots \big\rangle & \subset & \mathrm{A}; \\ \mathrm{A} \diamond \big(\mathrm{A} \diamond \mathrm{A}\big) + \big(\mathrm{A} \diamond \mathrm{A}\big) \diamond \mathrm{A} & = & \big\langle e_{k+2(k-l)}, \ldots, e_n, \ldots \big\rangle & \subset & \mathrm{A} \diamond \mathrm{A}. \end{array}$$

Applying mathematical induction, we find that

$$\mathrm{A}^{\diamond t} := \sum_{i+j=t} \mathrm{A}^{\diamond i} \diamond \mathrm{A}^{\diamond j} = \big\langle e_{k+(t-1)(k-l)}, \dots, e_n, \dots \big\rangle.$$

Due to $l < k$, we have that $\mathrm{A}^{\diamond t_1} \subset \mathrm{A}^{\diamond t_2}$, if $t_1 > t_2$. It is easy to see that $\bigcap_{n=1}^{\infty} \mathrm{A}^{\diamond n} = \big\{0\big\}$, i.e., A is pro-nilpotent. □

2.3 Conservative IDD algebras

In 1972, Kantor introduced conservative algebras as a generalization of Jordan algebras (also, see surveys about the study of conservative algebras and superalgebras [17, 23]).

Definition 15 (see [14])**.** A vector space V with a multiplication $\cdot$ is called a *conservative algebra* if there is a new multiplication $* : \mathrm{V} \times \mathrm{V} \to \mathrm{V}$ such that

$$\big[L^{\cdot}_b, [L^{\cdot}_a, \cdot]\big] \ = \ -\big[L^{\cdot}_{a*b}, \cdot\big], \ \text{ where } \ \big[L^{\cdot}_c, \mathrm{F}\big](x, y) \ = \ c \cdot \mathrm{F}(x, y) - \mathrm{F}(c \cdot x, y) - \mathrm{F}(x, c \cdot y).$$

In other words, the following identity holds for all $a, b, x, y \in \mathrm{V}$:

$$\begin{aligned} & b\big(a(xy) - (ax)y - x(ay)\big) - a\big((bx)y\big) + \big(a(bx)\big)y + (bx)(ay) - \\ & \quad - a\big(x(by)\big) + (ax)(by) + x\big(a(by)\big) \ = \ -(a * b)(xy) + \big((a * b)x\big)y + x\big((a * b)y\big). \quad (1) \end{aligned}$$

Proposition 16. $\mathrm{IDD}(K^k_\infty, m, 0)$, *where* $m \in \mathbb{Z}$, *is a left commutative algebra.*

Proof. It follows from Proposition 5. □

Definition 17 (see [18])**.** An algebra A is called a generalized associative algebra if there exists one multiplication F defined on the same vector space, such that $a(bx) \ = \ F(a, b)x$.

Theorem 18. *For any* $m \in \mathbb{Z}$, $\mathrm{IDD}(K^k_\infty, m, 0)$ *is conservative with an additional multiplication given by*

$$e_i * e_j = \tfrac{(i+j-2m)!}{(i+j-m)!} \tfrac{i!}{(i-m)!} \tfrac{j!}{(j-m)!} e_{i+j-m}.$$

Proof. By a direct computation, we obtain that $\mathrm{IDD}(K^k_\infty, m, 0)$ is a generalized associative algebra with additional multiplication $e_i * e_j$ given in our statement. By Proposition 16, $\mathrm{IDD}(K^k_\infty, m, 0)$ is left commutative. Thanks to [18], each left-commutative generalized associative algebra is conservative, hence $\mathrm{IDD}(K^k_\infty, m, 0)$ is conservative. □

Remark 19. Cantarini and Kac classified all complex linearly compact commutative and anticommutative conservative simple superalgebras [9]. Algebras $\mathrm{IDD}(K^k_\infty, m, 0)$, where $k < m$, give a wide class of examples of non-(anti)commutative conservative simple algebras. It is known that $\mathrm{IDD}(K^0_\infty, 1, 0)$ is a Novikov algebra, and it seems that it firstly appeared in [12].

3 Derivation of IDD algebras of rank 1

3.1 Derivations of $\mathrm{IDD}(K_{\mathbf{n}}^{\times}, 1, 0)$

Theorem 20. *If $1 \le n < \infty$, then $\mathfrak{Der}\left(\mathrm{IDD}(K_n^0, 1, 0)\right) = \langle \varphi, \phi \rangle$, where*

$$\varphi(e_i) = (i-1)e_i \text{ and } \phi(e_i) = ie_{i-1}.$$

Proof. Let $D \in \mathfrak{Der}\left(\mathrm{IDD}(K_n^0, 1, 0)\right)$, then we can suppose that

$$D(e_0) = \sum_{i=0}^{n} \alpha_i e_i,\ D(e_1) = \sum_{i=0}^{n} \beta_i e_i, \text{ and } D(e_2) = \sum_{i=0}^{n} \gamma_i e_i.$$

Firstly, we have

$$\begin{array}{rcl} 0 & = & D(e_0 \diamond e_0) \ = \ D(e_0) \diamond e_0 + e_0 \diamond D(e_0) \\ & = & \left(\sum\limits_{i=0}^{n} \alpha_i e_i\right) \diamond e_0 + e_0 \diamond \left(\sum\limits_{i=0}^{n} \alpha_i e_i\right) \ = \ \sum\limits_{i=1}^{n} i\alpha_i e_{i-1}; \end{array}$$

that immediately gives $\alpha_k = 0$ for $1 \le k \le n$ and $D(e_0) = \alpha_0 e_0$. Secondly,

$$\begin{array}{rcl} D(e_0) & = & D(e_1 \diamond e_0) \ = \ D(e_1) \diamond e_0 + e_1 \diamond D(e_0) \\ & = & \left(\sum\limits_{i=0}^{n} \beta_i e_i\right) \diamond e_0 + e_1 \diamond (\alpha_0 e_0) \ = \ \sum\limits_{i=0}^{n} i\beta_i e_{i-1} + \alpha_0 e_0 \\ & = & (\alpha_0 + \beta_1)\, e_0 + \sum\limits_{i=2}^{n} i\beta_i e_{i-1}; \end{array}$$

that immediately gives $\beta_k = 0$ for $1 \le k \le n$, i.e., $D(e_1) = \beta_0 e_0$. Thirdly,

$$\begin{array}{rcl} 2D(e_1) & = & D(e_2 \diamond e_0) \ = \ D(e_2) \diamond e_0 + e_2 \diamond D(e_0) \\ & = & \left(\sum\limits_{i=0}^{n} \gamma_i e_i\right) \diamond e_0 + e_2 \diamond (\alpha_0 e_0) \\ & = & \sum\limits_{i=0}^{n} i\gamma_i e_{i-1} + 2\alpha_0 e_1 \ = \ \gamma_1 e_0 + 2(\alpha_0 + \gamma_2)e_1 + \sum\limits_{i=3}^{n} i\gamma_i e_{i-1}. \end{array}$$

that immediately gives $\gamma_1 = 2\beta_0,\ \gamma_2 = -\alpha_0,\ \gamma_k = 0$, for $3 \le k \le n$, i.e.,

$$D(e_2) \ = \ \gamma_0 e_0 + 2\beta_0 e_1 - \alpha_0 e_2.$$

Next,

$$D(e_2) \ = \ \tfrac{1}{2} D(e_2 \diamond e_1) \ = \ \tfrac{1}{2}\left((\gamma_0 e_0 + 2\beta_0 e_1 - \alpha_0 e_2) \diamond e_1 + e_2 \diamond (\beta_0 e_0)\right) \ = \ 2\beta_0 e_1 - \alpha_0 e_2;$$

that immediately gives $\gamma_0 = 0$, i.e., $D(e_0) = \alpha_0 e_0,\ D(e_1) = \beta_0 e_0,\ D(e_2) = 2\beta_0 e_1 - \alpha_0 e_2$.

It is easy to see that $\varphi(e_i) = (i-1)e_i$ and $\phi(e_i) = ie_{i-1}$ are derivations of $\mathrm{IDD}(K_n^0, 1, 0)$:

$$\begin{array}{rcl} \varphi(e_i \diamond e_j) & = & i(i+j-2)e_{i+j-1} \ = \ i(i-1)e_{i+j-1} + i(j-1)e_{i+j-1} \\ & = & (i-1)e_i \diamond e_j + (j-1)e_i \diamond e_j \ = \ \varphi(e_i) \diamond e_j + e_i \diamond \varphi(e_j) \end{array}$$

and

$$\begin{array}{rcl}\phi(e_i \diamond e_j) & = & i(i+j-1)e_{i+j-2} \ = \ i(i-1)e_{i+j-2} + ije_{i+j-2} \\ & = & \big(ie_{i-1}\big) \diamond e_j + e_i \diamond \big(je_{j-1}\big) \ = \ \varphi(e_i) \diamond e_j + e_i \diamond \varphi(e_j).\end{array}$$

Then replacing D by $D + \alpha_0\varphi - \beta_0\phi$ we can suppose that $D(e_1) = D(e_2) = 0$.

Furthermore, using the induction method, we prove that $D(e_k) = 0$ for $k \geq 2$. To do this, let us consider

$$D(e_{k+1}) \ = \ \tfrac{1}{2}D(e_2 \diamond e_k) \ = \ \tfrac{1}{2}\big(D(e_2) \diamond e_k + e_2 \diamond D(e_k)\big) \ = \ 0.$$

It follows that each derivation of $\mathrm{IDD}(K_n^0, 1, 0)$ is a linear combination of φ and ϕ, that completes the proof of the statement. □

Corollary 21. $\mathfrak{Der}\left(\mathrm{IDD}(K_\infty^0, 1, 0)\right) = \langle \varphi, \phi \rangle$, *where* $\varphi(e_i) = (i-1)e_i$ *and* $\phi(e_i) = ie_{i-1}$.

Theorem 22. *If* $1 \leq n < \infty$, *then* $\mathfrak{Der}\big(\mathrm{IDD}(K_n^1, 1, 0)\big) = \big\langle \varphi \big\rangle$, *where* $\varphi(e_i) = (i-1)e_i$.

Proof. Let $D \in \mathfrak{Der}\big(\mathrm{IDD}(K_n^1, 1, 0)\big)$, then $D(e_1) = \sum\limits_{i=1}^{n} \alpha_i e_i$ and $D(e_2) = \sum\limits_{i=1}^{n} \beta_i e_i$. Firstly,

$$D\left(e_1\right) \ = \ D\left(e_1 \diamond e_1\right) \ = \ D\left(e_1\right) \diamond e_1 + e_1 \diamond \left(e_1\right) \ = \ \sum_{i=1}^{n} \left(i+1\right) \alpha_i e_i,$$

that immediately gives $\alpha_k = 0$ for $1 \leq k \leq n$ and $D\left(e_1\right) = 0$. Secondly, we have

$$\begin{array}{rcl}2\sum\limits_{i=1}^{n} \beta_i e_i & = & 2D(e_2) \ = \ D(e_2 \diamond e_1) \ = \ D(e_2) \diamond e_1 + e_2 \diamond D(e_1) \\ & = & \left(\sum\limits_{i=1}^{n} \beta_i e_i\right) e_1 \ = \ \sum\limits_{i=1}^{n} i\beta_i e_i,\end{array}$$

that immediately gives $\beta_1 = 0$, $\beta_k = 0$ for $3 \leq k \leq n$ and $D(e_2) = \beta_2 e_2$.

It is easy to see that $\varphi(e_i) = (i-1)e_i$ is a derivation of $\mathrm{IDD}(K_n^1, 1, 0)$:

$$\begin{array}{rcl}\varphi(e_i \diamond e_j) & = & i(i+j-2)e_{i+j-1} \ = \ i(i-1)e_{i+j-1} + i(j-1)e_{i+j-1} \ = \\ & = & (i-1)e_i \diamond e_j + (j-1)e_i \diamond e_j \ = \ \varphi(e_i) \diamond e_j + e_i \diamond \varphi(e_j).\end{array}$$

Then replacing D by $D - \beta_2\varphi$ we can suppose that $D(e_1) \ = \ D(e_2) \ = \ 0$. Furthermore, using the induction method, we prove that $D(e_k) = 0$ for $k \geq 2$. To do this, let us consider

$$D(e_{k+1}) \ = \ \tfrac{1}{2}D(e_2 \diamond e_k) \ = \ \tfrac{1}{2}\big(D(e_2) \diamond e_k + e_2 \diamond D(e_k)\big) \ = \ 0.$$

It follows that each derivation of $\mathrm{IDD}(K_n^1, 1, 0)$ lies in the linear span of φ, which completes the proof. □

3.2 Derivations of $\mathrm{IDD}(K_{\mathbf{n}}^{\times}, 0, -1)$

Theorem 23. *If $1 \le n < \infty$, then $\mathfrak{Der}\big(\mathrm{IDD}(K_n^0, 0, -1)\big) = \langle \varphi_i \rangle_{0 \le i \le n}$, where*

$$\varphi_i(e_k) = (i+1+k)\, e_{k+i}, \text{ for } k \le n-i.$$

Proof. Let $D \in \mathfrak{Der}\big(\mathrm{IDD}(K_n^0, 0, -1)\big)$, then we can say that $D(e_0) = \sum\limits_{i=0}^{n} \alpha_i e_i$. Firstly,

$$\begin{aligned} D(e_1) &= D(e_0 \diamond e_0) = \left(\sum_{i=0}^{n} \alpha_i e_i\right) \diamond e_0 + e_0 \diamond \left(\sum_{i=0}^{n} \alpha_i e_i\right) = \\ &= \sum_{i=0}^{n-1} \alpha_i e_{i+1} + \sum_{i=0}^{n-1} \tfrac{1}{i+1} \alpha_i e_{i+1} = \sum_{i=0}^{n-1} \tfrac{i+2}{i+1} \alpha_i e_{i+1}. \end{aligned}$$

It is easy to see that $\varphi_i(e_k) = (i+k+1)\, e_{i+k}$ is a derivation of $\mathrm{IDD}(K_n^0, 0, -1)$:

$$\begin{aligned} \varphi_k(e_i \diamond e_j) &= \tfrac{k+i+j+2}{j+1} e_{i+j+k+1} = \tfrac{i+k+1}{j+1} e_{i+j+k+1} + \tfrac{j+k+1}{j+k+1} e_{i+j+k+1} = \\ &= (i+k+1)\, e_{i+k} \diamond e_j + (j+k+1)\, e_i \diamond e_{j+k} = \varphi_k(e_i) \diamond e_j + e_i \diamond \varphi_k(e_j). \end{aligned}$$

Hence, we can replace D by $D - \sum\limits_{i=0}^{n} \alpha_i \varphi_i$ and suppose that $D(e_0) = D(e_1) = 0$.

Furthermore, using the induction method, we prove that $D(e_k) = 0$ for $k \ge 2$. To do this, let us consider

$$D(e_{k+1}) = kD(e_1 \diamond e_{k-1}) = k\big(D(e_1) \diamond e_{k-1} + e_1 \diamond D(e_{k-1})\big) = 0.$$

It follows that each derivation of $\mathrm{IDD}(K_n^0, 0, -1)$ is a linear combination of φ_i, that completes the proof of the statement. □

Corollary 24. $\mathfrak{Der}\big(\mathrm{IDD}(K_\infty^0, 0, -1)\big) = \langle \varphi_i \rangle_{i \ge 0}$, *where* $\varphi_i(e_k) = (i+1+k)\, e_{k+i}$.

