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On the behavior of free boundaries to generalized two-phase Stefan problems for parabolic partial differential equation systems

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Abstract. Recently, we have proposed a new free boundary problem representing the bread baking process in a hot oven. Unknown functions in this problem are the position of the evaporation front, the temperature field, and the water content. In solving this problem, we observed two difficulties: the growth rate of the free boundary depends on the water content, and the boundary condition for the water content involves the temperature. In this paper, by improving the regularity of solutions, we overcome these difficulties and establish the existence of a solution locally in time and its uniqueness. Moreover, under some sign conditions for initial data, we derive a result on the maximal interval of existence for solutions.

Contents

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1 Introduction

On the bread baking process, several mathematical results have been investigated in [12,15,16]. In particular, in [14], the free boundary problem was proposed as a mathematical model for the process and considered in the enthalpy formulation. The enthalpy formulation allows the treatment of the multidimensional domains. However, it does not describe the behavior of the free boundary directly. Therefore, in our previous result [2], we derived the following one-dimensional free boundary problem and established existence and uniqueness theorems only for its approximation problems, due to certain mathematical difficulties that will be discussed later in this section. The aim of the present paper is to show the existence, uniqueness, and the behavior of the solutions to the free boundary problem under suitable assumptions. The problem is to find a triplet (e, u, w) of the evaporation front $x = e(t)$ for $0 < t < T$, the temperature field $u = u(t, x)$ and the water content $w = w(t, x)$ for $(t, x) \in Q(T) := (0, T) \times (0, 1)$, where $T > 0$ is a given time, t is the time variable and x indicates the position. Here, we note that u is given by $u = \theta - \theta_c$, where θ is the temperature and θ_c is a positive constant indicating the phase transition from water to air. In this model we assume that the bread occupies the one-dimensional interval $(0, 1)$, and the common domain of u and w consists of the crumb region

$$Q_l(T, e) = \{(t, x) \mid 0 < x < e(t) \text{ for } 0 < t < T\}$$

and the crust region

$$Q_a(T, e) = \{(t, x) \mid e(t) < x < 1 \text{ for } 0 < t < T\}.$$

(see [2] for details). Also, the triplet (e, u, w) satisfies:

$$c_l \frac{\partial u}{\partial t} = k_l \frac{\partial^2 u}{\partial x^2} \text{ in } Q_l(T, e), \quad c_a \frac{\partial u}{\partial t} = k_a \frac{\partial^2 u}{\partial x^2} \text{ in } Q_a(T, e), \tag{1}$$

$$\frac{\partial w}{\partial t} = d_l \frac{\partial^2 w}{\partial x^2} \text{ in } Q_l(T, e), \quad \frac{\partial w}{\partial t} = d_a \frac{\partial^2 w}{\partial x^2} \text{ in } Q_a(T, e), \tag{2}$$

$$\frac{\partial u}{\partial x}(t, 0) = 0, \quad u(t, e(t)) = 0 \text{ for } 0 < t < T, \tag{3}$$

$$-k_a \frac{\partial u}{\partial x}(t, 1) = h(u(t, 1) + \theta_c - u_b(t)) + \sigma((u(t, 1) + \theta_c)^4 - u_b(t)^4) \text{ for } 0 < t < T, \tag{4}$$

$$-d_a\frac{\partial w}{\partial x}(t,1) = b_1 p(u(t,1)+\theta) - b_2 p(u_b(t)) \text{ for } 0 < t < T, \tag{5}$$

$$lw(t,e(t))e'(t) = k_l\frac{\partial u}{\partial x}(t,e(t)-) - k_a\frac{\partial u}{\partial x}(t,e(t)+) \text{ for } 0 < t < T, \tag{6}$$

$$\frac{\partial w}{\partial x}(t,0) = 0, \quad d_l\frac{\partial w}{\partial x}(t,e(t)-) = d_a\frac{\partial w}{\partial x}(t,e(t)+) \text{ for } 0 < t < T, \tag{7}$$

$$e(0) = e_0, u(0) = u_0, w(0) = w_0 \text{ on } [0,1], \tag{8}$$

where c_l and c_a are the specific heats, k_l and k_a are the thermal conductivities, and d_l and d_a are the water diffusion coefficients in the crumb and the crust regions, respectively. In addition, h and σ are non-negative constants corresponding to the heat transfer constant at the boundary at $x = 1$ and the Stefan–Boltzman constant, respectively, u_b is a given function on $[0,T]$ and indicates the temperature of the oven, b_1 and b_2 are positive constants and p is a continuous function on $\mathbb{R}$. Moreover, l is the latent heat, e_0 is the initial position of the free boundary, and u_0 and w_0 are the initial temperature field and the water content, respectively. Throughout this paper P denotes the system (1)-(8).

We refer [2] for the physical background in detail. Here, we explain the derivation of the system (1)-(8), briefly. Equation (1) is the standard heat equation and equation (2) describes the conservation law of the mass of water. The homogeneous Neumann boundary conditions (3) and (7) at $x = 0$ follow from the symmetry of the bread, and the boundary conditions (4) and (5) at $x = 1$ are assumed in [14]. In this paper we call the equation (6) the generalized Stefan condition, since the water content w appears as the coefficient of the time derivative of the free boundary. Moreover, (7) is called the transmission condition.

First, we show features of the system P as follows. As mentioned above, the generalized Stefan condition contains the water content on the free boundary in the coefficient of the growth rate. Accordingly, in order to discuss the solvability we need an estimate for the minimum value of the water content w. Also, the boundary value $u(\cdot,1)$ appears in the boundary condition for w. For existence of strong solutions for the two-phase Stefan problem we suppose that $u_0 \in W^{1,2}(0,1)$ as usual. However, in this case the regularity $u_t(\cdot,1) \in L^2(0,T)$ is not guaranteed, and then existence of a strong solution w is not proved. To address these difficulties, in [2] we approximate the boundary condition (5) by applying the mollifier to $u(\cdot,1)$ and establish existence and uniqueness based on the fixed-point argument.

The first main result of this paper is to obtain strong solutions of P under the high regularity $u_0 \in W^{2,2}(0,1)$. In the proof, the uniform estimate for time difference is essential (see Lemma 4.4). Our proof is due to [1, 2, 7].

Moreover, we show a result on the behavior on the free boundary. To present the result, briefly, let T^* be a maximal time for the existence of the solution. In our problem P one of the following conditions holds.

(a) $T^* = \infty$

(b) $T^* < \infty$, $e(t) \to 0$ as $t \uparrow T^*$

(c) $T^* < \infty$, $e(t) \to 1$ as $t \uparrow T^*$

This type of behavior result was already obtained by [11] and [8]. To show the behavior of the free boundary we assume that the temperature u_b in the oven is constant and the initial value w_0 is strictly positive on the domain $(0,1)$.

2 Main Results

First in this section, we define a solution to P and a theorem concerned with the uniqueness and the existence of the solution.

Definition 2.1. For given time T a triplet (e,u,w) of functions e, u and w is a solution of P on $[0,T_0]$ for $0 < T_0 \le T$ if (e,u,w) satisfies the following conditions:

(S1) $e \in W^{1,\infty}(0,T_0)$ and $0 < e < 1$ on $[0,T_0]$.

(S2) $u \in W^{1,2}(0,T_0;L^2(0,1)) \cap L^\infty(0,T_0;W^{1,2}(0,1))$, $u_{xx} \in L^2(Q_l(T_0,e))$, $L^2(Q_a(T_0,e))$, and $u_x(\cdot,e(\cdot)\pm) \in L^\infty(0,T_0)$.

(S3) $u(\cdot,1) \in W^{1,2}(0,T_0)$.

(S4) $w \in W^{1,2}(0,T_0;L^2(0,1)) \cap L^\infty(0,T_0;W^{1,2}(0,1))$, $w_{xx} \in L^2(Q_l(T_0,e))$, $L^2(Q_a(T_0,e))$, $w(t,e(t)) \ge \delta_1$ for any $t \in [0,T_0]$, where δ_1 is some positive constant.

(S5) (1)-(8) hold.

Before showing Theorem 2.2, we state the assumptions.

(A1) $p \in C^1(\mathbb{R})$, $0 \le p, p' \le M_p$ on $\mathbb{R}$, where M_p is a positive constant.

(A2) $0 < e_0 < 1$, $u_0 \in W^{1,\infty}(0,1)$, $u_0 \ge 0$ on $[e_0,1]$, $u_0 \le 0$ on $[0,e_0]$.

(A3) $w_0 \in W^{1,2}(0,1)$, $w_0(e_0) > 0$.

(A4) $u_b \in W^{1,2}(0,T)$, $u_b \ge \theta_c$ on $[0,T]$.

(A5) $u_0 \in W^{2,2}(e_0,1)$, $u_{0x}(0) = 0$,
$-k_a u_{0x}(1) = h(u_0(1) + \theta - u_b(0)) + \sigma((u_0(1)+\theta)^4 - u_b(0)^4)$.

Theorem 2.2. *Assume (A1)–(A5). Then P has at least one solution (e,u,w) on $[0,T_0]$ for some $T_0 \in (0,T]$. Also, for any $0 < T_0 \le T$, P admits at most one solution on $[0,T_0]$ in the sense of Definition 2.1.*

Remark 2.3. The physical relevance of (A1)–(A5) is as follows. The function $p = p(\theta)$ indicates the water vapor pressure at temperature θ, namely, its specific form is

$$p(\theta) = k_p \exp\left(A_p - \frac{B_p}{\theta + C_p}\right) \text{ for } \theta > 0,$$

where k_p, A_p, B_p and C_p are positive constants. Accordingly, (A1) is physically valid. In this paper, we assume that there exist crumb, crust and evaporation front at the initial. Under this assumption, (A2) holds. Since we consider that every crumbs and crusts contain a non-zero amount of water, (A3) is satisfied. On (A4), when we bake bread, the oven temperature is about 200°C, that is, the temperature u_b oh the hot air is greater than θ_c. Finally, (A5) is imposed for mathematical reasons.

Remark 2.4. We already established existence and uniqueness of solutions to approximate problems under (A1)–(A4) in [2]. In this paper by assuming (A5), we succeed to obtain uniform estimates for the time difference $\frac{u(t)-u(t-\Delta t)}{\Delta t}$ with $\Delta t > 0$, which leads to the regularity $u(\cdot, 1) \in W^{1,2}(0, T)$. This regularity allows us to prove existence of a function w satisfying the water diffusion system. However, due to dependence on the free boundary, it is not easy to show estimates for the difference of solutions to the water diffusion system. In order to overcome this difficulty, we employ the L^∞-estimate for u_x given in [10]. Furthermore, the application of the Gagliardo-Nirenberg inequality is also the key of our proof to Theorem 2.2 (see Lemma 3.5)

Definition 2.5. For $T^* \leq \infty$, we call that $[0, T^*)$ is the maximal interval of existence of the solution of P, if there exists a solution on $[0, T]$ for any $0 < T < T^*$, and do not exist on $[0, T^*]$ for $T^* < \infty$. For $T^* = \infty$, P has a solution on $[0, T]$ for any $T > 0$.

Theorem 2.6. *Assume (A1)–(A3) and (A5). If $b_1 \leq b_2$, u_b is a positive constant such that $u_b \geq u_0 + \theta_c$, and $w_0 > 0$ on $[0, 1]$. For $0 < T^* \leq \infty$, let $[0, T^*)$ be the maximal interval of existence of the solution (u, w, e) to P. Then, one and only one of the following cases (a), (b) and (c) always hold:*

(a) $T^* = \infty$.

(b) $T^* < \infty$ *and* $\lim_{t \uparrow T^*} e(t) = 0$.

(c) $T^* < \infty$ *and* $\lim_{t \uparrow T^*} e(t) = 1$.

In our proofs of Theorems 2.2 and 2.6, Proposition 2.7 which guarantees the integrability of the time derivative for the boundary function plays a very important role.

Proposition 2.7. *Under the same assumption as in Theorem 2.2, let (e, u, w) be a solution of P on $[0, T]$. Then, $u(\cdot, 1) \in W^{1,2}(0, T)$. Moreover, if $e \leq 1 - \delta$ on $[0, T]$ for some $\delta > 0$,* *then there exists a positive constant R_b depending only on $\delta > 0$, $|u_b|_{W^{1,2}(0,T)}$, $|u_{0xx}|_{L^2(1-\delta,1)}$ and $|u_0|_{W^{1,2}(0,1)}$ such that*

$$|u(\cdot,1)|_{W^{1,2}(0,T)} \le R_b \text{ and } |u_t(t)|_{L^2(1-\delta/2,1)} \le R_b \text{ for } 0 \le t \le T.$$

We shall prove this proposition in Section 4.