Theorem 25. *If $3 \le n < \infty$, then $\mathfrak{Der}\big(\mathrm{IDD}(K_n^1, 0, -1)\big) = \langle \varphi_i, \phi_1, \phi_2 \rangle_{1 \le i \le n}$, where*

φ_i	$\varphi_i(e_k)$	$=$	$(i+k)\, e_{k+i-1},$	$k \le 1+n-i$
ϕ_1	$\phi_1(e_2)$	$=$	e_{n-1}	
ϕ_2	$\phi_2(e_2)$	$=$	e_n	

Proof. Let $D \in \mathfrak{Der}\big(\mathrm{IDD}(K_n^1, 0, -1)\big)$, then we can say that

$$D(e_1) = \sum_{i=1}^{n} \alpha_i e_i \text{ and } D(e_2) = \sum_{i=1}^{n} \beta_i e_i.$$

Firstly,

$$D(e_3) = 2D(e_1 \diamond e_1) = 2\left(\left(\sum_{i=1}^{n} \alpha_i e_i\right) \diamond e_1 + e_1 \diamond \left(\sum_{i=1}^{n} \alpha_i e_i\right)\right) =$$

$$= \sum_{i=1}^{n-2} \alpha_i e_{i+2} + \sum_{i=1}^{n-2} \tfrac{2}{i+1}\alpha_i e_{i+2} \; = \; \sum_{i=1}^{n-2} \tfrac{i+3}{i+1}\alpha_i e_{i+2}.$$

Secondly,

$$\begin{aligned} D(e_4) &= 3D(e_1 \diamond e_2) \; = \; 3\left(\left(\sum_{i=1}^{n} \alpha_i e_i\right) \diamond e_2 + e_1 \diamond \left(\sum_{i=1}^{n} \beta_i e_i\right)\right) \; = \\ &= \sum_{i=1}^{n-3} \alpha_i e_{i+3} + \sum_{i=1}^{n-2} \tfrac{3}{i+1}\beta_i e_{i+2} \; = \; \tfrac{3}{2}\beta_1 e_3 + \sum_{i=1}^{n-3} \left(\alpha_i + \tfrac{3}{i+2}\beta_{i+1}\right) e_{i+3}. \end{aligned}$$

Thirdly,

$$\begin{aligned} D(e_4) &= 2D(e_2 \diamond e_1) \; = \; 2\left(\left(\sum_{i=1}^{n} \beta_i e_i\right) \diamond e_1 + e_2 \diamond \left(\sum_{i=1}^{n} \alpha_i e_i\right)\right) \; = \\ &= \sum_{i=1}^{n-2} \beta_i e_{i+2} + \sum_{i=1}^{n-3} \tfrac{2}{i+1}\alpha_i e_{i+3} \; = \; \beta_1 e_3 + \sum_{i=1}^{n-3} \left(\beta_{i+1} + \tfrac{2}{i+1}\alpha_i\right) e_{i+3}. \end{aligned}$$

Hence, comparing these two expressions for $D(e_4)$, we have that

$$\beta_1 = 0 \text{ and } \beta_k = \tfrac{k+1}{k}\alpha_{k-1}, \quad 3 \leq k \leq n-2, \text{ i.e.,}$$

$$D(e_2) \; = \; \beta_2 e_2 + \sum_{i=3}^{n-2} \tfrac{i+1}{i}\alpha_{i-1} e_i + \beta_{n-1} e_{n-1} + \beta_n e_n \text{ and}$$

$$D(e_4) \; = \; (\alpha_1 + \beta_2)\, e_4 + \sum_{i=2}^{n-3} \tfrac{i+4}{i+1}\alpha_i e_{i+3}.$$

Now, we will consider two different expressions of $D(e_5)$:

$$\begin{aligned} D(e_5) &= 4D(e_1 \diamond e_3) \; = \; 4\left(\left(\sum_{i=1}^{n} \alpha_i e_i\right) \diamond e_3 + e_1 \diamond \left(\sum_{i=1}^{n-2} \tfrac{i+3}{i+1}\alpha_i e_{i+2}\right)\right) \; = \\ &= \sum_{i=1}^{n-4} \alpha_i e_{i+4} + \sum_{i=1}^{n-4} \tfrac{4}{i+1}\alpha_i e_{i+4} \; = \; 3\alpha_1 e_5 + \sum_{i=2}^{n-4} \tfrac{i+5}{i+1}\alpha_i e_{i+4}; \\ D(e_5) &= 3D(e_2 \diamond e_2) \\ &= 3\Bigg(\left(\beta_2 e_2 + \sum_{i=3}^{n-2} \tfrac{i+1}{i}\alpha_{i-1} e_i + \beta_{n-1} e_{n-1} + \beta_n e_n\right) \diamond e_2 \\ &\qquad + e_2 \diamond \left(\beta_2 e_2 + \sum_{i=3}^{n-2} \tfrac{i+1}{i}\alpha_{i-1} e_i + \beta_{n-1} e_{n-1} + \beta_n e_n\right)\Bigg) \\ &= \beta_2 e_5 + \sum_{i=3}^{n-3} \tfrac{i+1}{i}\alpha_{i-1} e_{i+3} + \beta_2 e_5 + \sum_{i=3}^{n-3} \tfrac{3}{i}\alpha_{i-1} e_{i+3} \\ &= 2\beta_2 e_5 + \sum_{i=2}^{n-4} \tfrac{i+5}{i+1}\alpha_i e_{i+4}. \end{aligned}$$

The last gives $\beta_2 = \frac{3}{2}\alpha_1$, i.e.

$$D(e_2) \; = \; \sum_{i=1}^{n-3} \tfrac{i+2}{i+1}\alpha_i e_{i+1} + \beta_{n-1} e_{n-1} + \beta_n e_n \text{ and } D(e_4) \; = \; \sum_{i=1}^{n-3} \tfrac{i+4}{i+1}\alpha_i e_{i+3}.$$

Let us consider linear mappings $\phi_1(e_2) = e_{n-1}$ and $\phi_2(e_2) = e_n$. Obviously,

$$\phi_1\big(\mathrm{IDD}(K_n^1, 0, -1)\big) + \phi_2\big(\mathrm{IDD}(K_n^1, 0, -1)\big) \subseteq \mathrm{Ann}\big(\mathrm{IDD}(K_n^1, 0, -1)\big),$$

and $e_2 \notin \big(\mathrm{IDD}(K_n^1, 0, -1)\big)^2$, hence ϕ_1 and ϕ_2 are derivations. It is easy to see that linear mappings φ_i defined by

$$\varphi_i(e_k) = (i+k)\, e_{k+i-1}, \text{ where } k \le 1+n-i$$

are also derivations of $\mathrm{IDD}(K_n^1, 0, -1)$:

$$\begin{aligned} \varphi_k(e_i \diamond e_j) &= \tfrac{k+i+j+1}{k+1} e_{i+k+j} = \tfrac{k+i}{j+1} e_{k+i+j} + \tfrac{k+j}{k+j} e_{k+i+j} = \\ &= (k+i)\, e_{k+i-1} \diamond e_j + (k+j)\, e_i \diamond e_{k+j-1} = \varphi_k(e_i) \diamond e_j + e_i \diamond \varphi_k(e_j). \end{aligned}$$

Hence, we can replace D by $D - \sum\limits_{i=1}^{n} \alpha_i \varphi_i - (\beta_{n-1} - n\alpha_{n-2})\, \phi_1 - (\beta_n - (n-1)\, \alpha_{n-1})\, \phi_2$, and suppose that $D(e_1) = D(e_2) = D(e_3) = D(e_4) = 0$.

Furthermore, using the induction method, we prove that $D(e_k) = 0$ for $k \ge 3$. To do this, let us consider

$$D(e_{k+1}) = kD(e_1 \diamond e_{k-1}) = k\big(D(e_1) \diamond e_{k-1} + e_1 \diamond D(e_{k-1})\big) = 0.$$

It follows that each derivation of $\mathrm{IDD}(K_n^1, 0, -1)$ is a linear combination of φ_i, ϕ_1 and ϕ_2, that completes the proof of the statement. □

Corollary 26. $\mathfrak{Der}\big(\mathrm{IDD}(K_\infty^1, 0, -1)\big) = \big\langle \varphi_i \big\rangle_{i \ge 1}$, *where* $\varphi_i(e_k) = (i+k)\, e_{k+i-1}$.

4 Derivation of IDD algebras of rank 2

4.1 Derivations of $\mathrm{IDD}(K_n^\times, 2, 0)$

Proposition 27. $\mathfrak{Der}\big(\mathrm{IDD}(K_2^0, 2, 0)\big) = \big\langle \varphi_i, \phi_i, \psi_i \big\rangle_{0 \le i \le 1}$ *where*

φ_i	$\varphi_i(e_0) = e_i$	$0 \le i \le 1$
ϕ_i	$\phi_i(e_1) = e_i$	$0 \le i \le 1$
ψ_i	$\psi_i(e_2) = e_i$	$0 \le i \le 1$

Theorem 28. *If* $3 \le n < \infty$, *then* $\mathfrak{Der}\big(\mathrm{IDD}(K_n^0, 2, 0)\big) = \big\langle \varphi, \phi \big\rangle$, *where*

$$\varphi(e_i) = (i-2)e_i \text{ and } \phi(e_i) = ie_{i-1}.$$

Proof. Let $D \in \mathfrak{Der}\big(\mathrm{IDD}(K_n^0, 2, 0)\big)$, then we can suppose that

$$D(e_0) = \sum_{i=0}^{n} \alpha_i e_i,\ D(e_1) = \sum_{i=0}^{n} \beta_i e_i,\ D(e_2) = \sum_{i=0}^{n} \gamma_i e_i,\ D(e_3) = \sum_{i=0}^{n} \lambda_i e_i.$$

Firstly,

$$0 \ = \ D(e_0 \diamond e_0) \ = \ \left(\sum_{i=0}^{n} \alpha_i e_i\right) \diamond e_0 + e_0 \diamond \left(\sum_{i=0}^{n} \alpha_i e_i\right) \ = \ \sum_{i=0}^{n} i(i-1)\alpha_i e_{i-2},$$

that immediately gives $\alpha_k = 0$, for $2 \leq k \leq n$, i.e., $D(e_0) = \alpha_0 e_0 + \alpha_1 e_1$. Secondly,

$$0 \ = \ D(e_1 \diamond e_0) \ = \ \left(\sum_{i=0}^{n} \beta_i e_i\right) \diamond e_0 + e_1 \diamond \left(\sum_{i=0}^{n} \alpha_i e_i\right) \ = \ \sum_{i=0}^{n} i(i-1)\beta_i e_{i-2},$$

that immediately gives $\beta_k = 0$, for $2 \leq k \leq n$, i.e., $D(e_1) = \beta_0 e_0 + \beta_1 e_1$. Thirdly,

$$\begin{array}{rcl} D(e_0) & = & \frac{1}{2} D(e_2 \diamond e_0) \ = \ \frac{1}{2}\left(\left(\sum\limits_{i=0}^{n} \gamma_i e_i\right) \diamond e_0 + e_2 \diamond (\alpha_0 e_0 + \alpha_1 e_1)\right) \ = \\ & = & \frac{1}{2} \sum\limits_{i=0}^{n} i(i-1)\gamma_i e_{i-2} + \alpha_0 e_0 + \alpha_1 e_1, \end{array}$$

that immediately gives $\gamma_k = 0$, for $2 \leq k \leq n$, i.e., $D(e_2) = \gamma_0 e_0 + \gamma_1 e_1$. Next,

$$\begin{array}{rcl} D(e_1) & = & \frac{1}{6} D(e_3 \diamond e_0) \ = \ \frac{1}{6}\left(\left(\sum\limits_{i=0}^{n} \lambda_i e_i\right) \diamond e_0 + e_3 \diamond (\alpha_0 e_0 + \alpha_1 e_1)\right) \ = \\ & = & \frac{1}{3}\lambda_2 e_0 + (\alpha_0 + \lambda_3)\, e_1 + (\alpha_1 + 2\lambda_4)\, e_2 + \frac{1}{6} \sum\limits_{i=5}^{n} i(i-1)\lambda_i e_{i-2}, \end{array}$$

that immediately gives $\lambda_2 = 3\beta_0$, $\lambda_3 = -\alpha_0 + \beta_1$, $\lambda_4 = -\frac{1}{2}\alpha_1$, and $\lambda_k = 0$, for $5 \leq k \leq n$, i.e.,

$$D(e_3) = \lambda_0 e_0 + \lambda_1 e_1 + 3\beta_0 e_2 + (-\alpha_0 + \beta_1)\, e_3 - \tfrac{1}{2}\alpha_1 e_4.$$

Finally,

$$\begin{array}{rcl} D(e_2) & = & \frac{1}{6} D(e_3 \diamond e_1) \ = \ \frac{1}{6}\Big(\big(\lambda_0 e_0 + \lambda_1 e_1 + 3\beta_0 e_2 + (-\alpha_0 + \beta_1)\, e_3 - \frac{1}{2}\alpha_1 e_4\big) \diamond e_1 + \\ & & + e_3 \diamond (\beta_0 e_0 + \beta_1 e_1)\Big) \ = \ \beta_0 e_1 + (-\alpha_0 + \beta_1)\, e_2 - \alpha_1 e_3 + \beta_0 e_1 + \beta_1 e_2 \ = \\ & = & 2\beta_0 e_1 + (-\alpha_0 + \beta_1)\, e_2 - \alpha_1 e_3, \end{array}$$

that immediately gives $\gamma_0 = 0$, $\gamma_1 = 2\beta_0$, $\beta_1 = \frac{1}{2}\alpha_0$, and $\alpha_1 = 0$.

$$\begin{array}{rcl} D(e_3) & = & \frac{1}{6} D(e_3 \diamond e_2) \ = \ \frac{1}{6}\Big(\big(\lambda_0 e_0 + \lambda_1 e_1 + 3\beta_0 e_2 - \frac{1}{2}\alpha_0 e_3\big) \diamond e_2 + e_3 \diamond (2\beta_0 e_1)\Big) \ = \\ & = & 3\beta_0 e_2 - \frac{1}{2}\alpha_0 e_3, \end{array}$$

that immediately gives $\lambda_0 = 0$ and $\lambda_1 = 0$, i.e.,

$$D(e_0) = \alpha_0 e_0, \ \ D(e_1) = \beta_0 e_0 + \tfrac{1}{2}\alpha_0 e_1, \ \ D(e_2) = 2\beta_0 e_1 \ \text{ and } \ D(e_3) = 3\beta_0 e_2 - \tfrac{1}{2}\alpha_0 e_3.$$

It is easy to see that $\varphi(e_i) = (i-2)e_i$ and $\phi(e_i) = ie_{i-1}$ are derivations of $\mathrm{IDD}(K_n^0, 2, 0)$:

$$\begin{array}{rcl} \varphi(e_i \diamond e_j) & = & i(i-1)(i+j-4)e_{i+j-2} \ = \ i(i-1)\left((i-2)e_{i+j-2} + (j-2)e_{i+j-2}\right) \ = \\ & = & \big((i-2)e_i\big) \diamond e_j + e_i \diamond \big((j-2)e_j\big) \ = \ \varphi(e_i) \diamond e_j + e_i \diamond \varphi(e_j); \\ \phi(e_i \diamond e_j) & = & i(i-1)(i+j)e_{i+j-3} \ = \ i(i-1)ie_{i+j-3} + i(i-1)je_{i+j-3} \ = \\ & = & \big(ie_{i-1}\big) \diamond e_j + e_i \diamond \big(je_{j-1}\big) \ = \ \varphi(e_i) \diamond e_j + e_i \diamond \varphi(e_j). \end{array}$$

Then replacing D by $D+\alpha_0\varphi-\beta_0\phi$ we can suppose that $D(e_1) = D(e_2) = D(e_3) = 0$.

Furthermore, using the induction method, we prove that $D(e_k)=0$ for $k\geq 2$. To do this, let us consider

$$D(e_{k+1}) = \tfrac{1}{6}D(e_3\diamond e_k) = \tfrac{1}{6}\big(D(e_3)\diamond e_k+e_3\diamond D(e_k)\big) = 0.$$

It follows that each derivation of $\mathrm{IDD}(K_n^0,2,0)$ is a linear combination of φ and ϕ, that completes the proof of the statement. □

Corollary 29. *$\mathfrak{Der}\big(\mathrm{IDD}(K_\infty^0,2,0)\big)=\langle\varphi,\phi\rangle$, where $\varphi(e_i)=(i-2)e_i$ and $\phi(e_i)=ie_{i-1}$.*

Proposition 30. *$\mathfrak{Der}\big(\mathrm{IDD}(K_2^1,2,0)\big)=\langle\varphi,\phi\rangle$ where $\varphi(e_1)=e_1$ and $\phi(e_2)=e_1$.*

Theorem 31. *If $3\leq n<\infty$, then $\mathfrak{Der}\big(\mathrm{IDD}(K_n^1,2,0)\big)=\langle\varphi\rangle$, where $\varphi(e_i)=(i-2)e_i$.*

Proof. Let $D\in\mathfrak{Der}\big(\mathrm{IDD}(K_n^1,2,0)\big)$, then we can suppose that

$$D(e_1)=\sum_{i=1}^{n}\alpha_ie_i,\ D(e_2)=\sum_{i=1}^{n}\beta_ie_i,\text{ and }D(e_3)=\sum_{i=1}^{n}\gamma_ie_i.$$

Firstly,

$$0 = D(e_1\diamond e_1) = \left(\sum_{i=1}^{n}\alpha_ie_i\right)\diamond e_1+e_1\diamond\left(\sum_{i=1}^{n}\alpha_ie_i\right)=\sum_{i=1}^{n}i(i-1)\alpha_ie_{i-1};$$

that immediately gives $\alpha_k=0$ for $2\leq k\leq n$, i.e., $D(e_1)=\alpha_1e_1$. Secondly,

$$D(e_1) = \tfrac{1}{2}D(e_2\diamond e_1) = \tfrac{1}{2}\left(\left(\sum_{i=1}^{n}\beta_ie_i\right)\diamond e_1+e_2\diamond(\alpha_1e_1)\right) = \tfrac{1}{2}\sum_{i=1}^{n}i(i-1)\beta_ie_{i-1}+\alpha_1e_1;$$

that immediately gives $\beta_k=0$ for $2\leq k\leq n$, i.e.,

$$\begin{aligned}D(e_2) &= \tfrac{1}{6}D(e_3\diamond e_1) = \tfrac{1}{6}\left(\left(\sum_{i=1}^{n}\gamma_ie_i\right)\diamond e_1+e_3\diamond(\alpha_1e_1)\right)\\ &= \tfrac{1}{6}\sum_{i=1}^{n}i(i-1)\gamma_ie_{i-1}+\alpha_1e_2 = \tfrac{1}{3}\gamma_2e_1+(\gamma_3+\alpha_1)\,e_2+\tfrac{1}{6}\sum_{i=4}^{n}i(i-1)\gamma_ie_{i-1};\end{aligned}$$

that immediately gives $\gamma_2=3\beta_1,\gamma_3=-\alpha_1$, and $\gamma_k=0$ for $4\leq k\leq n$, i.e.,

$$D(e_3)=\gamma_1e_1+3\beta_1e_2-\alpha_1e_3.$$

Next,

$$D(e_3) = \tfrac{1}{6}D(e_3\diamond e_2)=\tfrac{1}{6}\big((\gamma_1e_1+3\beta_1e_2-\alpha_1e_3)\diamond e_2+e_3\diamond(\beta_1e_1)\big)=2\beta_1e_2-\alpha_1e_3;$$

that immediately gives $\beta_1=0$ and $\gamma_1=0$, i.e., $D(e_1)=\alpha_1e_1,\ \ D(e_2)=0,\ \ D(e_3)=-\alpha_1e_3$.

It is easy to see that $\varphi(e_i)=(i-2)e_i$ is a derivation of $\mathrm{IDD}(K_n^1,2,0)$:

$$\begin{aligned}\varphi(e_i\diamond e_j) &= i(i-1)(i+j-4)e_{i+j-2} = i(i-1)\left((i-2)e_{i+j-2}+(j-2)e_{i+j-2}\right) =\\ &= \big((i-2)e_i\big)\diamond e_j+e_i\diamond\big((j-2)e_j\big) = \varphi(e_i)\diamond e_j+e_i\diamond\varphi(e_j).\end{aligned}$$

Then replacing D by $D+\alpha_1\varphi$ we can suppose that $D(e_1) \ = \ D(e_2) = \ D(e_3) \ = \ 0.$

Furthermore, using the induction method, we prove that $D(e_k)=0$ for $k\geq 2$. To do this, let us consider

$$D(e_{k+1}) \ = \ \tfrac{1}{6}D(e_3\diamond e_k) \ = \ \tfrac{1}{6}\big(D(e_3)\diamond e_k+e_3\diamond D(e_k)\big) \ = \ 0.$$

It follows that each derivation of $\mathrm{IDD}(K^1_n,2,0)$ is a linear combination of φ, that completes the proof of the statement. □

Corollary 32. *$\mathfrak{Der}\big(\mathrm{IDD}(K^1_\infty,2,0)\big)=\langle\varphi,\phi\rangle$, where $\varphi(e_i)=(i-2)e_i$.*

4.2 Derivations of $\mathrm{IDD}(K^\times_{\mathfrak{n}},1,1)$

Theorem 33. *If $1\leq n<\infty$, then $\mathfrak{Der}\big(\mathrm{IDD}(K^0_n,1,1)\big)=\langle\varphi,\phi\rangle$, where*

$$\varphi(e_i)=(i-2)e_i \text{ and } \phi(e_i)=ie_{i-1}.$$

Proof. Let $D\in\mathfrak{Der}\big(\mathrm{IDD}(K^0_n,1,1)\big)$, then we can say that

$$D(e_0)=\sum_{i=0}^{n}\alpha_ie_i,\ D(e_1)=\sum_{i=0}^{n}\beta_ie_i,\ D(e_2)=\sum_{i=0}^{n}\gamma_ie_i, \text{ and } D(e_3)=\sum_{i=0}^{n}\lambda_ie_i.$$

Firstly,

$$0 \ = \ D(e_0\diamond e_1) \ = \ \left(\sum_{i=0}^{n}\alpha_ie_i\right)\diamond e_1+e_0\diamond\left(\sum_{i=0}^{n}\beta_ie_i\right) \ = \ \sum_{i=0}^{n}i\alpha_ie_{i-1},$$

that immediately gives $\alpha_k=0$, for $1\leq k\leq n$, i.e., $D(e_0)=\alpha_0e_0$. Secondly,

$$D(e_0) \ = \ D(e_1\diamond e_1) \ = \ \left(\sum_{i=0}^{n}\beta_ie_i\right)\diamond e_1+e_1\diamond\left(\sum_{i=0}^{n}\beta_ie_i\right) \ = \ \sum_{i=0}^{n}2i\beta_ie_{i-1},$$

that immediately gives $\beta_1=\frac{1}{2}\alpha_0$ and $\beta_k=0$, for $2\leq k\leq n$, i.e., $D(e_1)=\beta_0e_0+\frac{1}{2}\alpha_0e_1$. Thirdly,

$$\begin{aligned} D(e_1) \ &= \ \tfrac{1}{2}D(e_2\diamond e_1) \ = \ \tfrac{1}{2}\left(\left(\sum_{i=0}^{n}\gamma_ie_i\right)\diamond e_1+e_2\diamond\left(\beta_0e_0+\tfrac{1}{2}\alpha_0e_1\right)\right)\\ &= \ \tfrac{1}{2}\sum_{i=0}^{n}i\gamma_ie_{i-1}+\alpha_0e_1 \ = \ \tfrac{1}{2}\gamma_1e_0+\left(\tfrac{1}{2}\alpha_0+\gamma_2\right)e_1+\sum_{i=3}^{n}\tfrac{i}{2}\gamma_ie_{i-1}; \end{aligned}$$

that immediately gives $\gamma_1=2\beta_0$ and $\gamma_k=0$, for $2\leq k\leq n$, i.e., $D(e_2)=\gamma_0e_0+2\beta_0e_1$. Next,

$$\begin{aligned} D(e_2) \ &= \ \tfrac{1}{4}D(e_2\diamond e_2) \ = \ \tfrac{1}{4}\left((\gamma_0e_0+2\beta_0e_1)\diamond e_2+e_2\diamond(\gamma_0e_0+2\beta_0e_1)\right) \ = \ 2\beta_0e_1;\\ D(e_2) \ &= \ \tfrac{1}{3}D(e_3\diamond e_1) \ = \ \tfrac{1}{3}\left(\left(\sum_{i=0}^{n}\lambda_ie_i\right)\diamond e_1+e_3\diamond\left(\beta_0e_0+\tfrac{1}{2}\alpha_0e_1\right)\right) \ = \\ &= \ \tfrac{1}{3}\left(\sum_{i=0}^{n}i\lambda_ie_{i-1}+\tfrac{3}{2}\alpha_0e_2\right) \ = \ \tfrac{\lambda_1}{3}e_0+\tfrac{2}{3}\lambda_2e_1+\left(\tfrac{1}{2}\alpha_0+\lambda_3\right)e_2+\sum_{i=4}^{n}\tfrac{i}{3}\lambda_ie_{i-1}; \end{aligned}$$

that immediately gives $\gamma_0 = 0$, $\lambda_1 = 0$, $\lambda_2 = 3\beta_0$, $\lambda_3 = -\frac{1}{2}\alpha_0$, and $\lambda_k = 0$, for $4 \le k \le n$. Finally,

$$\begin{array}{rcl} D(e_3) & = & \frac{1}{6}D(e_3 \diamond e_2) \ = \ \frac{1}{6}\left(\left(\lambda_0 e_0 + 3\beta_0 e_2 - \frac{1}{2}\alpha_0 e_3\right) \diamond e_2 + e_3 \diamond (2\beta_0 e_1)\right) \ = \\ & = & 3\beta_0 e_2 - \frac{1}{2}\alpha_0 e_3, \end{array}$$

i.e., $\lambda_0 = 0$. Then we obtain

$$D(e_0) = \alpha_0 e_0, \ \ D(e_1) = \beta_0 e_0 + \tfrac{1}{2}\alpha_0 e_1, \ \ D(e_2) = 2\beta_0 e_1 \ \text{ and } \ D(e_3) = 3\beta_0 e_2 - \tfrac{1}{2}\alpha_0 e_3.$$

It is easy to see that $\varphi(e_i) = (i-2)e_i$ and $\phi(e_i) = ie_{i-1}$ are derivations of $\mathrm{IDD}(K_n^0, 1, 1)$:

$$\begin{array}{rcl} \varphi(e_i \diamond e_j) & = & ij(i+j-4)e_{i+j-2} \ = \ ij(i-2)e_{i+j-2} + ij(j-2)e_{i+j-2} \ = \\ & = & \big((i-2)e_i\big) \diamond e_j + e_i \diamond \big((j-2)e_j\big) \ = \ \varphi(e_i) \diamond e_j + e_i \diamond \varphi(e_j). \\ \phi(e_i \diamond e_j) & = & ij(i+j)e_{i+j-3} \ = \ ijie_{i+j-3} + ijje_{i+j-3} \ = \\ & = & \big(ie_{i-1}\big) \diamond e_j + e_i \diamond \big(je_{j-1}\big) \ = \ \varphi(e_i) \diamond e_j + e_i \diamond \varphi(e_j). \end{array}$$

Then replacing D by $D + \alpha_0\varphi - \beta_0\phi$ we can suppose that $D(e_1) = D(e_2) = D(e_3) = 0$.

Furthermore, using the induction method, we prove that $D(e_k) = 0$ for $k \ge 2$. To do this, let us consider

$$D(e_{k+1}) \ = \ \tfrac{1}{3k}D(e_3 \diamond e_k) \ = \ \tfrac{1}{3k}\big(D(e_3) \diamond e_k + e_3 \diamond D(e_k)\big) \ = \ 0.$$

It follows that each derivation of $\mathrm{IDD}(K_n^0, 1, 1)$ is a linear combination of φ and ϕ, which completes the proof of the statement. □

Corollary 34. $\mathfrak{Der}\big(\mathrm{IDD}(K_\infty^0, 1, 1)\big) = \langle \varphi, \phi \rangle$, *where* $\varphi(e_i) = (i-2)e_i$ *and* $\phi(e_i) = ie_{i-1}$.

Theorem 35. *If* $2 \le n < \infty$, *then* $\mathfrak{Der}\big(\mathrm{IDD}(K_n^1, 1, 1)\big) = \langle \varphi, \phi \rangle$, *where*

$$\varphi(e_i) = (i-2)e_i \ \textit{and} \ \phi(e_i) = (1 - \delta_{1,i})ie_{i-1}.$$

Proof. Let $D \in \mathfrak{Der}\big(\mathrm{IDD}(K_n^1, 1, 1)\big)$, then we can say that

$$D(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \ \ D(e_2) = \sum_{i=1}^{n} \beta_i e_i, \ \text{ and } D(e_3) = \sum_{i=1}^{n} \gamma_i e_i.$$

Firstly,

$$\begin{array}{rcl} 0 & = & D(e_1 \diamond e_1) \ = \ \left(\sum\limits_{i=1}^{n} \alpha_i e_i\right) \diamond e_1 + e_1 \diamond \left(\sum\limits_{i=1}^{n} \alpha_i e_i\right) \ = \\ & = & \sum\limits_{i=2}^{n} i\alpha_i e_{i-1} + \sum\limits_{i=2}^{n} i\alpha_i e_{i-1} \ = \ 2\sum\limits_{i=2}^{n} i\alpha_i e_{i-1}; \end{array}$$

that immediately gives $D(e_1) = \alpha_1 e_1$. Secondly,

$$D(e_1) \ = \ \tfrac{1}{2}D(e_1 \diamond e_2) \ = \ \tfrac{1}{2}\left((\alpha_1 e_1) \diamond e_2 + e_1 \diamond \left(\sum_{i=1}^{n} \beta_i e_i\right)\right) \ = \ \alpha_1 e_1 + \sum_{i=2}^{n} \tfrac{i}{2}\beta_i e_{i-1};$$

that immediately gives $D(e_2) \ = \ \beta_1 e_1$. Thirdly,

$$\begin{aligned} D(e_2) &= \tfrac{1}{3}D(e_1 \diamond e_3) \ = \ \tfrac{1}{3}\Big((\alpha_1 e_1)\diamond e_3 + e_1 \diamond \Big(\sum_{i=1}^{n}\gamma_i e_i\Big)\Big) \ = \\ &= \alpha_1 e_2 + \sum_{i=2}^{n}\tfrac{i}{3}\gamma_i e_{i-1} \ = \ \tfrac{2}{3}\gamma_2 e_1 + (\alpha_1+\gamma_3)e_2 + \sum_{i=4}^{n}\tfrac{i}{3}\gamma_i e_{i-1}; \end{aligned}$$

that immediately gives $D(e_3) \ = \ \gamma_1 e_1 + \frac{3}{2}\beta_1 e_2 - \alpha_1 e_3$. On the other hand, we have

$$\begin{aligned} D(e_3) &= \tfrac{1}{6}D(e_2 \diamond e_3) \ = \ \tfrac{1}{6}\big((\beta_1 e_1)\diamond e_3 + e_2 \diamond \big(\gamma_1 e_1 + \tfrac{3}{2}\beta_1 e_2 - \alpha_1 e_3\big)\big) \ = \\ &= \tfrac{1}{2}\beta_1 e_2 + \tfrac{1}{3}\gamma_1 e_1 + \beta_1 e_2 - \alpha_1 e_3 \ = \ \tfrac{1}{3}\gamma_1 e_1 + \tfrac{3}{2}\beta_1 e_2 - \alpha_1 e_3; \end{aligned}$$

i.e. $\gamma_1 = 0$. It is easy to see that linear mappings φ and ϕ defined by

$$\varphi(e_i) = (i-2)e_i \text{ and } \phi(e_i) = (1-\delta_{1,i})ie_{i-1}$$

are derivations of $\mathrm{IDD}(K_n^1, 1, 1)$:

$$\begin{aligned} \varphi(e_i \diamond e_j) &= ij(i+j-4)e_{i+j-2} \ = \ ij(i-2)e_{i+j-2} + ij(j-2)e_{i+j-2} \ = \\ &= \big((i-2)e_i\big)\diamond e_j + e_i \diamond \big((j-2)e_j\big) \ = \ \varphi(e_i)\diamond e_j + e_i \diamond \varphi(e_j); \\ \phi(e_i \diamond e_j) &= ij(1-\delta_{1,i+j-2})(i+j-2)e_{i+j-3} \ = \\ &= (1-\delta_{1,i})i(i-1)je_{i+j-3} + (1-\delta_{1,j})ji(j-1)e_{i+j-3} \ = \\ &= \big((1-\delta_{1,i})ie_{i-1}\big)\diamond e_j + e_i \diamond \big((1-\delta_{1,j})je_{j-1}\big) \ = \ \phi(e_i)\diamond e_j + e_i \diamond \phi(e_j). \end{aligned}$$

We can replace D by $D + \alpha_1\varphi - \frac{\beta_1}{2}\phi$ and suppose that $D(e_1) \ = \ D(e_2) \ = \ D(e_3) \ = \ 0$.