3 Known results

First, we give a list of useful inequalities.

Lemma 3.1. *For $a_1 < a_2$, there exists a positive constant $C_0 = C_0(a_1, a_2)$ such that*

$$|z| \le C_0(|z_x|^{1/2}_{L^2(a_1,a_2)}|z|^{1/2}_{L^2(a_1,a_2)} + |z|_{L^2(a_1,a_2)}) \text{ on } [a_1,a_2] \text{ for } z \in W^{1,2}(a_1,a_2),$$
$$|z(a_2)| \le C_0 |z_x|^{1/2}_{L^2(a_1,a_2)}|z|^{1/2}_{L^2(a_1,a_2)} \text{ for } z \in W^{1,2}(a_1,a_2) \text{ with } z(a_1) = 0.$$

Next, we consider auxiliary problems

$$\mathrm{AP}(e) := \{(1)-(5),(7),(8)\},$$
$$\mathrm{AP1}(e,u_0) := \{(1),(3),(4),(9)\}, \quad \text{and} \quad \mathrm{AP2}(e,b) := \{(2),(11),(10)\}$$

for given $e \in C([0,T])$ with $0 < e < 1$ on $[0,T]$ and $b \in C([0,T])$ as a preliminary step. Here, (9), (11) and (10) are as follows:

$$u(0) = u_0 \text{ on } [0,1], \tag{9}$$
$$\frac{\partial w}{\partial x}(t,0) = 0, \quad -d_a\frac{\partial w}{\partial x}(t,1) = b_1 p(b(t)) - b_2 p(u_b(t)) \text{ for } 0 < t < T, \tag{10}$$
$$w(0) = w_0 \text{ on } [0,1]. \tag{11}$$

For the problems $\mathrm{AP1}(e,u_0)$ and $\mathrm{AP2}(e,b)$ we have already obtained the following lemmas.

Lemma 3.2. *(cf. [2, Proposition 3.1 and Lemma 3.8]) Let $T > 0$ and $e \in W^{1,2}(0,T)$ satisfying $0 < e < 1$ on $[0,T]$. If (A2) and (A4) hold, then AP1(e, u_0) has one and only one solution u satisfying (S2). Moreover, $u \le 0$ in $Q_l(T,e)$ and $u \ge 0$ in $Q_a(T,e)$.*

Lemma 3.3. *Let $T > 0$, $e \in W^{1,2}(0,T)$ with $0 < e < 1$ on $[0,T]$ and $w_0 \in W^{1,2}(0,1)$. If $b \in W^{1,2}(0,T)$, then AP2(e,b) has a unique solution*

$$w \in W^{1,2}(0,T;L^2(0,1)) \cap L^\infty(0,T;W^{1,2}(0,1)), \quad w_{xx} \in L^2(Q_l(T,e)),\ L^2(Q_a(T,e)).$$

We can prove Lemma 3.3 in the similar way to that of [2, Proposition 3.2], on account of $b \in W^{1,2}(0,T)$. Thus, we omit the proof of the lemma.

Due to the results obtained in [2], we provide some uniform estimates for solutions of $\mathrm{AP1}(e,u_0)$. Here, we put

$$K(\delta,e_0,M,T_0) = \left\{e \in W^{1,3}(0,T) \mid e(0) = e_0, \delta \le e \le 1-\delta \text{ on } [0,T_0], \int_0^{T_0} |e'|^3\,dt \le M\right\},$$

where $\delta \in (0,1)$, $e_0 \in [\delta, 1-\delta]$, $M > 0$ and $0 < T_0 \le T$.

Lemma 3.4 (cf. Lemmas 4.2, 4.4 and 4.5 in [2]). *Let $\delta > 0$, $e_0 \in [\delta, 1-\delta]$, $M > 0$, $T > 0$ and $e \in K(\delta, e_0, M, T)$ and assume (A2) and (A4). If u is a solution of AP1(e, u_0), then for some $C_1 = C_1(\delta, M, T, |u_0|_{W^{1,2}(0,1)}) > 0$ such that*

$$\int_0^1 |u(t,x)|^2 \, dx + \int_0^t\int_0^1 |u_x(\tau,x)|^2 \, dx d\tau \le C_1 \text{ for } 0 \le t \le T, \tag{12}$$

and

$$\int_0^1 |u_x(t,x)|^2 \, dx + \int_0^t\int_0^1 |u_\tau(\tau,x)|^2 \, dx d\tau \le C_1 \text{ for } 0 \le t \le T. \tag{13}$$

Moreover, there exists a positive constant $C_2 = C_2(\delta, M, T, |u_0|_{W^{1,\infty}(0,1)})$ such that

$$|u_x(t, e(t)-)|\,, |u_x(t, e(t)+)| \le C_2 \text{ for } 0 \le t \le T. \tag{14}$$

Proof. In [2], by putting

$$\overline{u}(\cdot, y) = \begin{cases} u(\cdot, ye) \text{ for } 0 \le y \le 1, \\ u(\cdot, e + (y-1)(1-e)) \text{ for } 1 \le y \le 2 \text{ on } [0,T], \end{cases}$$

we obtained the existence of a positive constant $R_1 = R_1\left(\delta, M, T, |u_0|_{W^{1,2}(0,1)}\right)$ such that

$$\frac{c_*}{2}\int_0^2 |\overline{u}(t,y)|^2 \, dy + \frac{k_a}{2}\int_0^t\int_0^2 |\overline{u}_y(\tau,y)|^2 \, dy d\tau \le R_1 \text{ for } 0 \le t \le T, \tag{15}$$

$$\begin{aligned} &\frac{k_l}{2e(t)}\int_0^1 |\overline{u}_y(t,y)|^2 \, dy + \frac{k_a}{2(1-e(t))}\int_1^2 |\overline{u}_y(t,y)|^2 \, dy \\ &\quad + \frac{\delta c_a}{2}\int_0^t\int_0^2 |\overline{u}_\tau(\tau,y)|^2 \, dy d\tau \le R_1 \text{ for } 0 \le t \le T, \end{aligned} \tag{16}$$

where $c_* = \min\{c_l, c_a\}$. It is clear that $\overline{u}_y(t,y) = e(t)u_x(t, ye(t))$ for $0 \le y \le 1$ and

$$\overline{u}_y(t,y) = (1-e(t))u_x(t, e(t) + (y-1)(1-e(t))) \text{ for } 1 \le y \le 2,$$

and $0 \le t \le T$. Thus, we get (12) and (13). Moreover, (14) is a direct consequence of Lemma 4.5 in [2]. □

In the rest of this section we recall some results covered with estimates for the difference of solutions to AP1(e, u_0).

Lemma 3.5. *Let $\delta > 0$, $M > 0$, $T > 0$, $e_{0i} \in [\delta, 1-\delta]$ and $e_i \in K(\delta, e_{0i}, M, T)$ for $i = 1, 2$ and assume (A2). If u_i is a solution of AP1(e_i, u_0) for $i = 1, 2$, then, for some positive constants $C_3 = C_3(\delta, M, T, |u_0|_{W^{1,2}(0,1)})$ it holds that*

$$\int_0^1 |u_1(t) - u_2(t)|^2 dx + \int_0^t\int_0^1 |u_{1x}(\tau) - u_{2x}(\tau)|^2 dxd\tau \\ \leq C_3 \int_0^t (|e_1 - e_2|^2 + |e_1' - e_2'|^2) d\tau \text{ for } 0 \leq t \leq T. \tag{17}$$

Moreover, there exists a positive constant $C_4 = C_4(\delta, M, T, |u_0|_{W^{1,\infty}(0,1)})$ such that

$$\int_0^t |u_{1x}(\tau, e_1(\tau)-) - u_{2x}(\tau, e_2(\tau)-)|^3 d\tau \\ + \int_0^t |u_{1x}(\tau, e_1(\tau)+) - u_{2x}(\tau, e_2(\tau)+)|^3 d\tau \\ \leq C_4 t^{1/4} \int_0^t (|e_1 - e_2|^3 + |e_1' - e_2'|^3) d\tau \text{ for } 0 \leq t \leq T. \tag{18}$$

Proof. Let

$$\overline{u}_i(\cdot, y) = \begin{cases} u_i(\cdot, ye_i) \text{ if } 0 \leq y \leq 1, \\ u_i(\cdot, e_i + (y-1)(1-e_i)) \text{ if } 1 \leq y \leq 2, \end{cases} \text{ on } [0,T] \text{ for } i = 1, 2.$$

Then, Lemma 5.1 in [2] guarantees existence of a positive constant $R_2 = R_2(\delta, M, T)$ such that

$$\frac{c_l}{2} \int_0^1 |\overline{u}_1(t,y) - \overline{u}_2(t,y)|^2 dy + \frac{k_l}{2} \int_0^t\int_0^1 (\overline{u}_{1y}(\tau,y) - \overline{u}_{2y}(\tau,y))^2 dyd\tau \\ \leq R_2 \int_0^t (|e_1(\tau) - e_2(\tau)|^2 + |e_1'(\tau) - e_2'(\tau)|^2) d\tau, \tag{19}$$
$$\frac{c_a}{2} \int_1^2 |\overline{u}_1(t,y) - \overline{u}_2(t,y)|^2 dy + \frac{k_a}{2} \int_0^t\int_1^2 (\overline{u}_{1y}(\tau,y) - \overline{u}_{2y}(\tau,y))^2 dyd\tau \\ \leq R_2 \int_0^t (|e_1(\tau) - e_2(\tau)|^2 + |e_1'(\tau) - e_2'(\tau)|^2) d\tau \text{ for } 0 \leq t \leq T.$$

Hence, we can obtain (17), easily.

Next, in order to prove (18) we put $q(y) = -(y-1)^2 + 1$ for $y \in [0, 2]$ and $\overline{v}_i = q\overline{u}_i$ on $Q(T)$ for $i = 1, 2$. Due to Lemmas 5.3 and 5.4 in [2], there exists a positive constant $R_3 = R_3(\delta, M, T)$ such that

$$\frac{k_l}{2} \int_0^1 |\overline{v}_{1y}(t) - \overline{v}_{2y}(t)|^2 dy + \frac{c_l \delta^2}{4} \int_0^t\int_0^1 |\overline{v}_{1t} - \overline{v}_{2t}|^2 dyd\tau$$

$$\le R_3\int_0^t\left(|e_1-e_2|^2+|e_1'-e_2'|^2\right)d\tau, \tag{20}$$

$$\frac{k_a}{2}\int_1^2|\overline{v}_{1y}(t)-\overline{v}_{2y}(t)|^2\,dy+\frac{c_a\delta^2}{4}\int_0^t\!\!\int_1^2|\overline{v}_{1t}-\overline{v}_{2t}|^2\,dyd\tau$$
$$\le R_3\int_0^t\left(|e_1-e_2|^2+|e_1'-e_2'|^2\right)d\tau$$

for $0\le t\le T$.