Furthermore, using the induction method, we prove that $D(e_k) = 0$ for $k \geq 2$. To do this, let us consider

$$D(e_{k+1}) \ = \ \tfrac{1}{3k}D(e_k \diamond e_3) \ = \ \tfrac{1}{3k}\big(D(e_k)\diamond e_3 + e_k \diamond D(e_3)\big) \ = \ 0.$$

It follows that each derivation of $\mathrm{IDD}(K_n^1, 2, 0)$ is a linear combination of φ, that completes the proof of the statement. □

Corollary 36. $\mathfrak{Der}\big(\mathrm{IDD}(K_\infty^1, 1, 1)\big) = \langle \varphi, \phi \rangle$, *where*

$$\varphi(e_i) = (i-2)e_i \text{ and } \phi(e_i) = (1-\delta_{1,i})ie_{i-1}.$$

4.3 Derivations of $\mathrm{IDD}(K_{\mathfrak{n}}^{\times}, 1, -1)$

Theorem 37. *If* $1 \leq n < \infty$, *then* $\mathfrak{Der}\big(\mathrm{IDD}(K_n^0, 1, -1)\big) = \langle \varphi \rangle$, *where* $\varphi(e_i) = ie_i$.

Proof. Let $D \in \mathfrak{Der}\big(\mathrm{IDD}(K_n^0, 1, -1)\big)$, then $D(e_0) = \sum\limits_{i=0}^{n}\alpha_i e_i$ and $D(e_1) = \sum\limits_{i=0}^{n}\beta_i e_i$. Firstly,

$$0 \ = \ D(e_0 \diamond e_0) \ = \ \Big(\sum_{i=0}^{n}\alpha_i e_i\Big)\diamond e_0 + e_0 \diamond \Big(\sum_{i=0}^{n}\alpha_i e_i\Big) \ = \ \sum_{i=0}^{n} i\alpha_i e_i,$$

that immediately gives $\alpha_k = 0$ for $1 \le k \le n$, i.e., $D(e_0) = \alpha_0 e_0$. Secondly,

$$\begin{aligned} D(e_1) &= D(e_1 \diamond e_0) = \Big(\sum_{i=0}^{n} \beta_i e_i\Big) \diamond e_0 + e_1 \diamond (\alpha_0 e_0) = \sum_{i=0}^{n} i\beta_i e_i + \alpha_0 e_1 = \\ &= (\beta_1 + \alpha_0)\, e_1 + \sum_{i=2}^{n} i\beta_i e_i, \end{aligned}$$

that immediately gives $\beta_0 = 0$, $\alpha_0 = 0$, and $\beta_k = 0$ for $2 \le k \le n$, i.e.,

$$D(e_0) = 0 \text{ and } D(e_1) = \beta_1 e_1.$$

It is easy to see that $\varphi(e_i) = ie_i$ is a derivation of $\mathrm{IDD}(K_n^0, 1, -1)$:

$$\begin{aligned} \varphi(e_i \diamond e_j) &= \tfrac{i}{j+1}(i+j)e_{i+j} = \tfrac{i^2}{j+1}e_{i+j} + \tfrac{ij}{j+1}e_{i+j} = \\ &= \big(ie_i\big) \diamond e_j + e_i \diamond \big(je_j\big) = \varphi(e_i) \diamond e_j + e_i \diamond \varphi(e_j). \end{aligned}$$

Then replacing D by $D - \beta_1\varphi$ we can suppose that $D(e_0) = D(e_1) = 0$.

Furthermore, using the induction method, we prove that $D(e_k) = 0$ for $k \ge 2$. To do this, let us consider

$$D(e_{k+1}) = (k+1)\, D(e_1 \diamond e_k) = (k+1)\big(D(e_1) \diamond e_k + e_1 \diamond D(e_k)\big) = 0.$$

It follows that each derivation of $\mathrm{IDD}(K_n^0, 1, -1)$ is a linear combination of φ, that completes the proof of the statement. □

Corollary 38. *$\mathfrak{Der}\big(\mathrm{IDD}(K_\infty^0, 1, -1)\big) = \langle\varphi\rangle$, where $\varphi(e_i) = ie_i$.*

Proposition 39. *$\mathfrak{Der}\big(\mathrm{IDD}(K_2^1, 1, -1)\big) = \langle\varphi_1, \varphi_2\rangle$, where*

φ_2	$\varphi_2(e_1) = e_2$
φ_1	$\varphi_1(e_1) = e_1,\ \varphi_1(e_2) = 2e_2$

Theorem 40. *If $3 \le n < \infty$, then $\mathfrak{Der}\big(\mathrm{IDD}(K_n^1, 1, -1)\big) = \langle\varphi,\ \phi_1,\ \phi_2\rangle$, where*

φ	$\varphi(e_i) = ie_i$
ϕ_1	$\phi_1(e_1) = ne_{n-1},\ \phi_1(e_2) = (n^2 - n + 2)e_n$
ϕ_2	$\phi_2(e_1) = e_n$

Proof. Let $D \in \mathfrak{Der}\big(\mathrm{IDD}(K_n^1, 1, -1)\big)$, then we can say that $D(e_1) = \sum_{i=1}^{n} \alpha_i e_i$. Firstly,

$$\begin{aligned} D(e_2) &= 2D(e_1 \diamond e_1) = 2\Big(\Big(\sum_{i=1}^{n} \alpha_i e_i\Big) \diamond e_1 + e_1 \diamond \Big(\sum_{i=1}^{n} \alpha_i e_i\Big)\Big) = \\ &= \sum_{i=1}^{n-1} i\alpha_i e_{i+1} + \sum_{i=1}^{n-1} \tfrac{2}{i+1}\alpha_i e_{i+1} = \sum_{i=1}^{n-1} \tfrac{i^2+i+2}{i+1}\alpha_i e_{i+1}; \end{aligned}$$

$$\begin{array}{rcl} D(e_3) & = & 3D(e_1 \diamond e_2) \ = \ 3\left(\left(\sum\limits_{i=1}^{n}\alpha_i e_i\right)\diamond e_2 + e_1 \diamond \left(\sum\limits_{i=1}^{n-1}\frac{i^2+i+2}{i+1}\alpha_i e_{i+1}\right)\right) \ = \\ & = & \sum\limits_{i=1}^{n-2} i\alpha_i e_{i+2} + \sum\limits_{i=1}^{n-2}\frac{3(i^2+i+2)}{(i+1)(i+2)}\alpha_i e_{i+2} \ = \ \sum\limits_{i=1}^{n-2}\frac{i^3+6i^2+5i+6}{(i+1)(i+2)}\alpha_i e_{i+2}. \end{array}$$

On the other hand:

$$\begin{array}{rcl} D(e_3) & = & D(e_2 \diamond e_1) \ = \ \left(\sum\limits_{i=1}^{n-1}\frac{i^2+i+2}{i+1}\alpha_i e_{i+1}\right)\diamond e_1 + e_2 \diamond \left(\sum\limits_{i=1}^{n}\alpha_i e_i\right) = \\ & = & \sum\limits_{i=1}^{n-2}\frac{i^2+i+2}{2}\alpha_i e_{i+2} + \sum\limits_{i=1}^{n-2}\frac{2}{i+1}\alpha_i e_{i+2} \ = \ \sum\limits_{i=1}^{n-2}\frac{i^3+2i^2+3i+6}{2(i+1)}\alpha_i e_{i+2}. \end{array}$$

Comparing these two presentations of $D(e_3)$, we have that

$$i\big(i^3+2i^2-5i+2\big)\alpha_i \ = \ 0 \text{ for } 1 \leq i \leq n-2.$$

The roots of the equation $i\big(i^3+2i^2-5i+2\big)=0$ are $0, \ 1$, or $\frac{1}{2}\big(-3\pm\sqrt{17}\big)$. Then, we obtain $\alpha_2=\alpha_3=\ldots=\alpha_{n-2}=0$. Summarizing, we obtain:

$$D(e_1)=\alpha_1 e_1+\alpha_{n-1}e_{n-1}+\alpha_n e_n, \ \ D(e_2)=2\alpha_1 e_2+\tfrac{n^2-n+2}{n}\alpha_{n-1}e_n, \ \text{ and } \ D(e_3)=3\alpha_1 e_3.$$

It is easy to see that linear mappings $\varphi, \ \phi_1$, and ϕ_2 defined by

$$\varphi(e_i) \ = \ ie_i, \ \ \phi_1(e_1) \ = \ ne_{n-1}, \ \ \phi_1(e_2) \ = \ (n^2-n+2)e_n, \ \ \phi_2(e_1) \ = \ e_n$$

are derivations of $\mathrm{IDD}(K_n^1,1,-1)$:

$$\begin{array}{rcl} \varphi(e_i\diamond e_j) & = & \frac{i}{j+1}(i+j)e_{i+j} \ = \ i\frac{i}{j+1}e_{i+j}+j\frac{i}{j+1}e_{i+j} \ = \\ & = & \big(ie_i\big)\diamond e_j+e_i\diamond\big(je_j\big) \ = \ \varphi(e_i)\diamond e_j+e_i\diamond\varphi(e_j); \\ \phi_1(e_1\diamond e_1) & = & \frac{1}{2}\phi_1(e_2) \ = \ \frac{n^2-n+2}{2}e_n \ = \ \frac{n^2-n}{2n}e_n+\frac{n}{n}e_n \ = \\ & = & (ne_{n-1})\diamond e_1+e_1\diamond(ne_{n-1}) \ = \ \phi_1(e_1)\diamond e_1+e_1\diamond\phi_1(e_1). \end{array}$$

We can replace D by $D-\alpha_1\varphi-\frac{1}{n}\alpha_{n-1}\phi_1-\alpha_n\phi_2$ and suppose that $D(e_1) \ = \ D(e_2) \ = \ 0.$

Furthermore, using the induction method, we can prove that $D(e_k)=0$, for $k\geq 3$. To do this, let us consider

$$D(e_{k+1}) \ = \ \tfrac{2}{k}D(e_k\diamond e_1) \ = \ \tfrac{2}{k}\big(D(e_k)\diamond e_1+e_k\diamond D(e_1)\big) \ = \ 0.$$

It follows that each derivation of $\mathrm{IDD}(K_n^1,1,-1)$ is a linear sum of φ, ϕ_1, and ϕ_2. That completes the proof of the statement. □

Corollary 41. *$\mathfrak{Der}\big(\mathrm{IDD}(K_\infty^1,1,-1)\big)=\langle\varphi\rangle$, where $\varphi(e_i) \ = ie_i$.*

4.4 Derivations of $\mathrm{IDD}(K_{\mathfrak{n}}^{\times}, -1, -1)$

Proposition 42. $\mathfrak{Der}\big(\mathrm{IDD}(K_2^0, -1, -1)\big) = \big\langle \varphi_i, \phi_j \big\rangle_{0 \le i \le 2}^{1 \le j \le 2}$, *where*

φ_i	$\varphi_i(e_0) \ = \ e_i, \ \varphi_i(e_2) \ = \ 2\delta_{0,i}e_2$	$0 \le i \le 2$
ϕ_j	$\phi_j(e_1) \ = \ e_j$	$1 \le j \le 2$

Proposition 43. $\mathfrak{Der}\big(\mathrm{IDD}(K_3^0, -1, -1)\big) = \big\langle \varphi_i, \phi_j \big\rangle_{1 \le i \le 3}^{1 \le j \le 3}$, *where*

φ_i	$\varphi_i(e_0) = e_i, \ \ \varphi_1(e_2) = e_3$	$1 \le i \le 3$
ϕ_j	$\phi_j(e_1) = e_j, \ \ \phi_1(e_3) = e_3$	$1 \le j \le 3$

Proposition 44. $\mathfrak{Der}\big(\mathrm{IDD}(K_4^0, -1, -1)\big) = \big\langle \varphi_i, \phi_j \big\rangle_{0 \le i \le 4}^{2 \le j \le 4}$, *where*

φ_4	$\varphi_4(e_0) = e_4$
φ_3	$\varphi_3(e_0) = e_3$
φ_2	$\varphi_2(e_0) = 3e_2, \ \varphi_2(e_2) = 2e_4$
φ_1	$\varphi_1(e_0) = 2e_1, \ \varphi_1(e_2) = 2e_3, \ \varphi_1(e_3) = e_4,$
φ_0	$\varphi_0(e_0) = 2e_0, \ \varphi_0(e_1) = 3e_1, \ \varphi_0(e_2) = 4e_2, \ \varphi_0(e_3) = 5e_3, \ \varphi_0(e_4) = 6e_4$
ϕ_4	$\phi_4(e_1) = e_4$
ϕ_3	$\phi_3(e_1) = e_3$
ϕ_2	$\phi_2(e_1) = 3e_2, \ \phi_2(e_3) = 2e_4$

Theorem 45. *If* $5 \le n < \infty$, *then* $\mathfrak{Der}\big(\mathrm{IDD}(K_n^0, -1, -1)\big) = \big\langle \varphi, \phi_i, \psi_j \big\rangle_{0 \le i \le 4}^{0 \le j \le 2}$, *where*

φ	$\varphi(e_i) \ = \ (i+2)\, e_i, \ 0 \le i \le n$
ϕ_4	$\phi_4(e_0) = 4(n-1)(n-3)e_{n-4}, \ \phi_4(e_1) = (n-2)(n+5)e_{n-3},$ $\phi_4(e_2) = 8(n-1)e_{n-2}, \ \phi_4(e_3) = 6(n+1)e_{n-1}, \ \phi_4(e_4) = 4(n+5)e_n$
ϕ_3	$\phi_3(e_0) = (n-2)e_{n-3}, \ \phi_3(e_2) = 2e_{n-1}, \ \phi_3(e_3) = e_n$
ϕ_2	$\phi_2(e_0) = (n-1)e_{n-2}, \ \phi_2(e_2) = 2e_n$
ϕ_1	$\phi_1(e_0) = e_{n-1}$
ϕ_0	$\phi_0(e_0) = e_n$
ψ_2	$\psi_2(e_1) = (n-1)e_{n-2}, \psi_2(e_3) = e_n$
ψ_1	$\psi_1(e_1) = e_{n-1}$
ψ_0	$\psi_0(e_1) = e_n$

Proof. Let $D \in \mathfrak{Der}\big(\mathrm{IDD}(K_n^0, -1, -1)\big)$, then $D(e_0) = \sum\limits_{i=0}^{n} \alpha_i e_i$ and $D(e_1) = \sum\limits_{i=0}^{n} \beta_i e_i$. Firstly,

$$D(e_2) \ = \ D(e_0 \diamond e_0) \ = \ \left(\sum_{i=0}^{n} \alpha_i e_i\right) \diamond e_0 + e_0 \diamond \left(\sum_{i=0}^{n} \alpha_i e_i\right) \ =$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-2} \tfrac{1}{i+1}\alpha_i e_{i+2} + \sum_{i=0}^{n-2} \tfrac{1}{i+1}\alpha_i e_{i+2} \ = \ \sum_{i=0}^{n-2} \tfrac{2}{i+1}\alpha_i e_{i+2};\\
D(e_3) &= 2D(e_0 \diamond e_1) \ = \ 2\left(\left(\sum_{i=0}^{n} \alpha_i e_i\right) \diamond e_1 + e_0 \diamond \left(\sum_{i=0}^{n} \beta_i e_i\right)\right) \ = \\
&= \sum_{i=0}^{n-3} \tfrac{1}{i+1}\alpha_i e_{i+3} + \sum_{i=0}^{n-2} \tfrac{2}{i+1}\beta_i e_{i+2} \ = \ 2\beta_0 e_2 + \sum_{i=0}^{n-3} \left(\tfrac{1}{i+1}\alpha_i + \tfrac{2}{i+2}\beta_{i+1}\right) e_{i+3}.
\end{aligned}
$$

Secondly, we find two expressions for $D(e_4)$:

$$
\begin{aligned}
D(e_4) &= 3D(e_0 \diamond e_2) \ = \ 3\left(\left(\sum_{i=0}^{n} \alpha_i e_i\right) \diamond e_2 + e_0 \diamond \left(\sum_{i=0}^{n-2} \tfrac{2}{i+1}\alpha_i e_{i+2}\right)\right) \ = \\
&= \sum_{i=0}^{n-4} \tfrac{1}{i+1}\alpha_i e_{i+4} + \sum_{i=0}^{n-4} \tfrac{6}{(i+1)(i+3)}\alpha_i e_{i+4} \ = \ \sum_{i=0}^{n-4} \tfrac{i+9}{(i+1)(i+3)}\alpha_i e_{i+4};\\
D(e_4) &= 4D(e_1 \diamond e_1) \ = \ 4\left(\left(\sum_{i=0}^{n} \beta_i e_i\right) \diamond e_1 + e_1 \diamond \left(\sum_{i=0}^{n} \beta_i e_i\right)\right) \ = \\
&= \sum_{i=0}^{n-3} \tfrac{2}{i+1}\beta_i e_{i+3} + \sum_{i=0}^{n-3} \tfrac{2}{i+1}\beta_i e_{i+3} \ = \ 4\beta_0 e_3 + \sum_{i=0}^{n-4} \tfrac{4}{i+2}\beta_{i+1} e_{i+4};
\end{aligned}
$$

that immediately gives $\beta_0 = 0$ and $\beta_k = \frac{(k+1)(k+8)}{4k(k+2)}\alpha_{k-1}$ for $1 \le k \le n-3$, i.e.,

$$
\begin{aligned}
D(e_1) &= \sum_{i=1}^{n-3} \tfrac{(i+1)(i+8)}{4i(i+2)}\alpha_{i-1} e_i + \beta_{n-2}e_{n-2} + \beta_{n-1}e_{n-1} + \beta_n e_n;\\
D(e_3) &= \sum_{i=0}^{n-4} \tfrac{3(i+5)}{2(i+1)(i+3)}\alpha_i e_{i+3} + (\tfrac{1}{n-2}\alpha_{n-3} + \tfrac{2}{n-1}\beta_{n-2})e_n.
\end{aligned}
$$

Thirdly, we find different expressions for $D(e_5)$:

$$
\begin{aligned}
D(e_5) &= 4D(e_0 \diamond e_3) \ = \ 4\Big(\left(\sum_{i=0}^{n} \alpha_i e_i\right) \diamond e_3 + e_0 \diamond \left(\sum_{i=0}^{n-4} \tfrac{3(i+5)}{2(i+1)(i+3)}\alpha_i e_{i+3}\right)\Big) \ = \\
&= \sum_{i=0}^{n-5} \tfrac{1}{i+1}\alpha_i e_{i+5} + \sum_{i=0}^{n-5} \tfrac{6(i+5)}{(i+1)(i+3)(i+4)}\alpha_i e_{i+5}\\
&= \sum_{i=0}^{n-5} \tfrac{(i+6)(i+7)}{(i+1)(i+3)(i+4)}\alpha_i e_{i+5};
\end{aligned}
$$

$$
\begin{aligned}
D(e_5) &= 6D(e_1 \diamond e_2) \ = \ 6\big(D(e_1) \diamond e_2 + e_1 \diamond D(e_2)\big) \ = \\
&= 6\Big(\left(\sum_{i=1}^{n-3} \tfrac{(i+1)(i+8)}{4i(i+2)}\alpha_{i-1} e_i + \beta_{n-2}e_{n-2} + \beta_{n-1}e_{n-1} + \beta_n e_n\right) \diamond e_2 +\\
&\quad e_1 \diamond \left(\sum_{i=0}^{n-2} \tfrac{2}{i+1}\alpha_i e_{i+2}\right)\Big) \ = \\
&= \sum_{i=1}^{n-4} \tfrac{i+8}{2i(i+2)}\alpha_{i-1} e_{i+4} + \sum_{i=0}^{n-5} \tfrac{6}{(i+1)(i+3)}\alpha_i e_{i+5} \ = \\
&= \sum_{i=0}^{n-5} \tfrac{i+21}{2(i+1)(i+3)}\alpha_i e_{i+5}.
\end{aligned}
$$

Comparing the present expressions, we obtain $\alpha_k = 0$ for $1 \leq k \leq n-5$; i.e.,

$$\begin{array}{rcl} D(e_0) & = & \alpha_0 e_0 + \alpha_{n-4}e_{n-4} + \alpha_{n-3}e_{n-3} + \alpha_{n-2}e_{n-2} + \alpha_{n-1}e_{n-1} + \alpha_n e_n; \\ D(e_1) & = & \frac{3}{2}\alpha_0 e_1 + \frac{(n-2)(n+5)}{4(n-1)(n-3)}\alpha_{n-4}e_{n-3} + \beta_{n-2}e_{n-2} + \beta_{n-1}e_{n-1} + \beta_n e_n; \\ D(e_2) & = & 2\alpha_0 e_2 + \frac{2}{n-3}\alpha_{n-4}e_{n-2} + \frac{2}{n-2}\alpha_{n-3}e_{n-1} + \frac{2}{n-1}\alpha_{n-2}e_n; \\ D(e_3) & = & \frac{5}{2}\alpha_0 e_3 + \frac{3(n+1)}{2(n-1)(n-3)}\alpha_{n-4}e_{n-1} + \left(\frac{1}{n-2}\alpha_{n-3} + \frac{1}{n-1}\beta_{n-2}\right)e_n; \\ D(e_4) & = & 3\alpha_0 e_4 + \frac{n+5}{(n-1)(n-3)}\alpha_{n-4}e_n; \\ D(e_5) & = & \frac{7}{2}\alpha_0 e_5, \text{ and } D(e_6) \;=\; 4\alpha_0 e_6. \end{array}$$

Furthermore, using the induction method, we can prove that $D(e_k) = \frac{k+2}{2}\alpha_0 e_k$, for $5 \leq k \leq n$. To do this, let us consider

$$\begin{array}{rcl} D(e_{k+2}) & = & (k+1)D(e_k \diamond e_0) \;=\; (k+1)\big(D(e_k)\diamond e_0 + e_k \diamond D(e_0)\big) \;= \\ & = & (k+1)\left(\left(\frac{k+2}{2}\alpha_0 e_k\right)\diamond e_0 + e_k \diamond \left(\alpha_0 e_0 + \sum\limits_{i=n-4}^{n}\alpha_i e_i\right)\right) \;=\; \frac{k+4}{2}\alpha_0 e_{k+2}. \end{array}$$

Combining all the obtained results, we see that D is a linear combination of mappings $\{\varphi, \phi_i, \psi_j\}_{0\leq i\leq 4}^{0\leq j\leq 2}$. □

Corollary 46. $\mathfrak{Der}\big(\mathrm{IDD}(K^0_\infty, -1, -1)\big) = \langle \varphi \rangle$, *where* $\varphi(e_i) = (i+2)\, e_i$.

Proposition 47. $\mathfrak{Der}\big(\mathrm{IDD}(K^1_4, -1, -1)\big) = \langle \varphi_i, \phi_j, \psi_j \rangle_{1\leq i\leq 4}^{2\leq j\leq 4}$, *where*

φ_i	$\varphi_i(e_k) = \delta_{1,k}e_i + 2\delta_{1,i}\delta_{4,k}e_4$	$1 \leq i \leq 4$
ϕ_j	$\phi_j(e_2) = e_j$	$2 \leq j \leq 4$
ψ_j	$\psi_j(e_3) = e_j$	$2 \leq j \leq 4$

Proposition 48. $\mathfrak{Der}\big(\mathrm{IDD}(K^1_5, -1, -1)\big) = \langle \varphi_i, \phi_j, \psi_k \rangle_{1\leq i\leq 5;\ 2\leq j\leq 5}^{3\leq k\leq 5}$, *where*

φ_5	$\varphi_5(e_1) = e_5$
φ_4	$\varphi_4(e_1) = e_4$
φ_3	$\varphi_3(e_1) = e_3$
φ_2	$\varphi_2(e_1) = 3e_2, \quad \varphi_2(e_4) = 4e_5$
φ_1	$\varphi_1(e_1) = e_1, \quad \varphi_1(e_4) = 2e_4, \quad \varphi_1(e_5) = e_5$
ϕ_5	$\phi_5(e_2) = e_5$
ϕ_4	$\phi_4(e_2) = e_4$
ϕ_3	$\phi_3(e_2) = e_3$
ϕ_2	$\phi_2(e_2) = e_2, \quad \phi_2(e_5) = e_5,$
ψ_5	$\psi_5(e_3) = e_5$
ψ_4	$\psi_4(e_3) = e_4$
ψ_3	$\psi_3(e_3) = e_3$

Proposition 49. $\mathfrak{Der}\big(\mathrm{IDD}(K_6^1,-1,-1)\big)=\big\langle \varphi_i,\phi_j,\psi_k\big\rangle_{1\leq i\leq 6;\ 2\leq j\leq 6}^{4\leq k\leq 6}$, *where*

φ_6	$\varphi_6(e_1)=e_6$
φ_5	$\varphi_5(e_1)=e_5$
φ_4	$\varphi_4(e_1)=e_4$
φ_3	$\varphi_3(e_1)=e_3,\ \ \varphi_3(e_4)=e_6$
φ_2	$\varphi_2(e_1)=3e_2,\ \ \varphi_2(e_4)=4e_5,\ \ \varphi_2(e_5)=2e_6$
φ_1	$\varphi_1(e_1)=e_1,\ \ \varphi_1(e_3)=-e_3,\ \ \varphi_1(e_4)=2e_4,\ \ \varphi_1(e_5)=e_5$
ϕ_6	$\phi_6(e_2)=e_6$
ϕ_5	$\phi_5(e_2)=e_5$
ϕ_4	$\phi_4(e_2)=e_4$
ϕ_3	$\phi_3(e_2)=4e_3,\ \ \phi_3(e_5)=3e_6$
ϕ_2	$\phi_2(e_2)=e_2,\ \ \phi_2(e_3)=2e_3,\ \ \phi_2(e_5)=e_5,\ \ \phi_2(e_6)=2e_6$
ψ_6	$\psi_6(e_3)=e_6$
ψ_5	$\psi_5(e_3)=e_5$
ψ_4	$\psi_4(e_3)=e_4$

Proposition 50. $\mathfrak{Der}\big(\mathrm{IDD}(K_7^1,-1,-1)\big)=\big\langle \varphi_i,\phi_j,\psi_k\big\rangle_{1\leq i\leq 7;\ 3\leq j\leq 7}^{5\leq k\leq 7}$, *where*

φ_7	$\varphi_7(e_1)=e_7$
φ_6	$\varphi_6(e_1)=e_6$
φ_5	$\varphi_5(e_1)=e_5$
φ_4	$\varphi_4(e_1)=5e_4,\ \ \varphi_4(e_4)=4e_7$
φ_3	$\varphi_3(e_1)=2e_3,\ \ \varphi_3(e_4)=2e_6,\ \ \varphi_3(e_5)=e_7$
φ_2	$\varphi_2(e_1)=6e_2,\ \ \varphi_2(e_3)=5e_4,\ \ \varphi_2(e_4)=-8e_5,\ \ \varphi_2(e_5)=4e_6$
φ_1	$\varphi_1(e_1)=3e_1,\ \ \varphi_1(e_2)=4e_2,\ \ \varphi_1(e_3)=5e_3,\ \ \varphi_1(e_4)=2e_4,$ $\varphi_1(e_5)=7e_5,\ \ \varphi_1(e_6)=8e_6,\ \ \varphi_1(e_7)=3e_7$
ϕ_7	$\phi_7(e_2)=e_7$
ϕ_6	$\phi_6(e_2)=e_6$
ϕ_5	$\phi_5(e_2)=e_5$
ϕ_4	$\phi_4(e_2)=5e_4,\ \ \phi_4(e_5)=3e_7$
ϕ_3	$\phi_3(e_2)=8e_3,\ \ \phi_3(e_3)=15e_4,\ \ \phi_3(e_5)=6e_6,\ \ \phi_3(e_6)=12e_7$
ψ_7	$\psi_7(e_3)=e_7$
ψ_6	$\psi_6(e_3)=e_6$
ψ_5	$\psi_5(e_3)=e_5$

Theorem 51. *If* $8\leq n<\infty$, *then* $\mathfrak{Der}\big(\mathrm{IDD}(K_n^1,-1,-1)\big)=\big\langle \varphi,\phi_i,\psi_j,\pi_k\big\rangle_{0\leq i\leq 6;\ 0\leq j\leq 4}^{0\leq k\leq 2}$, *where*

φ	$\varphi(e_i) \ = \ (i+2)\, e_i, \ 1 \leq i \leq n$
ϕ_6	$\phi_6(e_1) = 18(n-2)(n-5)e_{n-6}, \ \phi_6(e_2) = 8(n-4)(n+3)e_{n-5},$ $\phi_6(e_3) = 3(n-3)(n+18)e_{n-4}, \ \ \phi_6(e_4) = 72(n-2)e_{n-3},$ $\phi_6(e_5) = 60ne_{n-2}, \ \phi_6(e_6) = 48(n+3)e_{n-1}, \ \phi_6(e_7) = 36(n+8)e_n$
ϕ_5	$\phi_5(e_1) = 2(n-4)e_{n-5}, \ \phi_5(e_3) = (2-n)e_{n-3}, \ \phi_5(e_4) = 8e_{n-2}, \ \phi_5(e_5) = 4e_{n-1}$
ϕ_4	$\phi_4(e_1) = (n-3)e_{n-4}, \ \phi_4(e_4) = 4e_{n-1}, \ \phi_4(e_5) = 2e_n$
ϕ_3	$\phi_3(e_1) = (n-2)e_{n-3}, \ \phi_3(e_4) = 4e_n$
ϕ_2	$\phi_2(e_1) = e_{n-2}$
ϕ_1	$\phi_1(e_1) = e_{n-1}$
ϕ_0	$\phi_0(e_1) = e_n$
ψ_4	$\psi_4(e_2) = 2(n-3)e_{n-4}, \ \psi_4(e_3) = 3(n-2)e_{n-3}, \ \psi_4(e_5) = -6e_{n-1}, \ \psi_4(e_6) = 12e_n$
ψ_3	$\psi_3(e_2) = (n-2)e_{n-3}, \psi_3(e_5) = 3e_n$
ψ_2	$\psi_2(e_2) = e_{n-2}$
ψ_1	$\psi_1(e_2) = e_{n-1}$
ψ_0	$\psi_0(e_2) = e_n$
π_2	$\pi_2(e_3) = e_{n-2}$
π_1	$\pi_1(e_3) = e_{n-1}$
π_0	$\pi_0(e_3) = e_n$