Here, we note that

$$c_l\frac{\partial\overline{v}_i}{\partial t}=\frac{k_l}{e_i^2}\left(\frac{\partial^2\overline{v}_i}{\partial y^2}-2\frac{\partial\overline{u}_i}{\partial y}q'-\overline{u}_iq''\right)+\frac{c_lye_i'}{e_i}\left(\frac{\partial\overline{v}_i}{\partial y}-\overline{u}_iq'\right)\text{ in }(0,T)\times(0,1), \tag{21}$$

$\max_{0\le y\le 1}|q'(y)|=2$ and $q''(y)=-2$ for $0\le y\le 1$. Hence, from (21) it follows that

$$\begin{aligned}
&|\overline{v}_{1yy}-\overline{v}_{2yy}|_{L^2(0,1)}\\
&=\left(\int_0^1\left|\frac{e_1^2}{k_l}c_l\overline{v}_{1t}+2\overline{u}_{1y}q'+\overline{u}_1q''-\frac{c_lye_1'e_1}{k_l}\overline{v}_{1y}+\frac{c_lye_1'e_1}{k_l}\overline{u}_1q'\right.\right.\\
&\qquad\left.\left.-\frac{e_2^2}{k_l}c_l\overline{v}_{2t}-2\overline{u}_{2y}q'-\overline{u}_2q''+\frac{c_lye_2'e_2}{k_l}\overline{v}_{2y}-\frac{c_lye_2'e_2}{k_l}\overline{u}_2q'\right|^2dy\right)^{\frac{1}{2}}\\
&\le R_4\left((e_1^2-e_2^2)^2\int_0^1|\overline{u}_{1t}|^2\,dy+\int_0^1|\overline{v}_{1t}-\overline{v}_{2t}|^2\,dy+\int_0^1|\overline{u}_{1y}-\overline{u}_{2y}|^2\,dy\right.\\
&\qquad+\int_0^1|\overline{u}_1-\overline{u}_2|^2\,dy+|e_2'|^2\int_0^1|\overline{v}_{1y}-\overline{v}_{2y}|^2\,dy\\
&\qquad+|e_2'|^2|e_1-e_2|^2\int_0^1|q'\overline{u}_1+q\overline{u}_{1y}|^2\,dy+|e_1'-e_2'|^2\int_0^1|\overline{v}_{1y}|^2\,dy\\
&\qquad\left.+|e_2'|^2\int_0^1|\overline{u}_1-\overline{u}_2|^2\,dy+|e_2'|^2|e_1-e_2|^2\int_0^1|\overline{u}_1|^2\,dy+|e_1'-e_2'|^2\int_0^1|\overline{u}_1|^2\,dy\right)^{\frac{1}{2}}\\
&\qquad\qquad\text{a.e. on }[0,T],
\end{aligned} \tag{22}$$

where R_4 is a positive constant. Integrating (22) with respect to the time variable, by Lemma 3.4, (19) and (20) we see that

$$\int_0^t\int_0^1|\overline{v}_{1yy}-\overline{v}_{2yy}|^2\,dyd\tau\le R_5\int_0^t(|e_1-e_2|^2+|e_1'-e_2'|^2)d\tau\text{ for }0\le t\le T,$$

where R_5 is a suitable positive constant. Since

$$u_i(t,x)=\overline{u}_i\left(t,\frac{x}{e_i(t)}\right)\text{ for }(t,x)\in Q_l(T,e_i)$$

and $i = 1, 2$, we have

$$\begin{aligned}
&|u_{1x}(t, e_1(t)-) - u_{2x}(t, e_2(t)-)| = \left|\frac{1}{e_1(t)}\overline{u}_{1y}(t, 1-) - \frac{1}{e_2(t)}\overline{u}_{2y}(t, 1-)\right| \\
&\leq \left|\left(\frac{1}{e_1(t)} - \frac{1}{e_2(t)}\right)\overline{u}_{1y}(t, 1-)\right| + \left|\frac{1}{e_2(t)}\left(\overline{u}_{1y}(t, 1-) - \overline{u}_{2y}(t, 1-)\right)\right| \\
&=: I_1(t) + I_2(t) \text{ for a.e. } t \in [0, T].
\end{aligned} \tag{23}$$

By (14), we see that

$$I_1(t) \leq \frac{e_1(t)}{\delta^2}|e_1(t) - e_2(t)|\,|u_{1x}(t, e_1(t)-)| \leq \frac{C_2}{\delta^2}|e_1(t) - e_2(t)| \text{ for a.e. } t \in [0, T]. \tag{24}$$

Also, thanks to the definition of $\overline{v}_i$ and Lemma 3.1, we infer that $\overline{u}_{iy}(t, 1-) = \overline{v}_{iy}(t, 1-)$ for a.e. $t \in [0, T]$ and $i = 1, 2$, and

$$\begin{aligned}
I_2(t) &\leq \frac{1}{\delta}|\overline{v}_{1y}(t, 1-) - \overline{v}_{2y}(t, 1-)| \\
&\leq \frac{C_0}{\delta}\left(|\overline{v}_{1yy}(t) - \overline{v}_{2yy}(t)|^{1/2}_{L^2(0,1)}|\overline{v}_{1y}(t) - \overline{v}_{2y}(t)|^{1/2}_{L^2(0,1)} + |\overline{v}_{1y}(t) - \overline{v}_{2y}(t)|_{L^2(0,1)}\right) \\
&\qquad\qquad\qquad\qquad \text{for a.e. } t \in [0, T].
\end{aligned} \tag{25}$$

On account of (19)-(25) and Lemma 3.4, we can choose positive constants R_6 and R_7 depending only on δ, M, T and $|u_0|_{W^{1,\infty}(0,1)}$ such that

$$\begin{aligned}
&\int_0^t |u_{1x}(\tau, e_1(\tau)-) - u_{2x}(\tau, e_2(\tau)-)|^3 d\tau \\
&\leq \left(\frac{C_2}{\delta^2}\right)^3 \int_0^t |e_1(\tau) - e_2(\tau)|^3 d\tau \\
&\quad + \left(\frac{2C_0}{\delta}\right)^3 \int_0^t \Big(|\overline{v}_{1yy}(\tau) - \overline{v}_{2yy}(\tau)|^{3/2}_{L^2(0,1)} |\overline{v}_{1y}(\tau) - \overline{v}_{2y}(\tau)|^{3/2}_{L^2(0,1)} \\
&\qquad + |\overline{v}_{1y}(\tau) - \overline{v}_{2y}(\tau)|^3_{L^2(0,1)} \Big) d\tau \\
&\leq \left(\frac{C_2}{\delta^2}\right)^3 \int_0^t |e_1(\tau) - e_2(\tau)|^3 d\tau \\
&\quad + \left(\frac{2C_0}{\delta}\right)^3 \Big(|\overline{v}_{1y} - \overline{v}_{2y}|^{3/2}_{L^\infty(0,t;L^2(0,1))} \int_0^t |\overline{v}_{1yy}(\tau) - \overline{v}_{2yy}(\tau)|^{3/2}_{L^2(0,1)} d\tau \\
&\qquad + |\overline{v}_{1y} - \overline{v}_{2y}|^3_{L^\infty(0,t;L^2(0,1))}\, t \Big) \\
&\leq \left(\frac{C_2}{\delta^2}\right)^3 t^3 \int_0^t |e_1' - e_2'|^3 d\tau \\
&\quad + \left(\frac{2C_0}{\delta}\right)^3 \Big(|\overline{v}_{1y} - \overline{v}_{2y}|^{3/2}_{L^\infty(0,t;L^2(0,1))} |\overline{v}_{1yy} - \overline{v}_{2yy}|^{3/2}_{L^2(0,t;L^2(0,1))}\, t^{1/4}
\end{aligned}$$

$$
\begin{aligned}
&\quad + |\overline{v}_{1y} - \overline{v}_{2y}|^3_{L^\infty(0,t;L^2(0,1))}\, t\Big) \\
&\le R_6 t^3 \int_0^t |e_1' - e_2'|^3\, d\tau + R_6 \left(\int_0^t (|e_1 - e_2|^2 + |e_1' - e_2'|^2) d\tau \right)^{3/2} t^{1/4} \text{ for } 0 \le t \le T.
\end{aligned}
$$

Hence, it holds that

$$
|u_{1x}(\cdot, e_1-) - u_{2x}(\cdot, e_2-)|^3_{L^3(0,t)} \le R_7 t^{1/4} \int_0^t (|e_1 - e_2|^3 + |e_1' - e_2'|^3) d\tau \text{ for } 0 \le t \le T.
$$

Similarly to above, we get

$$
\begin{aligned}
&\int_0^t |u_{1x}(\tau, e_1(t)+) - u_{2x}(\tau, e_2(\tau)+)|^3\, d\tau \\
&\le R_8 t^{1/4} \int_0^t (|e_1 - e_2|^3 + |e_1' - e_2'|^3) d\tau \text{ for } 0 \le t \le T,
\end{aligned}
$$

where R_8 is a positive constant. Thus, Lemma 3.5 has been proved. □

4 Proof of Proposition 2.7

To show Proposition 2.7, we introduce the function $\zeta \in C^\infty([0,1])$ which satisfies $0 < \zeta \le 1$ on $(1-\delta, 1]$, $\zeta = 0$ on $[0, 1-\delta]$, $\zeta(1) = 1$, and $\zeta'(1-\delta) = \zeta'(1) = 0$. Let $e \in K(\delta, e_0, M, T)$ and u be a solution of AP1(e, u_0) on $[0,T]$ for $T > 0$, and put $\tilde{u} = \zeta u$. Then, the following equation, the boundary condition and the initial condition hold:

$$
\begin{aligned}
c_a \tilde{u}_t &= k_a(\tilde{u}_{xx} - \zeta_{xx} u - 2\zeta_x u_x) \text{ in } (0,T) \times (1-\delta, 1), \qquad (26)\\
-k_a \frac{\partial \tilde{u}}{\partial x}(t, 1) &= h(\tilde{u}(t,1) + \theta - u_b(t)) + \sigma((\tilde{u}(t,1) + \theta)^4 - u_b(t)^4) \\
&= g(t, \tilde{u}(t,1)) \text{ for } 0 < t < T, \\
\tilde{u}(t, 1-\delta) &= 0 \text{ for } 0 < t < T, \\
\tilde{u}(0, x) &= \tilde{u}_0(x) := (\zeta u_0)(x) \text{ for } 1-\delta \le x \le 1,
\end{aligned}
$$

where g is the function defined in Section 2. The next lemma guarantees the existence of smooth approximations to u_0 and u, which will be used in the proof of Theorem 2.2.

Lemma 4.1. *Suppose that a pair (u_0, e_0) satisfies (A5) and let $0 < \delta < 1$, $\delta \le e_0 \le 1-\delta$, $M > 0$ and $T > 0$. If u is a solution of AP1(e, u_0) for $e \in K(\delta, e_0, M, T)$, then there exist approximate sequences $\{u_{0n}\}$ and $\{u_n\}$ to u_0 and u, respectively, such that*

$$
\begin{aligned}
&\{u_{0n}\} \subset C^\infty([1-\delta, 1]), u_{0n} \to u_0 \text{ in } W^{2,2}(1-\delta, 1) \text{ as } n \to \infty, \\
&\quad - k_a u_{0nx}(1) = h(u_{0n}(1) + \theta - u_b(0)) + \sigma((u_{0n}(1) + \theta)^4 - u_b(0)^4), \text{ and}
\end{aligned}
$$

$\{u_n\} \subset W^{1,2}(0,T;W^{2,2}(1-\delta,1))$ *for any* $n \in \mathbb{N}$,
$u_n \to u$ *in* $L^2(0,T;W^{2,2}(1-\delta,1))$ *as* $n \to \infty$,
$u_{nt} \to u_t$ *in* $L^2(0,T;L^2(1-\delta,1))$ *as* $n \to \infty$,
$u_n(t) = u_{0n}$ *on* $[1-\delta,1]$ *for* $t \le 0$.

Proof. First, we put $u(t) = u_0$ for $t < 0$ and $u_\nu(t) = \frac{1}{\nu}\int_{t-\nu}^t u d\tau$ in $L^2(1-\delta,1)$ for $t \le T$ and $\nu > 0$. Easily, we have $u_\nu(t) = u_0$ for $t \le 0$,

$$u_{\nu x} = \frac{1}{\nu}\int_{t-\nu}^t u_x d\tau \text{ and } u_{\nu xx} = \frac{1}{\nu}\int_{t-\nu}^t u_{xx} d\tau \text{ in } L^2(1-\delta,1) \text{ on } [0,T],$$

since $u \in L^2(0,T;W^{2,2}(1-\delta,1))$. For $t, t' \in [0,T]$ it is easy to see that

$$|u_\nu(t) - u_\nu(t')|_{L^2(1-\delta,1)} \le \frac{1}{\nu}\int_{-\nu}^0 |u(t+\tau) - u(t'+\tau)|_{L^2(1-\delta,1)}\, d\tau \to 0 \text{ as } t' \to t,$$

and it implies $u_\nu \in C([0,T];L^2(1-\delta,1))$.
Similarly, we can show that $u_\nu \in C([0,T];W^{2,2}(1-\delta,1))$. Also, we obtain

$$\int_0^T |u_\nu(t) - u(t)|^2_{L^2(1-\delta,1)}\, dt \le \frac{1}{\nu}\int_0^T \left(\int_{-\nu}^0 |u(t+\tau) - u(t)|_{L^2(1-\delta,1)}\, d\tau\right)^2 dt$$
$$\le \int_{-\nu}^0 \int_0^T |u(t+\tau) - u(t)|^2_{L^2(1-\delta,1)}\, dt d\tau \text{ for } \nu > 0.$$