Proof. Let $D \in \mathfrak{Der}\big(\mathrm{IDD}(K_n^1, -1, -1)\big)$, then we can say that

$$D(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \ D(e_2) = \sum_{i=1}^{n} \beta_i e_i, \text{ and } D(e_3) = \sum_{i=1}^{n} \gamma_i e_i.$$

Firstly, we find expressions for $D(e_4)$, $D(e_5)$, and $D(e_6)$:

$$\begin{aligned}
D(e_4) &= 4D(e_1 \diamond e_1) \ = \ 4\left(\left(\sum_{i=1}^{n} \alpha_i e_i\right) \diamond e_1 + e_1 \diamond \left(\sum_{i=1}^{n} \alpha_i e_i\right)\right) \\
&= \sum_{i=1}^{n-3} \tfrac{2}{i+1}\alpha_i e_{i+3} + \sum_{i=1}^{n-3} \tfrac{2}{i+1}\alpha_i e_{i+3} = \sum_{i=1}^{n-3} \tfrac{4}{i+1}\alpha_i e_{i+3}; \\
D(e_5) &= 6D(e_1 \diamond e_2) \ = \ 6\left(\left(\sum_{i=1}^{n} \alpha_i e_i\right) \diamond e_2 + e_1 \diamond \left(\sum_{i=1}^{n} \beta_i e_i\right)\right) \\
&= \sum_{i=1}^{n-4} \tfrac{2}{i+1}\alpha_i e_{i+4} + \sum_{i=1}^{n-3} \tfrac{3}{i+1}\beta_i e_{i+3} \\
&= \sum_{i=1}^{n-4} \tfrac{2}{i+1}\alpha_i e_{i+4} + \sum_{i=0}^{n-4} \tfrac{3}{i+2}\beta_{i+1} e_{i+4} \ = \ \tfrac{3}{2}\beta_1 e_4 + \sum_{i=1}^{n-4} \left(\tfrac{2}{i+1}\alpha_i + \tfrac{3}{i+2}\beta_{i+1}\right) e_{i+4}; \\
D(e_6) &= 8D(e_1 \diamond e_3) \ = \ 8\left(\left(\sum_{i=1}^{n} \alpha_i e_i\right) \diamond e_3 + e_1 \diamond \left(\sum_{i=1}^{n} \gamma_i e_i\right)\right) \\
&= \sum_{i=1}^{n-5} \tfrac{2}{i+1}\alpha_i e_{i+5} + \sum_{i=1}^{n-3} \tfrac{4}{i+1}\gamma_i e_{i+3} \\
&= \sum_{i=1}^{n-5} \tfrac{2}{i+1}\alpha_i e_{i+5} + \sum_{i=-1}^{n-5} \tfrac{4}{i+3}\gamma_{i+2} e_{i+5}
\end{aligned}$$

$$\begin{aligned}
&= 2\gamma_1 e_4 + \tfrac{4}{3}\gamma_2 e_5 + \sum_{i=1}^{n-5}\left(\tfrac{2}{i+1}\alpha_i + \tfrac{4}{i+3}\gamma_{i+2}\right) e_{i+5};\\
D(e_6) &= 9D(e_2 \diamond e_2) \;=\; 9\left(\left(\sum_{i=1}^{n}\beta_i e_i\right)\diamond e_2 + e_2 \diamond \left(\sum_{i=1}^{n}\beta_i e_i\right)\right)\\
&= \sum_{i=1}^{n-4}\tfrac{3}{i+1}\beta_i e_{i+4} + \sum_{i=1}^{n-4}\tfrac{3}{i+1}\beta_i e_{i+4}\\
&= \sum_{i=1}^{n-4}\tfrac{6}{i+1}\beta_i e_{i+4} = 3\beta_1 e_5 + \sum_{i=1}^{n-5}\tfrac{6}{i+2}\beta_{i+1} e_{i+5}.
\end{aligned}$$

Comparing two different expressions for $D(e_6)$, we have that

$$\gamma_1 = 0;\;\; \gamma_2 = \tfrac{9}{4}\beta_1;\;\; \gamma_k = \tfrac{3(k+1)}{2k}\beta_{k-1} - \tfrac{k+1}{2(k-1)}\alpha_{k-2},\; 3 \le k \le n-3;\;\; \text{i.e.}$$

$$D(e_3) \;=\; \tfrac{9}{4}\beta_1 e_2 + \sum_{i=3}^{n-3}\left(\tfrac{3(i+1)}{2i}\beta_{i-1} - \tfrac{i+1}{2(i-1)}\alpha_{i-2}\right) e_i + \gamma_{n-2}e_{n-2} + \gamma_{n-1}e_{n-1} + \gamma_n e_n.$$

In a similar way, we find two different expressions for $D(e_7)$:

$$\begin{aligned}
D(e_7) &= 10D(e_1 \diamond e_4) \;=\; 10\left(\left(\sum_{i=1}^{n}\alpha_i e_i\right)\diamond e_4 + e_1 \diamond \left(\sum_{i=1}^{n-3}\tfrac{4}{i+1}\alpha_i e_{i+3}\right)\right) \;=\\
&= \sum_{i=1}^{n-6}\tfrac{2}{i+1}\alpha_i e_{i+6} + \sum_{i=1}^{n-6}\tfrac{20}{(i+1)(i+4)}\alpha_i e_{i+6} \;=\; \sum_{i=1}^{n-6}\tfrac{2(i+14)}{(i+1)(i+4)}\alpha_i e_{i+6};\\
D(e_7) &= 12D(e_2 \diamond e_3) \;=\; 12\big(D(e_2)\diamond e_3 + e_2 \diamond D(e_3)\big) \;=\\
&= 12\left(\left(\sum_{i=1}^{n}\beta_i e_i\right)\diamond e_3 + \right.\\
&\left. + e_2 \diamond \left(\tfrac{9}{4}\beta_1 e_2 + \sum_{i=3}^{n-3}\left(\tfrac{3(i+1)}{2i}\beta_{i-1} - \tfrac{i+1}{2(i-1)}\alpha_{i-2}\right) e_i + \sum_{j=n-2}^{n}\gamma_j e_j\right)\right) \;=\\
&= \sum_{i=1}^{n-5}\tfrac{3}{i+1}\beta_i e_{i+5} + 3\beta_1 e_6 + \sum_{i=3}^{n-4}\tfrac{2}{i+1}\left(\tfrac{3(i+1)}{2i}\beta_{i-1} - \tfrac{6(i+1)}{i-1}\alpha_{i-2}\right) e_{i+4} \;=\\
&= \sum_{i=1}^{n-5}\tfrac{3}{i+1}\beta_i e_{i+5} + 3\beta_1 e_6 + \sum_{i=3}^{n-4}\left(\tfrac{6}{i}\beta_{i-1} - \tfrac{2}{i-1}\alpha_{i-2}\right) e_{i+4} =\\
&= \sum_{i=1}^{n-5}\tfrac{3}{i+1}\beta_i e_{i+5} + 3\beta_1 e_6 + \sum_{i=2}^{n-5}\left(\tfrac{6}{i+1}\beta_i - \tfrac{2}{i}\alpha_{i-1}\right) e_{i+5} =\\
&= \tfrac{3}{2}\beta_1 e_6 + \sum_{i=2}^{n-5}\tfrac{3}{i+1}\beta_i e_{i+5} + 3\beta_1 e_6 + \sum_{i=2}^{n-5}\left(\tfrac{6}{i+1}\beta_i - \tfrac{2}{i}\alpha_{i-1}\right) e_{i+5} =\\
&= \tfrac{9}{2}\beta_1 e_6 + \sum_{i=2}^{n-5}\left(\tfrac{9}{i+1}\beta_i - \tfrac{2}{i}\alpha_{i-1}\right) e_{i+5} \;=\\
&= \tfrac{9}{2}\beta_1 e_6 + \sum_{i=1}^{n-6}\left(\tfrac{9}{i+2}\beta_{i+1} - \tfrac{2}{i+1}\alpha_i\right) e_{i+6}.
\end{aligned}$$

Comparing two different expressions for $D(e_7)$, we have that $\beta_1 = 0$ and

$$\beta_k = \tfrac{4(k+1)(k+8)}{9k(k+3)}\alpha_{k-1}, \quad 2 \le k \le n-5;\;\; \text{i.e.,}$$

$$\begin{aligned}
D(e_2) &= \sum_{i=2}^{n-5} \tfrac{4(i+1)(i+8)}{9i(i+3)}\alpha_{i-1}e_i + \sum_{j=n-4}^{n} \beta_j e_j;\\
D(e_3) &= \sum_{i=3}^{n-4} \tfrac{(i+1)(i+22)}{6(i-1)(i+2)}\alpha_{i-2}e_i + \left(\tfrac{3(n-2)}{2(n-3)}\beta_{n-4} - \tfrac{n-2}{2(n-4)}\alpha_{n-5}\right)e_{n-3} + \sum_{j=n-2}^{n} \gamma_j e_j;\\
D(e_5) &= \sum_{i=1}^{n-6} \tfrac{10(i+6)}{3(i+1)(i+4)}\alpha_i e_{i+4} + \left(\tfrac{2}{n-4}\alpha_{n-5} - \tfrac{3}{n-3}\beta_{n-4}\right)e_{n-1} + \left(\tfrac{2}{n-3}\alpha_{n-4} - \tfrac{3}{n-2}\beta_{n-3}\right)e_n;\\
D(e_6) &= \sum_{i=1}^{n-6} \tfrac{8(i+9)}{3(i+1)(i+4)}\alpha_i e_{i+5} + \tfrac{6}{n-3}\beta_{n-4}e_n.
\end{aligned}$$

Next, we obtain two expressions for $D(e_8)$:

$$\begin{aligned}
D(e_8) &= 12D(e_1 \diamond e_5) = 12\left(\sum_{i=1}^{n}\alpha_i e_i\right)\diamond e_5 + 12e_1 \diamond \left(\sum_{i=1}^{n-6}\tfrac{10(i+6)}{3(i+1)(i+4)}\alpha_i e_{i+4} + \right.\\
&\quad \left. + \left(\tfrac{2\alpha_{n-5}}{n-4} - \tfrac{3\beta_{n-4}}{n-3}\right)e_{n-1} + \left(\tfrac{2\alpha_{n-4}}{n-3} - \tfrac{3\beta_{n-3}}{n-2}\right)e_n\right) =\\
&= \sum_{i=1}^{n-7}\tfrac{2}{i+1}\alpha_i e_{i+7} + \sum_{i=1}^{n-7}\tfrac{20(i+6)}{(i+1)(i+4)(i+5)}\alpha_i e_{i+7} = \sum_{i=1}^{n-7}\tfrac{2(i^2+19i+80)}{(i+1)(i+4)(i+5)}\alpha_i e_{i+7};\\
D(e_8) &= 16D(e_3 \diamond e_3) = 16\left(\sum_{i=3}^{n-4}\tfrac{(i+1)(i+22)}{6(i-1)(i+2)}\alpha_{i-2}e_i + \right.\\
&\quad \left. + \left(\tfrac{3(n-2)}{2(n-3)}\beta_{n-4} - \tfrac{n-2}{2(n-4)}\alpha_{n-5}\right)e_{n-3} + \sum_{j=n-2}^{n}\gamma_j e_j\right)\diamond e_3 +\\
&\quad +16e_3 \diamond \left(\sum_{i=3}^{n-4}\tfrac{(i+1)(i+22)}{6(i-1)(i+2)}\alpha_{i-2}e_i + \right.\\
&\quad \left. + \left(\tfrac{3(n-2)}{2(n-3)}\beta_{n-4} - \tfrac{n-2}{2(n-4)}\alpha_{n-5}\right)e_{n-3} + \sum_{j=n-2}^{n}\gamma_j e_j\right) =\\
&= \sum_{i=3}^{n-5}\tfrac{2(i+22)}{3(i-1)(i+2)}\alpha_{i-2}e_{i+5} + \sum_{i=3}^{n-5}\tfrac{2(i+22)}{3(i-1)(i+2)}\alpha_{i-2}e_{i+5}\\
&= \sum_{i=1}^{n-7}\tfrac{4(i+24)}{3(i+1)(i+4)}\alpha_i e_{i+7}.
\end{aligned}$$

Comparing the present expressions for $D(e_8)$, we have $\alpha_k = 0$, for $2 \le k \le n-7$, that gives

$$\begin{aligned}
D(e_1) &= \alpha_1 e_1 + \sum_{j=n-6}^{n}\alpha_j e_j;\\
D(e_2) &= \tfrac{4}{3}\alpha_1 e_2 + \tfrac{4(n-4)(n+3)}{9(n-2)(n-5)}\alpha_{n-6}e_{n-5} + \sum_{j=n-4}^{n}\beta_j e_j;\\
D(e_3) &= \tfrac{5}{3}\alpha_1 e_3 + \tfrac{(n-3)(n+18)}{6(n-2)(n-5)}\alpha_{n-6}e_{n-4} + \left(\tfrac{3(n-2)}{2(n-3)}\beta_{n-4} - \tfrac{n-2}{2(n-4)}\alpha_{n-5}\right)e_{n-3} + \sum_{j=n-2}^{n}\gamma_j e_j;\\
D(e_4) &= 2\alpha_1 e_4 + \tfrac{4}{n-5}\alpha_{n-6}e_{n-3} + \tfrac{4}{n-4}\alpha_{n-5}e_{n-2} + \tfrac{4}{n-3}\alpha_{n-4}e_{n-1} + \tfrac{4}{n-2}\alpha_{n-3}e_n;\\
D(e_5) &= \tfrac{7}{3}\alpha_1 e_5 + \tfrac{10n\alpha_{n-6}}{3(n-2)(n-5)}e_{n-2} + \left(\tfrac{2\alpha_{n-5}}{n-4} - \tfrac{3\beta_{n-4}}{n-3}\right)e_{n-1} + \left(\tfrac{2\alpha_{n-4}}{n-3} - \tfrac{3\beta_{n-3}}{n-2}\right)e_n;
\end{aligned}$$

$$\begin{array}{rcl} D(e_6) &=& \frac{8}{3}\alpha_1 e_6 + \frac{8(n+3)}{3(n-2)(n-5)}\alpha_{n-6}e_{n-1} + \frac{6}{n-3}\beta_{n-4}e_n; \\ D(e_7) &=& 3\alpha_1 e_7 + \frac{2(n+8)}{(n-2)(n-5)}\alpha_{n-6}e_n; \\ D(e_8) &=& \frac{10}{3}\alpha_1 e_8. \end{array}$$

Furthermore, using the induction method, we can prove that $D(e_k) = \frac{k+2}{3}\alpha_1 e_k$, for $8 \le k \le n$. To do this, let us consider

$$\begin{array}{rcl} D(e_{k+1}) &=& 2(k-1)D(e_1 \diamond e_{k-2}) \;=\; 2(k-1)\big(D(e_1)\diamond e_{k-2} + e_1 \diamond D(e_{k-2})\big) \;= \\ &=& 2(k-1)\left(\left(\alpha_1 e_1 + \sum\limits_{j=n-6}^{n}\alpha_j e_j\right)\diamond e_{k-2} + e_1 \diamond \left(\frac{k}{3}\alpha_1 e_{k-2}\right)\right) \;=\; \frac{k+3}{3}\alpha_1 e_{k+1}. \end{array}$$

Combining all the obtained results, we see that D is a linear combination of mappings $\{\varphi, \phi_i, \psi_j, \pi_k\}^{0\le k\le 2}_{0\le i\le 6;\ 0\le j\le 4}$. □

Corollary 52. *$\mathfrak{Der}\big(\mathrm{IDD}(K^1_\infty, -1, -1)\big) = \langle \varphi \rangle$, where $\varphi(e_i) = (i+2)\, e_i$.*