Consequently, we see that $u_\nu \to u$ in $L^2(0,T;L^2(1-\delta,1))$ as $\nu \to 0$. Similarly, we have $u_\nu \to u$ in $L^2(0,T;W^{2,2}(1-\delta,1))$ as $\nu \to 0$. Also, it holds that $u_{\nu t} \to u_t$ in $L^2(0,T;L^2(1-\delta,1))$ as $\nu \to 0$. Moreover, it is clear that $u_\nu(0) = u_0 \ge 0$ on $[1-\delta,1]$ and by (A5) $-k_a u_{\nu x}(0,1) = g(0,u_\nu(0,1))$. Therefore,

$$-k_a u_{\nu x}(t,1) = g(t,u_\nu(t,1)) \text{ for } t \in (-\infty,T].$$

Here, we note that $u_{\nu t} \in L^2(0,T;W^{2,2}(1-\delta,1))$ for $\nu > 0$. Next, let J_ε be a mollifier in $\mathbb{R}$ for $\varepsilon > 0$. For $\varepsilon > 0$ and $\nu > 0$ we put

$$U^0_{\nu\varepsilon}(t,x) = \int_{\mathbb{R}} u_{\nu xx}(t,y) J_\varepsilon(x-y) dy,$$
$$U^1_{\nu\varepsilon}(t,x) = -\int_x^1 U^0_{\nu\varepsilon}(t,\xi) d\xi + u_{\nu x}(t,1),$$
$$u_{\nu\varepsilon}(t,x) = -\int_x^1 U^1_{\nu\varepsilon}(t,\xi) d\xi + u_\nu(t,1) \text{ for } (t,x) \in (-\infty,T) \times (1-\delta,1).$$

By $u_{\nu xxt} \in L^2(0,T;L^2(1-\delta,1))$, we see that

$$U^0_{\nu\varepsilon t} = J_\varepsilon * u_{\nu xxt} \text{ on } (0,T) \times (1-\delta,1),$$

$$U^1_{\nu\varepsilon t}(t,x) = -\int_x^1 U^0_{\nu\varepsilon t}(t,\xi)d\xi + u_{\nu xt}(t,1) \text{ and } u_{\nu\varepsilon t}(t,x) = -\int_x^1 U^1_{\nu\varepsilon t}(t,\xi)d\xi + u_{\nu t}(t,1)$$

for $(t,x) \in (0,T) \times (1-\delta,1)$. Also, it holds that $u_{\nu\varepsilon}(t) = u_{\nu\varepsilon}(0)$ for $t \le 0$. Indeed, since $U^0_{\nu\varepsilon}(t) = J_\varepsilon * u_{\nu xx}(t) = J_\varepsilon * u_{0xx}$ for $t \le 0$, we have

$$U^1_{\nu\varepsilon}(t,x) = -\int_x^1 (J_\varepsilon * u_{0xx})d\xi + u_{0x}(1), \text{ and } u_{\nu\varepsilon}(t,x) = u_{\nu\varepsilon}(0,x)$$

for $(t,x) \in (-\infty,0] \times [0,1]$.

Moreover, it is clear that we have $U^0_{\nu\varepsilon}(t) \in C^\infty([1-\delta,1])$ for $t \in [0,T]$ and

$$\left|U^0_{\nu\varepsilon}(t) - U^0_{\nu\varepsilon}(t')\right|_{L^2(1-\delta,1)} \le |u_{\nu xx}(t) - u_{\nu xx}(t')|_{L^2(1-\delta,1)} \text{ for } t,t' \in [0,T].$$

Since $u_\nu \in C([0,T]; W^{2,2}(1-\delta,1))$, we have $U^0_{\nu\varepsilon} \in C([0,T]; L^2(1-\delta,1))$. Easily, we get

$$\left|U^0_{\nu\varepsilon}(t)\right|_{L^2(1-\delta,1)} \le |u_{\nu xx}(t)|_{L^2(1-\delta,1)} \text{ and } U^0_{\nu\varepsilon}(t) \to u_{\nu xx}(t)$$

in $L^2(1-\delta,1)$ as $\varepsilon \to 0$ for $0 \le t \le T$. By applying the dominated convergence theorem, we get $U^0_{\nu\varepsilon} \to u_{\nu xx}$ in $L^2(0,T; L^2(1-\delta,1))$ as $\varepsilon \to 0$. Clearly $u_{\nu\varepsilon x} = U^1_{\nu\varepsilon}$ and $u_{\nu\varepsilon xx} = U^0_{\nu\varepsilon}$ on $(0,T) \times (1-\delta,1)$, that is, $u_{\nu\varepsilon} \in C([0,T]; W^{2,2}(1-\delta,1))$ for any ν, $\varepsilon > 0$. Also, we have

$$\begin{aligned}|u_{\nu\varepsilon x}(t,x) - u_{\nu x}(t,x)| &= \left|\int_1^x \frac{\partial}{\partial\xi}(u_{\nu\varepsilon x}(t,\xi) - u_{\nu x}(t,\xi))d\xi\right| \\ &\le \int_{1-\delta}^1 |U^0_{\nu\varepsilon}(t) - u_{\nu xx}(t)|d\xi \text{ for a.e. } t \in [0,T].\end{aligned}$$

This shows

$$u_{\nu\varepsilon x} \to u_{\nu x} \text{ in } L^2(0,T; L^2(1-\delta,1)) \text{ as } \varepsilon \to 0.$$

Similarly, we have

$$u_{\nu\varepsilon} \to u_\nu \text{ in } L^2(0,T; L^2(1-\delta,1)) \text{ as } \varepsilon \to 0.$$

Easily, we see that

$$-k_a u_{\nu\varepsilon x}(0,1) = -k_a u_{\nu x}(0,1) = g(0, u_\nu(0,1)) = g(0, u_{\nu\varepsilon}(0,1)).$$

Also, we obtain

$$\begin{aligned}|u_{\nu\varepsilon t}(t,x) - u_{\nu t}(t,x)| &= \left|\int_x^1 (u_{\nu\varepsilon tx}(t,\xi) - u_{\nu tx}(t,\xi))d\xi\right| \\ &\le \int_0^1 |u_{\nu\varepsilon tx}(t,x) - u_{\nu tx}(t,x)|\, dx\end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 \int_x^1 |u_{\nu\varepsilon txx}(t,\xi) - u_{\nu txx}(t,\xi)|\, d\xi dx \\
&\le |u_{\nu\varepsilon txx}(t) - u_{\nu txx}(t)|_{L^2(1-\delta,1)} \\
&= |(J_\varepsilon * u_{\nu txx})(t) - u_{\nu txx}(t)|_{L^2(1-\delta,1)} \\
&\qquad\qquad\qquad \text{for } (t,x) \in (0,T)\times(1-\delta,1).
\end{aligned}$$

Hence, thanks to the basic property of the mollifier we have

$$u_{\nu\varepsilon t} \to u_{\nu t} \text{ in } L^2(0,T;L^2(1-\delta,1)) \text{ as } \varepsilon \to 0 \text{ for } \nu > 0.$$

Accordingly, for any $n \in \mathbb{N}$, there exist $\nu_n > 0$ and $\varepsilon_n > 0$ such that

$$|u_{\nu_n} - u|_{L^2(0,T;W^{2,2}(1-\delta,1))} \le \frac{1}{2n}, |u_{\nu_n t} - u_t|_{L^2(0,T;L^2(1-\delta,1))} \le \frac{1}{2n},$$

$$|u_{\nu_n} - u_{\nu_n\varepsilon_n}|_{L^2(0,T;W^{2,2}(1-\delta,1))} \le \frac{1}{2n} \text{ and } |u_{\nu_n\varepsilon_n t} - u_{\nu_n t}|_{L^2(0,T;L^2(1-\delta,1))} \le \frac{1}{2n}.$$

By putting $u_n = u_{\nu_n\varepsilon_n}$ for each n, we have

$$\begin{aligned}
&u_n \to u \text{ in } L^2(0,T;W^{2,2}(1-\delta,1)),\\
&u_{nt} \to u_t \text{ in } L^2(0,T;L^2(1-\delta,1)) \text{ as } n \to \infty.
\end{aligned}$$

Easily, we get $u_n(0) = u_{\nu_n\varepsilon_n}(0) \in C^\infty([1-\delta,1])$ and $-k_a u_{nx}(0,1) = g(0,u_n(0,1))$ for each n. By putting $u_{0n} = u_n(0)$ for n, we can prove Lemma 4.1. □

For any n let

$$\tilde{u}_n \in W^{1,2}(0,T;L^2(1-\delta,1)) \cap L^\infty(0,T;W^{1,2}(1-\delta,1))$$

be a solution of the following problem:

$$c_a \tilde{u}_{nt} = k_a(\tilde{u}_{nxx} - \zeta_{xx}u_n - 2\zeta_x u_{nx}) \text{ in } (0,T)\times(1-\delta,1), \tag{27}$$

$$\begin{aligned}
-k_a \frac{\partial \tilde{u}_n}{\partial x}(t,1) &= h([\tilde{u}_n(t,1)+\theta_c]^+ - u_b(t)) + \sigma(([\tilde{u}_n(t,1)+\theta_c]^+)^4 - u_b(t)^4)\\
&=: g_+(t,\tilde{u}_n(t,1)) \text{ for } 0 < t < T,
\end{aligned} \tag{28}$$

$$\tilde{u}_n(t,1-\delta) = 0 \text{ for } 0 < t < T, \tag{29}$$

$$\tilde{u}_n(0,x) = \tilde{u}_{0n}(x) = (\zeta u_{0n})(x) \text{ for } 1-\delta \le x \le 1. \tag{30}$$

Since the equation (27) is linear and the boundary condition (28) is monotone, for given u_n we can prove existence and uniqueness of the strong solution to the problem (27)-(30) in the similar way to AP1(e, u_0).

Lemma 4.2. *Under (A4) and (A5), let $\tilde{u}_n$ be the strong solution of (27)-(30). Then, there exists a constant $C > 0$ independent of n such that $|\tilde{u}_n(t,1)| \le C$ for $0 \le t \le T$.*

Proof. First, by multiplying $\tilde{u}_n$ on both sides of (27) and integrating it with x on $[1-\delta, 1]$, we have

$$\int_{1-\delta}^{1} c_a \tilde{u}_{nt}\tilde{u}_n dx = k_a \int_{1-\delta}^{1} \tilde{u}_{nxx}\tilde{u}_n dx - k_a \int_{1-\delta}^{1} \zeta_{xx} u_n \tilde{u}_n dx - 2k_a \int_{1-\delta}^{1} \zeta_x u_{nx}\tilde{u}_n dx$$
$$=: I_1 + I_2 + I_3 \text{ a.e. on } [0, T].$$

By elementary calculations, we obtain

$$\int_{1-\delta}^{1} c_a \tilde{u}_{nt}\tilde{u}_n dx = \frac{d}{dt}\frac{c_a}{2}\int_{1-\delta}^{1} \tilde{u}_n^2 dx,$$

and

$$I_1 = -g_+(\cdot, \tilde{u}_n(\cdot, 1))\tilde{u}_n(\cdot, 1) - k_a \int_{1-\delta}^{1} |\tilde{u}_{nx}|^2 dx \text{ a.e. on } [0, T].$$

Thanks to Young's inequality, we see that

$$I_2 \leq \frac{c_a}{4}\int_{1-\delta}^{1} \tilde{u}_n^2 dx + \frac{k_a^2}{c_a}|\zeta_{xx}|^2_{L^\infty(0,1)}\int_{1-\delta}^{1} u_n^2 dx,$$

and

$$I_3 \leq \frac{c_a}{4}\int_{1-\delta}^{1} \tilde{u}_n^2 dx + \frac{4k_a^2}{c_a}|\zeta_x|^2_{L^\infty(0,1)}\int_{1-\delta}^{1} |u_{nx}|^2 dx \text{ a.e. on } [0, T].$$

Hence, we have

$$\begin{aligned}
&\frac{d}{dt}\frac{c_a}{2}\int_{1-\delta}^{1} \tilde{u}_n^2 dx + k_a \int_{1-\delta}^{1} |\tilde{u}_{nx}|^2 dx \\
&\leq -g_+(\cdot, \tilde{u}_n(\cdot, 1))\tilde{u}_n(\cdot, 1) + \frac{c_a}{2}\int_{1-\delta}^{1} \tilde{u}_n^2 dx \\
&\quad + \frac{k_a^2}{c_a}|\zeta_{xx}|^2_{L^\infty(0,1)}\int_{1-\delta}^{1} u_n^2 dx + \frac{4k_a^2}{c_a}|\zeta_x|^2_{L^\infty(0,1)}\int_{1-\delta}^{1} |u_{nx}|^2 dx \text{ a.e. on } [0, T].
\end{aligned}$$

Here, we estimate the first term of the above inequality by the monotonicity and Young's inequality. Namely, we have

$$\begin{aligned}
&- g_+(\cdot, \tilde{u}_n(\cdot, 1))\tilde{u}_n(\cdot, 1) \\
&= -h[\tilde{u}_n(\cdot, 1) + \theta_c]^+(\tilde{u}_n(\cdot, 1) + \theta_c - \theta_c) + hu_b\tilde{u}_n(\cdot, 1) \\
&\quad - \sigma(([\tilde{u}_n(\cdot, 1) + \theta_c]^+)^4 - \theta_c^4)(\tilde{u}_n(\cdot, 1) + \theta_c - \theta_c) - \sigma\theta_c^4\tilde{u}_n(\cdot, 1) + \sigma u_b^4\tilde{u}_n(\cdot, 1) \\
&\leq h[\tilde{u}_n(\cdot, 1) + \theta_c]^+\theta_c + hu_b\tilde{u}_n(\cdot, 1) - \sigma\theta_c^4\tilde{u}_n(\cdot, 1) + \sigma u_b^4\tilde{u}_n(\cdot, 1) \\
&\leq C_1|\tilde{u}_n(\cdot, 1)| + h\theta_c^2 \text{ a.e. on } [0, T],
\end{aligned}$$

where $C_1 = h(\theta_c + \max_{0\le t\le T} |u_b|) + \sigma(\theta_c^4 + \max_{0\le t\le T} u_b^4)$. Moreover, Lemma 3.1 guarantees existence of a positive constant C_0 depending only on δ such that

$$\begin{aligned} & - g_+(\cdot, \tilde{u}_n(\cdot,1))\tilde{u}_n(\cdot,1) \\ & \le C_0 C_1 |\tilde{u}_{nx}|^{1/2}_{L^2(1-\delta,1)} |\tilde{u}_n|^{1/2}_{L^2(1-\delta,1)} + h\theta_c^2 \\ & \le \frac{k_a}{2} |\tilde{u}_{nx}|^2_{L^2(1-\delta,1)} + C_2(|\tilde{u}_n|^2_{L^2(1-\delta,1)} + 1) \text{ a.e. on } [0,T], \end{aligned}$$

where C_2 is a positive constant independent of n. Thus, it follows that

$$\begin{aligned} & \frac{d}{dt}\frac{c_a}{2}\int_{1-\delta}^1 \tilde{u}_n^2 dx + \frac{k_a}{2}\int_{1-\delta}^1 |\tilde{u}_{nx}|^2 dx \\ & \le C_3 \left(\int_{1-\delta}^1 \tilde{u}_n^2 dx + \int_{1-\delta}^1 u_n^2 dx + \int_{1-\delta}^1 |u_{nx}|^2 dx \right) \text{ a.e. on } [0,T], \end{aligned}$$

where C_3 is a positive constant independent of n. By applying Gronwall's inequality to above, there exists a positive constant C_4 such that

$$\frac{c_a}{2}\int_{1-\delta}^1 \tilde{u}_n(t,x)^2 dx + \frac{k_a}{2}\int_0^t \int_{1-\delta}^1 |\tilde{u}_{nx}(\tau,x)|^2 dx d\tau \le C_4 \text{ for } 0 \le t \le T.$$

Next, by multiplying $\tilde{u}_{nt}$ on both sides of (27) and integrating it with x on $[1-\delta, 1]$, we have

$$\begin{aligned} \int_{1-\delta}^1 c_a \tilde{u}_{nt}^2 dx = k_a \int_{1-\delta}^1 \tilde{u}_{nxx}\tilde{u}_{nt} dx - k_a \int_{1-\delta}^1 \zeta_{xx} u_n \tilde{u}_{nt} dx \\ - 2k_a \int_{1-\delta}^1 \zeta_x u_{nx} \tilde{u}_{nt} dx \text{ a.e. on } [0,T]. \end{aligned}$$

By integration by parts for the first term of the right hand side and applying Young's inequality, we see that

$$\begin{aligned} & c_a \int_{1-\delta}^1 |\tilde{u}_{nt}(t)|^2 dx + \frac{d}{dt}\left(\frac{k_a}{2}\int_{1-\delta}^1 |\tilde{u}_{nx}(t)|^2 dx + \hat{g}_+(t, \tilde{u}_n(t,1)) \right) \\ & = \frac{\partial \hat{g}_+}{\partial t}(t, \tilde{u}_n(t,1)) - k_a \int_{1-\delta}^1 \zeta_{xx} u_n(t) \tilde{u}_{nt}(t) dx - 2k_a \int_{1-\delta}^1 \zeta_x u_{nx}(t)\tilde{u}_{nt}(t) dx \\ & \le \frac{\partial \hat{g}_+}{\partial t}(t, \tilde{u}_n(t,1)) + \frac{c_a}{2}\int_{1-\delta}^1 |\tilde{u}_{nt}(t)|^2 dx + \frac{k_a^2}{c_a} |\zeta_{xx}|^2_{L^\infty(1-\delta,1)} \int_{1-\delta}^1 u_n(t)^2 dx \\ & \quad + \frac{4k_a^2}{c_a} |\zeta_x|^2_{L^\infty(1-\delta,1)} \int_{1-\delta}^1 |u_{nx}(t)|^2 dx \text{ for a.e. } t \in [0,T], \end{aligned}$$

where $\hat{g}_+(t,r) = \int_0^r g_+(t,\xi)d\xi$ for $r \in \mathbb{R}$. Here, we note that the differentiability of the function

$$t \to \frac{k_a}{2}\int_{1-\delta}^1 |\tilde{u}_{nx}(t)|^2 dx + \hat{g}_+(t, \tilde{u}_n(t,1))$$

can be proved in the similar way to that of Lemma 4.4 in [2]. By Lemma 4.1 we can take a positive constant C_5 independent of n such that

$$\begin{aligned}
&\frac{c_a}{2}\int_{1-\delta}^{1}|\tilde{u}_{nt}(t)|^2\,dx+\frac{d}{dt}\left(\frac{k_a}{2}\int_{1-\delta}^{1}|\tilde{u}_{nx}(t)|^2\,dx+\hat{g}_+(t,\tilde{u}_n(t,1))\right)\\
&\le\frac{\partial\hat{g}_+}{\partial t}(t,\tilde{u}_n(t,1))+C_5\\
&=-hu_b'(t)\tilde{u}_n(t,1)-4\sigma u_b^3(t)u_b'(t)\tilde{u}_n(t,1)+C_5\\
&\le C_6|u_b'(t)|\left(\int_{1-\delta}^{1}\tilde{u}_n(t)^2dx+\int_{1-\delta}^{1}|\tilde{u}_{nx}(t)|^2\,dx\right)+C_5 \text{ for a.e. } t\in[0,T],
\end{aligned}$$

where C_6 is a positive constant. Since $\hat{g}_+(t,r)\ge 0$ for $t\in[0,T]$ and $r\ge 0$, from Gronwall's inequality, it follows that

$$\begin{aligned}
&\frac{c_a}{2}\int_0^t\int_{1-\delta}^{1}|\tilde{u}_{nt}(\tau,x)|^2\,dxd\tau+\frac{k_a}{2}\int_{1-\delta}^{1}|\tilde{u}_{nx}(t,x)|^2\,dx+\hat{g}_+(t,\tilde{u}_n(t,1))\\
&\le C_7 \text{ for } 0\le t\le T,
\end{aligned}$$

where C_7 is a positive constant. Since $\tilde{u}_n\in L^\infty(0,T;W^{1,2}(1-\delta,1))$ and $\tilde{u}_n(t,1-\delta)=0$, on account of Lemma 4.4 we can prove Lemma 4.2, completely. □

Lemma 4.3. *Under (A4) and (A5), let $\tilde{u}_n$ be the strong solution of (27)-(30) for n and $\tilde{u}=\zeta u$ on $(0,T)\times(0,1)$. Then, $\tilde{u}_n(\cdot,1)\to\tilde{u}(\cdot,1)$ in $L^2(0,T)$ as $n\to\infty$.*

Proof. By multiplying $\tilde{u}_n-\tilde{u}$ on both sides of difference of (27) and (26) for n, we have

$$\begin{aligned}
&\int_{1-\delta}^{1}c_a(\tilde{u}_{nt}-\tilde{u}_t)(\tilde{u}_n-\tilde{u})dx\\
&=k_a\int_{1-\delta}^{1}(\tilde{u}_{nxx}-\tilde{u}_{xx})(\tilde{u}_n-\tilde{u})dx-k_a\int_{1-\delta}^{1}\zeta_{xx}(u_n-u)(\tilde{u}_n-\tilde{u})dx\\
&\quad-2k_a\int_{1-\delta}^{1}\zeta_x(u_{nx}-u_x)(\tilde{u}_n-\tilde{u})dx=:I_1+I_2+I_3 \text{ a.e. on } [0,T].
\end{aligned}$$

On the left hand side, we see that

$$\int_{1-\delta}^{1}c_a(\tilde{u}_{nt}-\tilde{u}_t)(\tilde{u}_n-\tilde{u})dx=\frac{d}{dt}\frac{c_a}{2}\int_{1-\delta}^{1}|\tilde{u}_n-\tilde{u}|^2\,dx \text{ a.e. on } [0,T].$$

Easily, we observe that

$$\begin{aligned}
I_1=&\,k_a(\tilde{u}_{nx}(\cdot,1)-\tilde{u}_x(\cdot,1))(\tilde{u}_n(\cdot,1)-\tilde{u}(\cdot,1))\\
&-k_a\int_{1-\delta}^{1}|\tilde{u}_{nx}-\tilde{u}_x|^2\,dx \text{ a.e. on } [0,T].
\end{aligned}$$

Thanks to Young's inequality, we have

$$I_2 \le \frac{c_a}{4}\int_{1-\delta}^{1} |\tilde{u}_n - \tilde{u}|^2\, dx + C_1 \int_{1-\delta}^{1} |u_n - u|^2\, dx \text{ a.e. on } [0,T],$$

and

$$I_3 \le \frac{c_a}{4}\int_{1-\delta}^{1} |\tilde{u}_n - \tilde{u}|^2\, dx + C_2 \int_{1-\delta}^{1} |u_{nx} - u_x|^2\, dx \text{ a.e. on } [0,T],$$

where C_1 and C_2 are positive constants. Hence, we obtain

$$\begin{aligned}
&\frac{d}{dt}\frac{c_a}{2}\int_{1-\delta}^{1} |\tilde{u}_n - \tilde{u}|^2\, dx + k_a \int_{1-\delta}^{1} |\tilde{u}_{nx} - \tilde{u}_x|^2\, dx \\
&\le k_a(\tilde{u}_{nx}(\cdot,1) - \tilde{u}_x(\cdot,1))(\tilde{u}_n(\cdot,1) - \tilde{u}(\cdot,1)) + \frac{c_a}{2}\int_{1-\delta}^{1} |\tilde{u}_n - \tilde{u}|^2\, dx \\
&\quad + C_1 \int_{1-\delta}^{1} |u_n - u|^2\, dx + C_2 \int_{1-\delta}^{1} |u_{nx} - u_x|^2\, dx \\
&= -(g_+(\cdot, \tilde{u}_n(\cdot,1)) - g_+(\cdot, \tilde{u}(\cdot,1)))(\tilde{u}_n(\cdot,1) - \tilde{u}(\cdot,1)) \\
&\quad + \frac{c_a}{2}\int_{1-\delta}^{1} |\tilde{u}_n - \tilde{u}|^2\, dx + C_1 \int_{1-\delta}^{1} |u_n - u|^2\, dx + C_2 \int_{1-\delta}^{1} |u_{nx} - u_x|^2\, dx \\
&\le \frac{c_a}{2}\int_{1-\delta}^{1} |\tilde{u}_n - \tilde{u}|^2\, dx + C_1 \int_{1-\delta}^{1} |u_n - u|^2\, dx + C_2 \int_{1-\delta}^{1} |u_{nx} - u_x|^2\, dx \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. on } [0,T].
\end{aligned}$$

In the last inequality we use the monotonicity of g_+. By applying Gronwall's inequality, we have

$$\begin{aligned}
&\frac{c_a}{2}\int_{1-\delta}^{1} |\tilde{u}_n(t,x) - \tilde{u}(t,x)|^2\, dx + k_a \int_0^t \int_{1-\delta}^{1} |\tilde{u}_{nx}(\tau,x) - \tilde{u}_x(\tau,x)|^2\, dxd\tau \\
&\le C_3 \left(\int_{1-\delta}^{1} |u_{0n} - u_0|^2\, dx + \int_0^t \int_{1-\delta}^{1} |u_n(\tau,x) - u(\tau,x)|^2\, dxd\tau \right) \text{ for } 0 \le t \le T,
\end{aligned}$$

where C_3 is a positive constant independent of n. By Lemma 4.1, it follows that

$$|\tilde{u}_n - \tilde{u}|_{L^2(0,T;W^{1,2}(1-\delta,1))} \to 0 \text{ as } n \to \infty.$$

Hence, we have

$$\begin{aligned}
&|\tilde{u}_n(\cdot,1) - \tilde{u}(\cdot,1)|^2_{L^2(0,T)} \\
&= \int_0^T \left| \int_{1-\delta}^{1} \frac{\partial}{\partial x}\left(x(\tilde{u}_n(t,x) - \tilde{u}(t,x))\right) dx \right|^2 dt \\
&\le 2\, |\tilde{u}_n - \tilde{u}|^2_{L^2(0,T;W^{1,2}(1-\delta,1))} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

This shows that Lemma 4.3 holds. □

The next lemma is a key in the proof of Theorem 2.2, since it guarantees the uniform estimate for the derivative of solutions on the boundary $x = 1$. In order to obtain the estimate we extend $\tilde{u}_n$ by

$$\tilde{u}_n(t) = \tilde{u}_{0n} + t\frac{k_a}{c_a}(\tilde{u}_{0nxx} - \zeta_{xx}u_{0n} - 2\zeta_x u_{0nx}) \text{ for } t < 0 \text{ and } n. \tag{31}$$

By Lemma 4.1, it is clear that $\tilde{u}_{nt} \in L^2(-1, T; L^2(1-\delta, 1))$ for each n.