4.5 Derivations of $\mathrm{IDD}(K^{\times}_{\mathfrak{n}}, 0, -2)$

Proposition 53. *$\mathfrak{Der}\big(\mathrm{IDD}(K^0_2, 0, -2)\big) = \langle \varphi_i, \phi_j \rangle^{1\le j\le 2}_{0\le i\le 2}$, where*

φ_i	$\varphi_i(e_0) = e_i,\ \varphi_i(e_2) = 2\delta_{0,i}e_2$	$0 \le i \le 2$
ϕ_j	$\phi_j(e_1) = e_j$	$1 \le j \le 2$

Proposition 54. *$\mathfrak{Der}\big(\mathrm{IDD}(K^0_3, 0, -2)\big) = \langle \varphi_i, \phi_j \rangle^{1\le j\le 3}_{0\le i\le 3}$, where*

φ_3	$\varphi_3(e_0) = e_3$
φ_2	$\varphi_2(e_0) = e_2$
φ_1	$\varphi_1(e_0) = 3e_1,\quad \varphi_1(e_2) = 4e_3$
φ_0	$\varphi_0(e_0) = e_0,\quad \varphi_0(e_2) = 2e_2,\quad \varphi_0(e_3) = e_3$
ϕ_3	$\phi_3(e_1) = e_3$
ϕ_2	$\phi_2(e_1) = e_2$
ϕ_1	$\phi_1(e_1) = e_1,\quad \phi_1(e_3) = e_3$

Theorem 55. *If $4 \le n < \infty$, then $\mathfrak{Der}\big(\mathrm{IDD}(K^0_n, 0, -2)\big) = \langle \varphi, \phi_i, \psi_j, \pi_k \rangle^{0\le j\le 1}_{0\le i\le 3}$, where*

φ	$\varphi(e_i) = (i+2)\, e_i,\ 0 \le i \le n$
ϕ_3	$\phi_3(e_0) = (n-1)(n^2-4)e_{n-3},\ \phi_3(e_1) = n^2(n-1)e_{n-2},$ $\phi_3(e_2) = (n+2)(n^2-3n+4)e_{n-1},\ \phi_3(e_3) = (n+1)(n^2-2n+4)e_n$
ϕ_2	$\phi_2(e_0) = n(n-1)e_{n-2},\ \phi_2(e_2) = (n^2-n+2)e_n$
ϕ_1	$\phi_1(e_0) = e_{n-1}$
ϕ_0	$\phi_0(e_0) = e_n$
ψ_1	$\psi_1(e_1) = e_{n-1}$

ψ_0	$\psi_0(e_1) = e_n$

Proof. Let $D \in \mathfrak{Der}\big(\mathrm{IDD}(K_n^0, 0, -2)\big)$, then $D(e_0) = \sum\limits_{i=0}^{n} \alpha_i e_i$ and $D(e_1) = \sum\limits_{i=0}^{n} \beta_i e_i$. Firstly,

$$\begin{array}{rcl}
D(e_2) & = & 2D(e_0 \diamond e_0) \ = \ 2\left(\left(\sum\limits_{i=0}^{n} \alpha_i e_i\right) \diamond e_0 + e_0 \diamond \left(\sum\limits_{i=0}^{n} \alpha_i e_i\right)\right) \ = \\
 & = & \sum\limits_{i=0}^{n-2} \alpha_i e_{i+2} + \sum\limits_{i=0}^{n-2} \frac{2}{(i+1)(i+2)} \alpha_i e_{i+2} \ = \ \sum\limits_{i=0}^{n-2} \frac{i^2+3i+4}{(i+1)(i+2)} \alpha_i e_{i+2}; \\
D(e_3) & = & 6D(e_0 \diamond e_1) \ = \ 6\left(\left(\sum\limits_{i=0}^{n} \alpha_i e_i\right) \diamond e_1 + e_0 \diamond \left(\sum\limits_{i=0}^{n} \beta_i e_i\right)\right) \ = \\
 & = & \sum\limits_{i=0}^{n-3} \alpha_i e_{i+3} + \sum\limits_{i=0}^{n-2} \frac{6}{(i+1)(i+2)} \beta_i e_{i+2} \ = \ 3\beta_0 e_2 + \sum\limits_{i=0}^{n-3} \left(\alpha_i + \frac{6}{(i+2)(i+3)} \beta_{i+1}\right) e_{i+3}; \\
D(e_3) & = & 2D(e_1 \diamond e_0) \ = \ 2\left(\left(\sum\limits_{i=0}^{n} \beta_i e_i\right) \diamond e_0 + e_1 \diamond \left(\sum\limits_{i=0}^{n} \alpha_i e_i\right)\right) \ = \\
 & = & \sum\limits_{i=0}^{n-2} \beta_i e_{i+2} + \sum\limits_{i=0}^{n-3} \frac{2}{(i+1)(i+2)} \alpha_i e_{i+3} \ = \ \beta_0 e_2 + \sum\limits_{i=0}^{n-3} \left(\beta_{i+1} + \frac{2}{(i+1)(i+2)} \alpha_i\right) e_{i+3};
\end{array}$$

that immediately gives $\beta_0 = 0$ and $\beta_k = \frac{(k+2)^2}{k(k+4)} \alpha_{k-1}$ for $2 \le k \le n-2$, i.e.,

$$\begin{array}{rcl}
D(e_1) & = & \beta_1 e_1 + \sum\limits_{i=2}^{n-2} \frac{(i+2)^2}{i(i+4)} \alpha_{i-1} e_i + \beta_{n-1} e_{n-1} + \beta_n e_n \text{ and} \\
D(e_3) & = & (\alpha_0 + \beta_1)\, e_3 + \sum\limits_{i=1}^{n-3} \frac{(i+4)(i^2+4i+7)}{(i+1)(i+2)(i+5)} \alpha_i e_{i+3}.
\end{array}$$

Secondly,

$$\begin{array}{rcl}
D(e_4) & = & 12D(e_0 \diamond e_2) \ = \ 12\left(\left(\sum\limits_{i=0}^{n} \alpha_i e_i\right) \diamond e_2 + e_0 \diamond \left(\sum\limits_{i=0}^{n-2} \frac{i^2+3i+4}{(i+1)(i+2)} \alpha_i e_{i+2}\right)\right) \ = \\
 & = & \sum\limits_{i=0}^{n-4} \alpha_i e_{i+4} + \sum\limits_{i=0}^{n-4} \frac{12(i^2+3i+4)}{(i+1)(i+2)(i+3)(i+4)} \alpha_i e_{i+4} \ = \sum\limits_{i=0}^{n-4} \frac{i^4+10i^3+47i^2+86i+72}{(i+1)(i+2)(i+3)(i+4)} \alpha_i e_{i+4}; \\
D(e_4) & = & 2D(e_2 \diamond e_0) \ = \ 2\left(\left(\sum\limits_{i=0}^{n-2} \frac{i^2+3i+4}{(i+1)(i+2)} \alpha_i e_{i+2}\right) \diamond e_0 + e_2 \diamond \left(\sum\limits_{i=0}^{n} \alpha_i e_i\right)\right) \ = \\
 & = & \sum\limits_{i=0}^{n-4} \frac{i^2+3i+4}{(i+1)(i+2)} \alpha_i e_{i+4} + \sum\limits_{i=0}^{n-4} \frac{2}{(i+1)(i+2)} \alpha_i e_{i+4} \ = \ \sum\limits_{i=0}^{n-4} \frac{i^2+3i+6}{(i+1)(i+2)} \alpha_i e_{i+4};
\end{array}$$

that immediately gives $\alpha_k = 0$ for $1 \le k \le n-4$, i.e.,

$$\begin{array}{rcl}
D(e_0) & = & \alpha_0 e_0 + \alpha_{n-3} e_{n-3} + \alpha_{n-2} e_{n-2} + \alpha_{n-1} e_{n-1} + \alpha_n e_n; \\
D(e_1) & = & \beta_1 e_1 + \frac{n^2}{n^2-4} \alpha_{n-3} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n; \\
D(e_2) & = & 2\alpha_0 e_2 + \frac{n^2-3n+4}{(n-1)(n-2)} \alpha_{n-3} e_{n-1} + \frac{n^2-n+2}{n(n-1)} \alpha_{n-2} e_n; \\
D(e_3) & = & (\alpha_0 + \beta_1)\, e_3 + \frac{(n+1)(n^2-2n+4)}{(n-1)(n^2-4)} \alpha_{n-3} e_n;
\end{array}$$

$$D(e_4) \;=\; 3\alpha_0 e_4.$$

On the other hand,

$$\begin{aligned} D(e_4) &= 6D(e_1 \diamond e_1) \;=\; 6\left(\left(\beta_1 e_1 + \tfrac{n^2\alpha_{n-3}}{n^2-4}e_{n-2} + \sum_{i=n-1}^{n}\beta_i e_i\right)\diamond e_1 + \right. \\ &\quad \left. + e_1 \diamond \left(\beta_1 e_1 + \tfrac{n^2\alpha_{n-3}}{n^2-4}e_{n-2} + \sum_{i=n-1}^{n}\beta_i e_i\right)\right) \;=\; 2\beta_1 e_4; \end{aligned}$$

i.e., $\beta_1 = \frac{3}{2}\alpha_0$. Then we obtain

$$\begin{aligned} D(e_1) &= \tfrac{3}{2}\alpha_0 e_1 + \tfrac{n^2}{n^2-4}\alpha_{n-3}e_{n-2} + \beta_{n-1}e_{n-1} + \beta_n e_n; \\ D(e_3) &= \tfrac{5}{2}\alpha_0 e_3 + \tfrac{(n+1)(n^2-2n+4)}{(n-1)(n^2-4)}\alpha_{n-3}e_n. \end{aligned}$$

Furthermore, using the induction method, we can prove that $D(e_k) = \frac{k+2}{2}\alpha_0 e_k$, for $4 \le k \le n$. To do this, let us consider

$$\begin{aligned} D(e_{k+2}) &= 2D(e_k \diamond e_0) \;=\; 2\big(D(e_k)\diamond e_0 + e_k \diamond D(e_0)\big) \;=\; \\ &= 2\left(\tfrac{k+2}{2}\alpha_0 e_k\right)\diamond e_0 + e_k \diamond \left(\alpha_0 e_0 + \sum_{i=n-3}^{n}\alpha_i e_i\right) \;=\; \tfrac{k+4}{4}\alpha_0 e_{k+2}. \end{aligned}$$

Combining all the obtained results, we see that D is a linear combination of mappings $\{\varphi, \phi_i, \psi_j\}_{0\le i\le 3;\ 0\le j\le 1}$. □

Corollary 56. $\mathfrak{Der}\big(\mathrm{IDD}(K^0_\infty, 0, -2)\big) = \langle\varphi\rangle$, *where* $\varphi(e_i) = (i+2)\,e_i$.

Proposition 57. $\mathfrak{Der}\big(\mathrm{IDD}(K^1_4, 0, -2)\big) = \langle\varphi_i, \phi_j, \psi_j\rangle^{2\le j\le 4}_{1\le i\le 4}$, *where*

φ_i	$\varphi_i(e_k) = \delta_{1,k}e_i + 2\delta_{1,i}\delta_{4,k}e_4$	$1 \le i \le 4$
ϕ_j	$\phi_j(e_2) = e_j$	$2 \le j \le 4$
ψ_j	$\psi_j(e_3) = e_j$	$2 \le j \le 4$

Proposition 58. $\mathfrak{Der}\big(\mathrm{IDD}(K^1_5, 0, -2)\big) = \langle\varphi_i, \phi_j, \psi_k\rangle^{3\le k\le 5}_{1\le i\le 5;\ 2\le j\le 5}$, *where*

φ_5	$\varphi_5(e_1) = e_5$
φ_4	$\varphi_4(e_1) = e_4$
φ_3	$\varphi_3(e_1) = e_3$
φ_2	$\varphi_2(e_1) = 2e_2,\ \ \varphi_2(e_4) = 3e_5$
φ_1	$\varphi_1(e_1) = e_1,\ \ \varphi_1(e_4) = 2e_4,\ \ \varphi_1(e_5) = e_5$
ϕ_5	$\phi_5(e_2) = e_5$
ϕ_4	$\phi_4(e_2) = e_4$
ϕ_3	$\phi_3(e_2) = e_3$

ϕ_2	$\phi_2(e_2) = e_2, \;\; \varphi_2(e_5) = e_5$
ψ_5	$\psi_5(e_3) = e_5$
ψ_4	$\psi_4(e_3) = e_4$
ψ_3	$\psi_3(e_3) = e_3$

Proposition 59. $\mathfrak{Der}\big(\mathrm{IDD}(K_6^1, 0, -2)\big) = \big\langle \varphi_i, \phi_j, \psi_k \big\rangle_{1 \leq i \leq 6; \; 2 \leq j \leq 6}^{4 \leq k \leq 6}$, *where*

φ_6	$\varphi_6(e_1) = e_6$
φ_5	$\varphi_5(e_1) = e_5$
φ_4	$\varphi_4(e_1) = e_4$
φ_3	$\varphi_3(e_1) = 10e_3, \;\; \varphi_3(e_4) = 13e_6$
φ_2	$\varphi_2(e_1) = 4e_2, \;\; \varphi_2(e_2) = 5e_3, \;\; \varphi_2(e_4) = 6e_5, \;\; \varphi_2(e_5) = 7e_6$
φ_1	$\varphi_1(e_1) = e_1, \;\; \varphi_1(e_3) = -e_3, \;\; \varphi_1(e_4) = 2e_4, \;\; \varphi_1(e_5) = e_5$
ϕ_6	$\phi_6(e_2) = e_6$
ϕ_5	$\phi_5(e_2) = e_5$
ϕ_4	$\phi_4(e_2) = e_4$
ϕ_3	$\phi_3(e_2) = e_3$
ϕ_2	$\phi_2(e_2) = e_2, \;\; \phi_2(e_3) = 2e_3, \;\; \phi_2(e_5) = e_5, \;\; \phi_2(e_6) = 2e_6$
ψ_6	$\psi_6(e_3) = e_6$
ψ_5	$\psi_5(e_3) = e_5$
ψ_4	$\psi_4(e_3) = e_4$

Proposition 60. $\mathfrak{Der}\big(\mathrm{IDD}(K_7^1, 0, -2)\big) = \big\langle \varphi_i, \phi_j, \psi_k \big\rangle_{1 \leq i \leq 7; \; 5 \leq j \leq 7}^{5 \leq k \leq 7}$, *where*

φ_7	$\varphi_7(e_1) = e_7$
φ_6	$\varphi_6(e_1) = e_6$
φ_5	$\varphi_5(e_1) = e_5$
φ_4	$\varphi_4(e_1) = 5e_4, \;\; \varphi_4(e_4) = 6e_7$
φ_3	$\varphi_3(e_1) = 30e_3, \;\; \varphi_3(e_2) = 35e_4, \;\; \varphi_3(e_4) = 39e_6, \;\; \varphi_3(e_5) = 44e_7$
φ_2	$\varphi_2(e_1) = 20e_2, \;\; \varphi_2(e_2) = 25e_3, \;\; \varphi_2(e_3) = 30e_4,$ $\varphi_2(e_4) = 30e_5, \;\; \varphi_2(e_5) = 28e_6, \;\; \varphi_2(e_6) = 40e_7$
φ_1	$\varphi_1(e_1) = 3e_1, \;\; \varphi_1(e_2) = 4e_2, \;\; \varphi_1(e_3) = 5e_3,$ $\varphi_1(e_4) = 6e_4, \;\; \varphi_1(e_5) = 7e_5, \;\; \varphi_1(e_6) = 8e_6, \;\; \varphi_1(e_7) = 9e_7$
ϕ_7	$\phi_7(e_2) = e_7$
ϕ_6	$\phi_6(e_2) = e_6$
ϕ_5	$\phi_5(e_2) = e_5$
ψ_7	$\psi_7(e_3) = e_7$
ψ_6	$\psi_6(e_3) = e_6$
ψ_5	$\psi_5(e_3) = e_5$