Lemma 4.4. *Assume (A4) and (A5), let u be a solution of $AP1(e, u_0)$ for $e \in (\delta, e_0, M, T)$, where $\delta > 0$, $\delta \leq e_0 \leq 1-\delta$, $M > 0$ and $T > 0$, and put $\tilde{u} = \zeta u$. Then, there exists a positive constant C' depending only on $\delta > 0$, $|u_b|_{W^{1,2}(0,T)}$, $|u_{0xx}|_{L^2(1-\delta,1)}$ and $|u_0|_{W^{1,2}(0,1)}$ such that*

$$|\tilde{u}_t(t)|^2_{L^2(1-\delta,1)} + \int_0^T |\tilde{u}_t(t)|_{W^{1,2}(1-\delta,1)}\, dt \leq C' \text{ for } 0 \leq t \leq T.$$

Proof. Let $\Delta t > 0$. In case $0 < t < \Delta t$, we see that

$$\begin{aligned}
&\int_{1-\delta}^1 c_a \frac{\tilde{u}_{nt}(t) - \tilde{u}_{nt}(t-\Delta t)}{\Delta t}\frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t}dx \\
&= \int_{1-\delta}^1 k_a \frac{\tilde{u}_{nxx}(t) - \tilde{u}_{0nxx}}{\Delta t}\frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t}dx \\
&\quad - \int_{1-\delta}^1 k_a \frac{\zeta_{xx}u_n(t) - \zeta_{xx}u_{0n}}{\Delta t}\frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t}dx \\
&\quad - 2\int_{1-\delta}^1 k_a \frac{\zeta_x u_{nx}(t) - \zeta_x u_{0nx}}{\Delta t}\frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t}dx \\
&=: I_1(t) + I_2(t) + I_3(t) \text{ for a.e. } t \in (0, \Delta t).
\end{aligned}$$

Easily, by integration by parts we observe that

$$\begin{aligned}
I_1(t) = &-\int_{1-\delta}^1 k_a \frac{\tilde{u}_{nx}(t) - \tilde{u}_{0nx}}{\Delta t}\frac{\tilde{u}_{nx}(t) - \tilde{u}_{nx}(t-\Delta t)}{\Delta t}dx \\
&+ k_a \frac{\tilde{u}_{nx}(t,1) - \tilde{u}_{0nx}(1)}{\Delta t}\frac{\tilde{u}_n(t,1) - \tilde{u}_n(t-\Delta t,1)}{\Delta t} \\
=: &\, I_{11}(t) + I_{12}(t) \text{ for a.e. } t \in (0, \Delta t).
\end{aligned}$$

By the definition (31) of $\tilde{u}_n$ and Young's inequality, we have

$$\begin{aligned}
I_{11}(t) \leq &-\frac{k_a}{2}\int_{1-\delta}^1 \left|\frac{\tilde{u}_{nx}(t) - \tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|^2 dx \\
&+ \frac{k_a^3}{2c_a^2}\frac{|t-\Delta t|^2}{(\Delta t)^2}\int_{1-\delta}^1 |\tilde{u}_{0nxxx} - (\zeta_{xx}u_{0n})_x - 2(\zeta_x u_{0nx})_x|^2\, dx \\
&\qquad \text{for a.e. } t \in (0, \Delta t).
\end{aligned}$$

Next, we estimate I_{12} with respect to n and Δt. Clearly, by (A5) and the definition of g_+, we have

$$\begin{aligned} I_{12}(t) &= -\frac{1}{\Delta t}\left(g_+(t,\tilde{u}_n(t,1)) - g_+(0,\tilde{u}_{0n}(1))\right)\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t} \\ &= -\frac{h}{\Delta t}\left([\tilde{u}_n(t,1)+\theta_c]^+ - [\tilde{u}_{0n}(1)+\theta_c]^+\right)\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t} \\ &\quad - \frac{\sigma}{\Delta t}\left(([\tilde{u}_n(t,1)+\theta_c]^+)^4 - ([\tilde{u}_{0n}(1)+\theta_c]^+)^4\right)\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t} \\ &\quad - \left\{\frac{h}{\Delta t}\left(u_b(t)-u_b(0)\right) + \frac{\sigma}{\Delta t}\left(u_b(t)^4 - u_b(0)^4\right)\right\}\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t} \\ &=: I_{12}^{(1)}(t) + I_{12}^{(2)}(t) + I_{12}^{(3)}(t) \text{ for a.e. } t \in (0,\Delta t), \end{aligned}$$

where h and σ are positive constants in the boundary condition. For the first term, thanks to

$$\tilde{u}_{0n}(1) = \tilde{u}_n(t-\Delta t,1) - (t-\Delta t)\frac{k_a}{c_a}\tilde{u}_{0nxx}(1)$$

for $t \in (0,\Delta t)$, the monotonicity and Lipschitz continuity of the positive part imply that

$$\begin{aligned} I_{12}^{(1)}(t) &= -\frac{h}{\Delta t}([\tilde{u}_n(t,1)+\theta_c]^+ - [\tilde{u}_n(t-\Delta t,1)+\theta_c]^+)\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t} \\ &\quad - \frac{h}{\Delta t}([\tilde{u}_n(t-\Delta t,1)+\theta_c]^+ - [\tilde{u}_{0n}(1)+\theta_c]^+)\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t} \\ &\le \frac{h}{\Delta t}|\tilde{u}_n(t-\Delta t,1) - \tilde{u}_{0n}(1)|\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right| \\ &\le \frac{C_0 k_a h|t-\Delta t|}{c_a\Delta t}|\tilde{u}_{0nxx}(1)|\left(\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|_{L^2(1-\delta,1)}\right. \\ &\quad \left. + \left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|_{L^2(1-\delta,1)}^{1/2}\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|_{L^2(1-\delta,1)}^{1/2}\right) \\ &\qquad\qquad \text{for a.e. } t \in (0,\Delta t). \end{aligned}$$

We note that Lemma 3.1 is applied in the last inequality. According to Young's inequality, it yields that

$$\begin{aligned} I_{12}^{(1)}(t) &\le \frac{k_a}{16}\left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|_{L^2(1-\delta,1)}^2 \\ &\quad + C_1\left(\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|_{L^2(1-\delta,1)}^2 + |\tilde{u}_{0nxx}(1)|^2\right) \text{ for a.e. } t \in (0,\Delta t), \end{aligned}$$

where C_1 is a positive constant independent of n and Δt. Similarly, on account of Lemma 4.2, we observe that

$$
\begin{aligned}
I_{12}^{(2)}(t) &= -\frac{\sigma}{\Delta t}\left(\left([\tilde{u}_n(t,1)+\theta_c]^+\right)^4 - \left([\tilde{u}_n(t-\Delta t,1)+\theta_c]^+\right)^4\right)\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t} \\
&\quad - \frac{\sigma}{\Delta t}\left(\left([\tilde{u}_n(t-\Delta t,1)+\theta_c]^+\right)^4 - \left([\tilde{u}_{0n}(1)+\theta_c]^+\right)^4\right)\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t} \\
&\le \frac{\sigma}{\Delta t}\left|\tilde{u}_n(t-\Delta t,1)-\tilde{u}_{0n}(1)\right|\left(|\tilde{u}_n(t-\Delta t,1)+\theta_c| + |\tilde{u}_{0n}(1)+\theta_c|\right) \\
&\quad \left(|\tilde{u}_n(t-\Delta t,1)+\theta_c|^2 + |\tilde{u}_{0n}(1)+\theta_c|^2\right)\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right| \\
&\le \frac{k_a}{16}\left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|^2_{L^2(1-\delta,1)} \\
&\quad + C_2\left(\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|^2_{L^2(1-\delta,1)} + |\tilde{u}_{0nxx}(1)|^2\right) \text{ for a.e. } t\in(0,\Delta t),
\end{aligned}
$$

where C_2 is a positive constant independent of n and Δt. Here, the last inequality holds, based on the boundedness of $\tilde{u}_{0n}(1)$ guaranteed by Lemma 4.1. On $I_{12}^{(3)}$, by putting $u_b(t) = u_b(0)$ for $t < 0$ and applying Young's inequality, we have

$$
\begin{aligned}
I_{12}^{(3)}(t) &\le \frac{h}{2}\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|^2 + \frac{h}{2}\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right|^2 \\
&\quad + 4\sigma M_b^3\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right| \\
&\le \frac{h}{2}\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|^2 + \frac{h}{2}\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right|^2 \\
&\quad + 8\sigma^2 M_b^6\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|^2 + \frac{1}{2}\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right|^2 \\
&\qquad \text{for a.e. } t\in(0,\Delta t),
\end{aligned}
$$

where $M_b = \max_{0\le t\le T}|u_b(t)|$. Thus, we get

$$
\begin{aligned}
I_1(t) &\le -\frac{3k_a}{8}\left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|^2_{L^2(1-\delta,1)} + C_3\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|^2 \\
&\quad + C_3\left(|\tilde{u}_{0n}|^2_{W^{1,3}(1-\delta,1)} + |(\zeta_{xx}u_{0n})_x|^2_{L^2(1-\delta,1)} + |(\zeta_x u_{0n})_x|^2_{L^2(1-\delta,1)}\right) \\
&\quad + C_3\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|^2_{L^2(1-\delta,1)} + C_3\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right|^2 \\
&\qquad \text{for a.e. } t\in(0,\Delta t),
\end{aligned}
$$

where C_3 is a positive constant independent of n and Δt.

As a next step, we deal with I_2 and I_3. First, by Lemma 4.2, we have $u_n(t) = u_{0n}$ for $t \le 0$. It is obvious that

$$
\begin{aligned}
I_2(t) &\le k_a |\zeta_{xx}|_{L^\infty(1-\delta,1)} \int_{1-\delta}^{1} \left| \frac{u_n(t) - u_{0n}}{\Delta t} \right| \left| \frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t} \right| dx \\
&\le C_4 \int_{1-\delta}^{1} \left| \frac{u_n(t) - u_n(t-\Delta t)}{\Delta t} \right|^2 dx + C_4 \int_{1-\delta}^{1} \left| \frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t} \right|^2 dx \\
&\qquad \text{for a.e. } t \in (0, \Delta t),
\end{aligned}
$$

where $C_4 = \frac{k_a}{2} |\zeta_{xx}|_{L^\infty(1-\delta,1)}$. Furthermore, thanks to $\zeta_x(1) = \zeta_x(1-\delta) = 0$ and Young's inequality, integration by parts implies that

$$
\begin{aligned}
I_3(t) &= 2k_a \int_{1-\delta}^{1} \zeta_{xx} \frac{u_n(t) - u_{0n}}{\Delta t} \frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t} dx \\
&\quad + 2k_a \int_{1-\delta}^{1} \zeta_x \frac{u_n(t) - u_{0n}}{\Delta t} \frac{\tilde{u}_{nx}(t) - \tilde{u}_{nx}(t-\Delta t)}{\Delta t} dx \\
&\le C_5 \int_{1-\delta}^{1} \left| \frac{u_n(t) - u_n(t-\Delta t)}{\Delta t} \right|^2 dx + C_5 \int_{1-\delta}^{1} \left| \frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t} \right|^2 dx \\
&\quad + \frac{k_a}{4} \int_{1-\delta}^{1} \left| \frac{\tilde{u}_{nx}(t) - \tilde{u}_{nx}(t-\Delta t)}{\Delta t} \right|^2 dx \text{ for a.e. } t \in (0, \Delta t),
\end{aligned}
$$

where C_5 is a positive constant independent of n and Δt. Consequently, we have

$$
\begin{aligned}
&\frac{d}{dt} \frac{c_a}{2} \int_{1-\delta}^{1} \left| \frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t} \right|^2 dx + \frac{k_a}{4} \int_{1-\delta}^{1} \left| \frac{\tilde{u}_{nx}(t) - \tilde{u}_{nx}(t-\Delta t)}{\Delta t} \right|^2 dx \\
&\le C_6 \left(|\tilde{u}_{0nxxx}|^2_{L^2(1-\delta,1)} + |(\zeta_{xx} u_{0n})_x|^2_{L^2(1-\delta,1)} + |(\zeta_x u_{0nx})_x|^2_{L^2(1-\delta,1)} \right) \\
&\quad + C_6 |\tilde{u}_{0nxx}(1)|^2 + C_6 \left| \frac{u_b(t) - u_b(t-\Delta t)}{\Delta t} \right|^2 + C_6 \int_{1-\delta}^{1} \left| \frac{u_n(t) - u_n(t-\Delta t)}{\Delta t} \right|^2 dx \\
&\quad + C_6 \int_{1-\delta}^{1} \left| \frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t} \right|^2 dx \text{ for a.e. } t \in (0, \Delta t),
\end{aligned} \tag{32}
$$

where C_6 is a positive constant.