Theorem 61. *If* $8 \leq n < \infty$, *then* $\mathfrak{Der}\big(\mathrm{IDD}(K_n^1, 0, -2)\big) = \big\langle \varphi, \phi_i, \psi_j, \pi_k \big\rangle_{0 \leq i \leq 4}^{0 \leq j,k \leq 2}$, *where*

φ	$\varphi(e_i) \;=\; (i+2)\, e_i,\;\; 1\leq i\leq n$
ϕ_4	$\phi_4(e_1)=(n-3)(n^2-4)e_{n-4},\;\; \phi_4(e_2)=n(n-1)(n-2)e_{n-3},$ $\phi_4(e_4)=(n+2)(n^2-5n+12)e_{n-1},\;\; \phi_4(e_5)=(n+1)(n^2-4n+12)e_n$
ϕ_3	$\phi_3(e_1)=(n-1)(n-2)e_{n-3},\;\; \phi_3(e_4)=(n^2-3n+8)e_n$
ϕ_2	$\phi_2(e_1)=e_{n-2}$
ϕ_1	$\phi_1(e_1)=e_{n-1}$
ϕ_0	$\phi_0(e_1)=e_n$
ψ_2	$\psi_2(e_2)=e_{n-2}$
ψ_1	$\psi_1(e_2)=e_{n-1}$
ψ_0	$\psi_0(e_2)=e_n$
π_2	$\pi_2(e_3)=e_{n-2}$
π_1	$\pi_1(e_3)=e_{n-1}$
π_0	$\pi_0(e_3)=e_n$

Proof. Let $D\in\mathfrak{Der}\big(\mathrm{IDD}(K_n^1,0,-2)\big)$, then we can say that

$$D(e_1)=\sum_{i=1}^{n}\alpha_ie_i,\;\; D(e_2)=\sum_{i=1}^{n}\beta_ie_i, \text{ and } D(e_3)=\sum_{i=1}^{n}\gamma_ie_i.$$

Following the ideas presented above, we find $D(e_4)$ and $D(e_5)$.

$$\begin{array}{rcl}
D(e_4) &=& 6D(e_1\diamond e_1) \;=\; 6\left(\left(\sum\limits_{i=1}^{n}\alpha_ie_i\right)\diamond e_1+e_1\diamond\left(\sum\limits_{i=1}^{n}\alpha_ie_i\right)\right) \;=\\
&=& \sum\limits_{i=1}^{n-3}\alpha_ie_{i+3}+\sum\limits_{i=1}^{n-3}\frac{6}{(i+1)(i+2)}\alpha_ie_{i+3} \;=\sum\limits_{i=1}^{n-3}\frac{i^2+3i+8}{(i+1)(i+2)}\alpha_ie_{i+3};\\
D(e_5) &=& 12D(e_1\diamond e_2) \;=\; 12\left(\left(\sum\limits_{i=1}^{n}\alpha_ie_i\right)\diamond e_2+e_1\diamond\left(\sum\limits_{i=1}^{n}\beta_ie_i\right)\right) \;=\\
&=& \sum\limits_{i=1}^{n-4}\alpha_ie_{i+4}+\sum\limits_{i=1}^{n-3}\frac{12}{(i+1)(i+2)}\beta_ie_{i+3} \;=\; 2\beta_1e_4+\sum\limits_{i=1}^{n-4}\left(\alpha_i+\frac{12}{(i+2)(i+3)}\beta_{i+1}\right)e_{i+4};\\
D(e_5) &=& 6D(e_2\diamond e_1) \;=\; 6\left(\left(\sum\limits_{i=1}^{n}\beta_ie_i\right)\diamond e_1+e_2\diamond\left(\sum\limits_{i=1}^{n}\alpha_ie_i\right)\right) \;=\\
&=& \sum\limits_{i=1}^{n-3}\beta_ie_{i+3}+\sum\limits_{i=1}^{n-4}\frac{6}{(i+1)(i+2)}\alpha_ie_{i+4} \;=\; \beta_1e_4+\sum\limits_{i=1}^{n-4}\left(\beta_{i+1}+\frac{6}{(i+1)(i+2)}\alpha_i\right)e_{i+4}.
\end{array}$$

Comparing two expressions of $D(e_5)$, we have

$$\beta_1=0;\;\; \beta_k=\frac{(k-2)(k+2)(k+3)}{k(k^2+3n-10)}\alpha_{k-1},\;\; 3\leq k\leq n-3; \text{ i.e.,}$$

$$\begin{array}{rcl}
D(e_2) &=& \beta_2e_2+\sum\limits_{i=3}^{n-3}\frac{(i-2)(i+2)(i+3)}{i(i^2+3i-10)}\alpha_{i-1}e_i+\beta_{n-2}e_{n-2}+\beta_{n-1}e_{n-1}+\beta_ne_n;\\
D(e_5) &=& (\alpha_1+\beta_2)\,e_5+\sum\limits_{i=2}^{n-4}\frac{(i+5)(i^2+4i+12)}{(i+1)(i+2)(i+6)}\alpha_ie_{i+4}.
\end{array}$$

Next, we find three different expressions of $D(e_6)$.

$$\begin{array}{rcl}
D(e_6) & = & 20D(e_1 \diamond e_3) \; = \; 20\left(\left(\sum\limits_{i=1}^{n}\alpha_i e_i\right)\diamond e_3 + e_1 \diamond \left(\sum\limits_{i=1}^{n}\gamma_i e_i\right)\right) \; = \\
& = & \sum\limits_{i=1}^{n-5}\alpha_i e_{i+5} + \sum\limits_{i=1}^{n-3}\frac{20}{(i+1)(i+2)}\gamma_i e_{i+3} \; = \; \sum\limits_{i=1}^{n-5}\alpha_i e_{i+5} + \sum\limits_{i=-1}^{n-5}\frac{20}{(i+3)(i+4)}\gamma_{i+2}e_{i+5} = \\
& = & \frac{10}{3}\gamma_1 e_4 + \frac{5}{3}\gamma_2 e_5 + \sum\limits_{i=1}^{n-5}\left(\alpha_i + \frac{20}{(i+3)(i+4)}\gamma_{i+2}\right)e_{i+5}; \\
D(e_6) & = & 6D(e_3 \diamond e_1) \; = \; 6\left(\left(\sum\limits_{i=1}^{n}\gamma_i e_i\right)\diamond e_1 + e_3 \diamond \left(\sum\limits_{i=1}^{n}\alpha_i e_i\right)\right) \; = \\
& = & \sum\limits_{i=1}^{n-3}\gamma_i e_{i+3} + \sum\limits_{i=1}^{n-5}\frac{6}{(i+1)(i+2)}\alpha_i e_{i+5} \; = \; \sum\limits_{i=-1}^{n-5}\gamma_{i+2}e_{i+5} + \sum\limits_{i=1}^{n-5}\frac{6}{(i+1)(i+2)}\alpha_i e_{i+5} \; = \\
& = & \gamma_1 e_4 + \gamma_2 e_5 + \sum\limits_{i=1}^{n-5}\left(\gamma_{i+2} + \frac{6}{(i+1)(i+2)}\alpha_i\right)e_{i+5}; \\
D(e_6) & = & 12D(e_2 \diamond e_2) \; = \; 12\big(D(e_2)\diamond e_2 + e_2 \diamond D(e_2)\big) \; = \\
& = & 12\left(\beta_2 e_2 + \sum\limits_{i=3}^{n-3}\frac{(i-2)(i+2)(i+3)}{i(i^2+3i-10)}\alpha_{i-1}e_i + \sum\limits_{j=n-2}^{n}\beta_j e_j\right)\diamond e_2 + \\
& & +12e_2 \diamond \left(\beta_2 e_2 + \sum\limits_{i=3}^{n-3}\frac{(i-2)(i+2)(i+3)}{i(i^2+3i-10)}\alpha_{i-1}e_i + \sum\limits_{j=n-2}^{n}\beta_j e_j\right) \; = \\
& = & \beta_2 e_6 + \sum\limits_{i=3}^{n-4}\frac{(i-2)(i+2)(i+3)}{i(i^2+3i-10)}\alpha_{i-1}e_{i+4} + \beta_2 e_6 + \sum\limits_{i=3}^{n-4}\frac{12(i-2)(i+3)}{i(i+1)(i^2+3i-10)}\alpha_{i-1}e_{i+4} \; = \\
& = & 2\beta_2 e_6 + \sum\limits_{i=3}^{n-4}\frac{(i+3)(i^2+3i+14)}{i(i+1)(i+5)}\alpha_{i-1}e_{i+4} \; = \; 2\beta_2 e_6 + \sum\limits_{i=7}^{n}\frac{(i-1)(i^2-5i+18)}{(i-4)(i-3)(i+1)}\alpha_{i-5}e_i.
\end{array}$$

Comparing the presented expressions, we have

$$\begin{array}{ll}
\alpha_k = 0, & 3 \le k \le n-5; \\
\gamma_k = \frac{(k+1)(k+2)^2}{k(k-1)(k+6)}\alpha_{k-2}, & 4 \le k \le n-3. \\
\gamma_1 = 0; \;\; \gamma_2 = 0; \;\; \gamma_3 = -\alpha_1 + 2\beta_2; &
\end{array}$$

The last gives

$$\begin{array}{rcl}
D(e_1) & = & \alpha_1 e_1 + \alpha_2 e_2 + \sum\limits_{j=n-4}^{n}\alpha_j e_j; \\
D(e_2) & = & \beta_2 e_2 + \frac{5}{4}\alpha_2 e_3 + \frac{n(n-1)}{(n+2)(n-3)}\alpha_{n-4}e_{n-3} + \sum\limits_{j=n-2}^{n}\beta_j e_j; \\
D(e_3) & = & (-\alpha_1 + 2\beta_2)\,e_3 + \frac{3}{2}\alpha_2 e_4 + \sum\limits_{j=n-2}^{n}\gamma_j e_j; \\
D(e_4) & = & 2\alpha_1 e_4 + \frac{3}{2}\alpha_2 e_5 + \frac{n^2-5n+12}{(n-2)(n-3)}\alpha_{n-4}e_{n-1} + \frac{n^2-3n+8}{(n-1)(n-2)}\alpha_{n-3}e_n; \\
D(e_5) & = & (\alpha_1 + \beta_2)\,e_5 + \frac{7}{4}\alpha_2 e_6 + \frac{(n+1)(n^2-4n+12)}{(n+2)(n-2)(n-3)}\alpha_{n-4}e_n; \\
D(e_6) & = & 2\beta_2 e_6 + 2\alpha_2 e_7;
\end{array}$$

$$\begin{array}{rcl}
D(e_7) & = & 30D(e_1 \diamond e_4) \ = \ 30\left(\alpha_1 e_1 + \alpha_2 e_2 + \sum\limits_{j=n-4}^{n} \alpha_j e_j\right) \diamond e_4 + \\
& & +30 e_1 \diamond \left(2\alpha_1 e_4 + \frac{3}{2}\alpha_2 e_5 + \frac{n^2-5n+12}{(n-2)(n-3)}\alpha_{n-4} e_{n-1} + \frac{n^2-3n+8}{(n-1)(n-2)}\alpha_{n-3} e_n\right) \ = \\
& = & \alpha_1 e_7 + \alpha_2 e_8 + 2\alpha_1 e_7 + \frac{15}{14}\alpha_2 e_8 \ = \ 3\alpha_1 e_7 + \frac{29}{14}\alpha_2 e_8; \\
D(e_7) & = & 6D(e_4 \diamond e_1) \ = \ 6\big(D(e_4) \diamond e_1 + e_4 \diamond D(e_1)\big) \ = \\
& = & 6\left(2\alpha_1 e_4 + \frac{3}{2}\alpha_2 e_5 + \frac{n^2-5n+12}{(n-2)(n-3)}\alpha_{n-4} e_{n-1} + \frac{n^2-3n+8}{(n-1)(n-2)}\alpha_{n-3} e_n\right) \diamond e_1 + \\
& & +6 e_4 \diamond \left(\alpha_1 e_1 + \alpha_2 e_2 + \sum\limits_{j=n-4}^{n} \alpha_j e_j\right) \ = \\
& = & 2\alpha_1 e_7 + \frac{3}{2}\alpha_2 e_8 + \alpha_1 e_7 + \frac{1}{2}\alpha_2 e_8 \ = \ 3\alpha_1 e_7 + 2\alpha_2 e_8; \\
D(e_7) & = & 20D(e_2 \diamond e_3) \ = \ 20\big(D(e_2) \diamond e_3 + e_2 \diamond D(e_3)\big) \ = \\
& = & 20\left(\beta_2 e_2 + \frac{5}{4}\alpha_2 e_3 + \frac{n(n-1)}{(n+2)(n-3)}\alpha_{n-4} e_{n-3} + \sum\limits_{j=n-2}^{n} \beta_j e_j\right) \diamond e_3 + \\
& & +20 e_2 \diamond \left((-\alpha_1 + 2\beta_2)\, e_3 + \frac{3}{2}\alpha_2 e_4 + \sum\limits_{j=n-2}^{n} \gamma_j e_j\right) \ = \\
& = & \beta_2 e_7 + \frac{5}{4}\alpha_2 e_8 + (2\beta_2 - \alpha_1) e_7 + \alpha_2 e_8 \ = \ (3\beta_2 - \alpha_1) e_7 + \frac{9}{4}\alpha_2 e_8.
\end{array}$$

That gives $\alpha_2 = 0$ and $\beta_2 = \frac{4}{3}\alpha_1$, i.e.,

$$\begin{array}{rcl}
D(e_1) & = & \alpha_1 e_1 + \sum\limits_{j=n-4}^{n} \alpha_j e_j; \\
D(e_2) & = & \frac{4}{3}\alpha_1 e_2 + \frac{n(n-1)}{(n+2)(n-3)}\alpha_{n-4} e_{n-3} + \sum\limits_{j=n-2}^{n} \beta_j e_j; \\
D(e_3) & = & \frac{5}{3}\alpha_1 e_3 + \sum\limits_{j=n-2}^{n} \gamma_j e_j; \\
D(e_4) & = & 2\alpha_1 e_4 + \frac{n^2-5n+12}{(n-2)(n-3)}\alpha_{n-4} e_{n-1} + \frac{n^2-3n+8}{(n-1)(n-2)}\alpha_{n-3} e_n; \\
D(e_5) & = & \frac{7}{3}\alpha_1 e_5 + \frac{(n+1)(n^2-4n+12)}{(n+2)(n-2)(n-3)}\alpha_{n-4} e_n; \\
D(e_6) & = & \frac{8}{3}\alpha_1 e_6; \\
D(e_7) & = & 3\alpha_1 e_7.
\end{array}$$

Furthermore, using the induction method, we can prove that $D(e_k) = \frac{k+2}{3}\alpha_1 e_k$, for $k \geq 8$. To do this, let us consider

$$\begin{array}{rcl}
D(e_{k+1}) & = & k(k-1)D(e_1 \diamond e_{k-2}) \ = \ k(k-1)\big(D(e_1) \diamond e_{k-2} + e_1 \diamond D(e_{k-2})\big) \ = \\
& = & k(k-1)\left(\left(\alpha_1 e_1 + \sum\limits_{j=n-4}^{n} \alpha_j e_j\right) \diamond e_{k-2} + e_1 \diamond \left(\frac{k}{3}\alpha_1 e_{k-2}\right)\right) \ = \ \frac{k+3}{3}\alpha_1 e_{k+1}.
\end{array}$$

Joining all the obtained results, we see that D is a linear combination of mappings $\{\varphi, \phi_i, \psi_j, \pi_k\}_{0\leq i\leq 4;\ 0\leq j\leq 2;\ 0\leq k\leq 2}$. □

Corollary 62. *$\mathfrak{Der}\big(\mathrm{IDD}(K^1_\infty, 0, -2)\big) = \langle \varphi \rangle$, where $\varphi(e_i) = (i+2)\, e_i$.*

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