Similarly to above, for $\Delta t < t < T$, we can get

$$
\begin{aligned}
&\frac{d}{dt} \frac{c_a}{2} \int_{1-\delta}^{1} \left| \frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t} \right|^2 dx + \frac{k_a}{2} \int_{1-\delta}^{1} \left| \frac{\tilde{u}_{nx}(t) - \tilde{u}_{nx}(t-\Delta t)}{\Delta t} \right|^2 dx \\
&\le C_7 \left| \frac{u_b(t) - u_b(t-\Delta t)}{\Delta t} \right|^2 + C_7 \int_{1-\delta}^{1} \left| \frac{u_n(t) - u_n(t-\Delta t)}{\Delta t} \right|^2 dx \\
&\quad + C_7 \int_{1-\delta}^{1} \left| \frac{\tilde{u}_n(t) - \tilde{u}_n(t-\Delta t)}{\Delta t} \right|^2 dx \text{ for a.e. } t \in (\Delta t, T),
\end{aligned} \tag{33}
$$

where C_7 is a positive constant. Indeed, we have

$$
\begin{aligned}
&\frac{d}{dt}\frac{c_a}{2}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|^2 dx\\
&=\int_{1-\delta}^{1} k_a\frac{\tilde{u}_{nxx}(t)-\tilde{u}_{nxx}(t-\Delta t)}{\Delta t}\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}dx\\
&\quad-\int_{1-\delta}^{1} k_a\frac{\zeta_{xx}u_n(t)-\zeta_{xx}u_n(t-\Delta t)}{\Delta t}\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}dx\\
&\quad-2\int_{1-\delta}^{1} k_a\frac{\zeta_{x}u_{nx}(t)-\zeta_{x}u_{nx}(t-\Delta t)}{\Delta t}\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}dx\\
&=: I_4(t)+I_5(t)+I_6(t) \text{ for a.e. } t\in(\Delta t,T).
\end{aligned}
$$

By integration by parts and (29), we see that

$$
\begin{aligned}
I_4(t)&=k_a\frac{\tilde{u}_{nx}(t,1)-\tilde{u}_{nx}(t-\Delta t,1)}{\Delta t}\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\\
&\quad-k_a\int_{1-\delta}^{1}\left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|^2 dx\\
&=: I_{41}(t)-k_a\int_{1-\delta}^{1}\left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|^2 dx \text{ for a.e. } t\in(\Delta t,T).
\end{aligned}
$$

We also have

$$
\begin{aligned}
I_{41}(t)&=-\frac{h}{\Delta t}([\tilde{u}_n(t,1)+\theta_c]^+-[\tilde{u}_n(t-\Delta t,1)+\theta_c]^+)\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\\
&\quad+\frac{h}{\Delta t}(u_b(t)-u_b(t-\Delta t))\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\\
&\quad-\frac{\sigma}{\Delta t}(([\tilde{u}_n(t,1)+\theta_c]^+)^4-([\tilde{u}_n(t-\Delta t,1)+\theta_c]^+)^4)\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\\
&\quad+\frac{\sigma}{\Delta t}(u_b(t)^4-u_b(t-\Delta t)^4)\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\\
&=: I_{41}^{(1)}(t)+I_{41}^{(2)}(t)+I_{41}^{(3)}(t)+I_{41}^{(4)}(t) \text{ for a.e. } t\in(\Delta t,T).
\end{aligned}
$$

Thanks to the Lipschitz continuity of the positive part, it follows that

$$
I_{41}^{(1)}(t)\le h\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right|^2,
$$

and

$$
\begin{aligned}
I_{41}^{(3)}(t)&\le\frac{\sigma}{\Delta t}((\tilde{u}_n(t,1)+\theta_c)^4-(\tilde{u}_n(t-\Delta t,1)+\theta_c)^4)\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\\
&\le 4\sigma(C+\theta_c)^2\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right|^2 \text{ for a.e. } t\in(\Delta t,T),
\end{aligned}
$$

where C is the positive constant defined by Lemma 4.2. Also, by applying Young's inequality, we have

$$I_{41}^{(2)}(t) \le \frac{h}{2}\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|^2 + \frac{h}{2}\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right|^2,$$

and

$$\begin{aligned} I_{41}^{(4)}(t) &\le 4\sigma M_b^3\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right| \\ &\le 8\sigma^2 M_b^6\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|^2 + \frac{1}{2}\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right|^2 \\ &\qquad \text{for a.e. } t \in (\Delta t, T), \end{aligned}$$

where $M_b = \max_{0\le t\le T}|u_b(t)|$. Hence, we obtain

$$\begin{aligned} I_4(t) \le\ & C_8\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|^2 + C_8\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right|^2 \\ & - k_a\int_{1-\delta}^{1}\left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|^2 dx \text{ for a.e. } t \in (\Delta t, T), \end{aligned}$$

where C_8 is a positive constant. Similarly to I_2 and I_3, we have

$$\begin{aligned} I_5(t) \le C_9\int_{1-\delta}^{1}\left|\frac{u_n(t)-u_n(t-\Delta t)}{\Delta t}\right|^2 dx + C_9\int_{1-\delta}^{1}\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|^2 dx \\ \text{for a.e. } t \in (\Delta t, T), \end{aligned}$$

and

$$\begin{aligned} I_6(t) \le\ & C_{10}\int_{1-\delta}^{1}\left|\frac{u_n(t)-u_n(t-\Delta t)}{\Delta t}\right|^2 dx + C_{10}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|^2 dx \\ & + \frac{k_a}{2}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|^2 dx \text{ for a.e. } t \in (\Delta t, T), \end{aligned}$$

where C_9 and C_{10} are positive constants. Accordingly, we have

$$\begin{aligned} & \frac{d}{dt}\frac{c_a}{2}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|^2 dx + \frac{k_a}{2}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|^2 dx \\ & \le C_{11}\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|^2 + C_{11}\left|\frac{\tilde{u}_n(t,1)-\tilde{u}_n(t-\Delta t,1)}{\Delta t}\right|^2 \\ & \quad + C_{11}\int_{1-\delta}^{1}\left|\frac{u_n(t)-u_n(t-\Delta t)}{\Delta t}\right|^2 dx + C_{11}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|^2 dx \\ & \qquad \text{for a.e. } t \in (\Delta t, T), \end{aligned}$$

where C_{11} is a positive constant. Due to Lemma 3.1 and Young's inequality, (33) holds.

From (32) and (33), there exists a positive constant C_{12} independent of n and Δt such that

$$\begin{aligned}
&\frac{d}{dt}\frac{c_a}{2}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|^2 dx+\frac{k_a}{4}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|^2 dx\\
&\leq C_{12}\chi_{\Delta t}(t)\alpha_{0n}+C_{12}\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|^2+C_{12}\int_{1-\delta}^{1}\left|\frac{u_n(t)-u_n(t-\Delta t)}{\Delta t}\right|^2 dx\\
&\quad +C_{12}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_n(t)-\tilde{u}_n(t-\Delta t)}{\Delta t}\right|^2 dx \text{ for a.e. } t\in[0,T], \qquad (34)
\end{aligned}$$

where

$$\chi_{\Delta t}(t)=\begin{cases} 1 \text{ for } 0<t<\Delta t,\\ 0 \text{ for } \Delta t<t<T,\end{cases}$$

and

$$\alpha_{0n}=|\tilde{u}_{0nxx}(1)|^2+|\tilde{u}_{0nxxx}|^2_{L^2(1-\delta,1)}+|(\zeta_{xx}u_{0n})_x|^2_{L^2(1-\delta,1)}+|(\zeta_x u_{0nx})_x|^2_{L^2(1-\delta,1)}.$$

By applying Gronwall's inequality to (34), we see that

$$\begin{aligned}
&\frac{c_a}{2}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_n(t_1)-\tilde{u}_n(t_1-\Delta t)}{\Delta t}\right|^2 dx+\frac{k_a}{4}\int_0^{t_1}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|^2 dxdt\\
&\leq \frac{c_a}{2}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_n(0)-\tilde{u}_n(-\Delta t)}{\Delta t}\right|^2 dx+C_{13}\alpha_{0n}\int_0^{t_1}\chi_{\Delta t}(t)dt\\
&\quad +C_{13}\int_0^{t_1}\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|^2 dt\\
&\quad +C_{13}\int_0^{t_1}\int_{1-\delta}^{1}\left|\frac{u_n(t)-u_n(t-\Delta t)}{\Delta t}\right|^2 dxdt \text{ for } t_1\in[0,T],
\end{aligned}$$

where C_{13} is a positive constant independent of n and Δt.
Because of $\tilde{u}_n(0)-\tilde{u}_n(-\Delta t)=\Delta t\frac{k_a}{c_a}\zeta u_{0nxx}$, we have

$$\begin{aligned}
&\frac{c_a}{2}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_n(t_1)-\tilde{u}_n(t_1-\Delta t)}{\Delta t}\right|^2 dx+\frac{k_a}{4}\int_0^{T}\int_{1-\delta}^{1}\left|\frac{\tilde{u}_{nx}(t)-\tilde{u}_{nx}(t-\Delta t)}{\Delta t}\right|^2 dxdt\\
&\leq \frac{k_a^2}{2c_a}\int_{1-\delta}^{1}|\zeta u_{0nxx}|^2 dx+C_{13}\alpha_{0n}\Delta t+C_{13}\int_0^{T}\left|\frac{u_b(t)-u_b(t-\Delta t)}{\Delta t}\right|^2 dt\\
&\quad +C_{13}\int_0^{T}\int_{1-\delta}^{1}\left|\frac{u_n(t)-u_n(t-\Delta t)}{\Delta t}\right|^2 dxdt \text{ for any } n \text{ and } \Delta t. \qquad (35)
\end{aligned}$$

This shows that $\{\frac{\tilde{u}_n-\tilde{u}_n(\cdot-\Delta t)}{\Delta t} : \Delta t > 0\}$ and $\{\frac{\tilde{u}_{nx}-\tilde{u}_{nx}(\cdot-\Delta t)}{\Delta t} : \Delta t > 0\}$ are bounded in $L^\infty(0,T;\, L^2(1-\delta,1))$ and $L^2(0,T;L^2(1-\delta,1))$, respectively. It is clear that

$$\frac{\tilde{u}_n-\tilde{u}_n(\cdot-\Delta t)}{\Delta t} \to \tilde{u}_{nt} \text{ weakly}* \text{ in } L^\infty(0,T;L^2(1-\delta,1))$$

and

$$\frac{\tilde{u}_{nx}-\tilde{u}_{nx}(\cdot-\Delta t)}{\Delta t} \to \tilde{u}_{nxt} \text{ weakly in } L^2(0,T;L^2(1-\delta,1)) \text{ as } \Delta t \to 0.$$

Accordingly, from (35) it follows that

$$\begin{aligned}
&\frac{c_a}{2}\int_{1-\delta}^1 |\tilde{u}_{nt}(t)|^2\,dx + \frac{k_a}{4}\int_0^t\int_{1-\delta}^1 |\tilde{u}_{nx\tau}|^2\,dxd\tau \\
&\le \frac{k_a^2}{2c_a}\int_{1-\delta}^1 |\zeta u_{0nxx}|^2\,dx + C_{13}\int_0^t |u_b'|^2\,d\tau + C_{13}\int_0^t\int_{1-\delta}^1 |u_{n\tau}|^2\,dxd\tau \\
&\hspace{20em}\text{for } t\in[0,T] \text{ and any } n.
\end{aligned}$$

By using Lemma 4.1, we observe that $\{\tilde{u}_{nxt}\}$ is bounded in $L^2(0,T;L^2(1-\delta,1))$. Hence, due to Lemma 4.3 it is obvious that $\tilde{u}_{nxt}\in L^2(0,T;L^2(1-\delta,1))$ and it satisfies

$$\begin{aligned}
&\frac{c_a}{2}\int_{1-\delta}^1 |\tilde{u}_t(t)|^2\,dx + \frac{k_a}{4}\int_0^t\int_{1-\delta}^1 |\tilde{u}_{x\tau}|^2\,dxd\tau \\
&\le \frac{k_a^2}{2c_a}\int_{1-\delta}^1 |\zeta u_{0xx}|^2\,dx + C_{13}\int_0^t |u_b'|^2\,d\tau + C_{13}\int_0^t\int_{1-\delta}^1 |u_\tau|^2\,dxd\tau \text{ for } 0\le t\le T.
\end{aligned}$$

Thus, Lemma 4.4 holds. □

Proof of Proposition 2.7. By Lemma 4.4, there exists a positive constant R_b such that

$$\int_{1-\delta}^1 |\tilde{u}_t(t)|^2\,dx + \int_0^t\int_{1-\delta}^1 |\tilde{u}_{x\tau}|^2\,dxd\tau \le R_b \text{ for } 0\le t\le T.$$

Hence, we have

$$\begin{aligned}
&\int_0^T |\tilde{u}_t(\cdot,1)|^2\,dt \\
&= \int_0^T\int_{1-\delta}^1 \frac{\partial}{\partial x}\left(\frac{1}{\delta}(x-(1-\delta))\tilde{u}_t(\cdot,x)^2\right)dxdt \\
&\le \left(\frac{1}{\delta}+1\right)\int_0^T\int_{1-\delta}^1 \tilde{u}_t(\cdot,x)^2dxdt + \int_0^T\int_{1-\delta}^1 \tilde{u}_{tx}(\cdot,x)^2dxdt \\
&\le \left(\frac{1}{\delta}+1\right)(R_bT+R_b).
\end{aligned}$$

This implies that $\tilde{u}(\cdot,1) \in W^{1,2}(0,T)$, namely, $u(\cdot,1) \in W^{1,2}(0,T)$. Moreover, it holds

$$\begin{aligned} |u_t(t)|_{L^2(1-\delta/2,1)} &\le \left|\frac{1}{\zeta}\tilde{u}_t(t)\right|_{L^2(1-\delta/2,1)} \\ &\le \frac{1}{d_0}|\tilde{u}_t(t)|_{L^2(1-\delta/2,1)} \text{ for } 0 \le t \le T, \end{aligned}$$

where $d_0 = \inf_{x\in[1-\delta/2,1]} \zeta(x)$. Thus, we have proved Proposition 2.7. □

5 Estimates for water contents

We give some uniform estimates for the water content in order to prove Theorem 2.2. The proofs of Lemmas 5.2 and 5.4 are similar to Lemmas 3.4 and 3.5, respectively. Thus, we omit the proofs.

Lemma 5.1 (cf. Proposition 3.3 in [2]). *Let $T > 0$, and assume that*

$$\begin{aligned} &e \in W^{1,2}(0,T), \quad 0 < e < 1 \quad on\ [0,T], \\ &u_0 \in W^{1,2}(0,1), \quad u_0 \ge 0 \quad on\ [e(0),1], \quad u_0 \le 0 \quad on\ [0,e(0)], \\ &u_b \in W^{1,2}(0,T), \quad u_b \ge \theta_c \ on\ [0,T], \\ &w_0 \in W^{1,2}(0,1). \end{aligned}$$

Then AP(e) has a unique solution (u,w) on $[0,T]$, where u and w satisfy {(S2), (S3)} and (S4), respectively.

Proof. Since $u(\cdot,1) \in W^{1,2}(0,T)$ as proved in Proposition 2.7, by applying Lemma 3.3, we can show this lemma. □

Lemma 5.2 (cf. Lemmas 4.3 and 4.6 in [2]). *Suppose the same assumption as in Lemma 3.4 and Lemma 5.1. Let (u,w) be a solution of $AP(e)$ on $[0,T]$ for $e \in K(\delta, e_0, M, T_0)$. Then, there exists a positive constant*

$$C_1 = C_1\left(\delta, M, T, |u_b|_{W^{1,2}(0,T)}, |u_{0xx}|_{L^2(1-\delta,1)}, |u_0|_{W^{1,2}(0,1)}, |w_0|_{W^{1,2}(0,1)}\right)$$

such that

$$\int_0^1 w(t,x)^2 dx + \int_0^t\int_0^1 |w_x(\tau,x)|^2 dx d\tau \le C_1 \text{ for } 0 \le t \le T,$$

and

$$\int_0^1 |w_x(t,x)|^2 dx + \int_0^t\int_0^1 |w_\tau(\tau,x)|^2 dx d\tau \le C_1 \text{ for } 0 \le t \le T. \tag{36}$$

Remark 5.3. Because of $u(\cdot,1) \in W^{1,2}(0,T)$ by Proposition 2.7, we can prove (36).

Lemma 5.4 (cf. Lemma 5.2 in [2]). *Let*

$$\delta > 0, M > 0, T > 0, e_0 \in [\delta, 1-\delta]$$

and $e_i \in K(\delta, e_0, M, T)$ *for* $i = 1, 2$ *and assume (A2) and (A3). If* (u_i, w_i) *is a solution of AP(*e_i*) for* $i = 1, 2$*, then, for some positive constant*

$$C_2 = C_2\left(\delta, M, T, |u_b|_{W^{1,2}(0,T)}, |u_{0xx}|_{L^2(1-\delta,1)}, |u_0|_{W^{1,2}(0,1)}, |w_0|_{W^{1,2}(0,1)}\right)$$

it holds that

$$\begin{aligned}&\int_0^1 |w_1(t,x) - w_2(t,x)|^2 dx + \int_0^t\int_0^1 |w_{1x}(\tau,x) - w_{2x}(\tau,x)|^2 dxd\tau \\ &\le C_2 \int_0^t \left(|e_1(\tau) - e_2(\tau)|^2 + |e_1'(\tau) - e_2'(\tau)|^2\right) d\tau \text{ for } 0 \le t \le T.\end{aligned}$$

6 Proof of Theorem 2.2

In this section, first, by applying Banach's fixed point theorem to a solution operator of AP(e), we prove the existence of a solution of P locally in time. As mentioned in Section 3, for proving (14), we need $e \in W^{1,3}(0,T)$. Let $K(\delta, e_0, M, T_0)$ be a subset of $W^{1,3}(0,T)$ given in Section 3, for $0 < \delta < e_0 < 1-\delta < 1$, $M > 0$ and $T > 0$. Moreover, for $T_0 \in (0, T]$, we define a solution operator $\Gamma : K(\delta, e_0, M, T_0) \to W^{1,3}(0,T_0)$ as follows:

$$\Gamma(e)(t) = e_0 + \int_0^t \frac{1}{lw(\tau, e(\tau))} \left(k_l u_x(\tau, e(\tau)-) - k_a u_x(\tau, e(\tau)+)\right) d\tau \tag{37}$$
$$\text{for } 0 \le t \le T \text{ and } e \in K(\delta, e_0, M, T_0),$$

where (u, w) is a solution of AP(e) on $[0, T]$. Clearly, the set $K(\delta, e_0, M, T_0)$ is closed in $W^{1,3}(0,T_0)$, and $\Gamma(e) \in W^{1,3}(0,T_0)$ for any $e \in K(\delta, e_0, M, T_0)$.

Lemma 6.1. *Let* $0 < \delta < 1$ *and* $M > 0$*. Suppose the same assumption as in the later part of Lemma 3.4 and* $w_0(e_0) \ge 2\delta_1$ *for some* $\delta_1 > 0$*, then there exists* $T_1 > 0$ *such that* $w(t, e(t)) \ge \delta_1$ *for any* $t \in [0, T_1]$*.*

Proof. First, it is easy to see that

$$w(t, e(t)) \ge w_0(e_0) - |w(t, e(t)) - w_0(e(t))| - |w_0(e(t)) - w_0(e_0)| \text{ for } 0 \le t \le T.$$

Here, we note that

$$\begin{aligned}&|w(t, e(t)) - w_0(e(t))|^2 \\ &= \frac{1}{e(t)} \int_0^{e(t)} \frac{\partial}{\partial x}(x(w(t,x) - w_0(x))^2) dx\end{aligned}$$

$$\le \frac{1}{\delta}\left(\int_0^1 |w(t)-w_0|^2\,dx + 2\int_0^1 |w_x(t)-w_{0x}||w(t)-w_0|dx\right)$$
$$\le \frac{1}{\delta}\left(t\int_0^t |w_\tau|^2_{L^2(0,1)}\,d\tau + 2t^{1/2}\left(\int_0^t |w_\tau|^2_{L^2(0,1)}\,d\tau\right)^{1/2}\left(|w_x(t)|_{L^2(0,1)} + |w_{0x}|_{L^2(0,1)}\right)\right)$$
$$\text{for } 0\le t\le T.$$

Hence, thanks to the assumption, Lemmas 5.2 and 5.4 imply that

$$\begin{aligned} w(t,e(t)) &\ge 2\delta_1 - R_1 t^{1/2} - |e(t)-e_0|^{1/2}\,|w_{0x}|_{L^2(0,1)} \\ &\ge 2\delta_1 - R_1 t^{1/2} - |e'|^{1/2}_{L^3(0,t)}\, t^{1/3}\,|w_{0x}|_{L^2(0,1)} \text{ for } 0\le t\le T, \end{aligned}$$

where R_1 is a positive constant. Thus, we can prove this lemma. □

Lemma 6.2. *Let $M>0$. Suppose the same assumption as in Lemma 6.1, then there exist $\delta>0$ and $T_2\in(0,T_1]$ such that $\delta\le\Gamma(e)(t)\le 1-\delta$ for $0\le t\le T_2$ and $e\in K(\delta,e_0,M,T)$.*

Proof. First, we choose $\delta>0$ such that $2\delta\le e_0\le 1-2\delta$. For any $e\in K(\delta,e_0,M,T)$, by applying Lemmas 3.4 and 6.1, we have

$$\begin{aligned} &|\Gamma(e)(t)-e_0| \\ &= \left|\int_0^t \frac{1}{lw(\tau,e(\tau))}\left(k_l u_x(\tau,e(\tau)-) - k_a u_x(\tau,e(\tau)+)\right)d\tau\right| \\ &\le \frac{k^* C_1}{l\delta_1} t \text{ for } 0\le t\le T_1, \end{aligned}$$

where $k^*=\max\{k_l,k_a\}$ and C_1 is a positive constant defined by (14) in Lemma 3.4. Hence, by taking $0<T_2\le T_1$ satisfying $\frac{k^*C_1}{l\delta_1}T_2\le\delta$, we obtain

$$\delta\le\Gamma(e)(t)\le 1-\delta \text{ for } 0\le t\le T_2.$$ □

Lemma 6.3. *Suppose the same assumption as in Lemma 6.2. Then, there exists T_3 in $(0,T_2]$ such that $\int_0^{T_3}\left|\frac{d}{dt}\Gamma(e)(t)\right|^3 dt\le M$ for $e\in K(\delta,e_0,M,T)$.*

Proof. By using Lemmas 3.4 and 6.1, we have

$$\begin{aligned} &\int_0^t \left|\frac{d}{dt}\Gamma(e)(t)\right|^3 dt = \int_0^t \left|\frac{1}{lw(\tau,e(\tau))}\left(k_l u_x(\tau,e(\tau)-) - k_a u_x(\tau,e(\tau)+)\right)\right|^3 d\tau \\ &\le \frac{1}{l^3\delta_1^3}\int_0^t \left\{ |k_l u_x(\tau,e(\tau)-)| + |k_a u_x(\tau,e(\tau)+)| \right\}^3 d\tau \le C_1 t, \end{aligned}$$

where C_1 is a positive constant. By choosing small $T_3\in(0,T_2]$ with $C_1T_3\le M$, we can prove this lemma. □

Lemma 6.4. *Under the same assumption as in Lemma 6.1, there exists* $\lambda \in (0,1)$ *and* $T_0 \in (0, T_3]$ *such that*

$$|\Gamma(e_1) - \Gamma(e_2)|_{W^{1,3}(0,T_0)} \leq \lambda \, |e_1 - e_2|_{W^{1,3}(0,T_0)} .$$

Proof. For $0 < T_4 \leq T_3$, we have

$$\begin{aligned}
&\left| \frac{d}{dt}(\Gamma(e_1) - \Gamma(e_2)) \right|_{L^3(0,T_4)} \\
&= \left| \frac{1}{l w_1(\cdot, e_1)} \left(k_l u_{1x}(\cdot, e_1-) - k_a u_{1x}(\cdot, e_1+) \right) \right. \\
&\qquad \left. - \frac{1}{l w_2(\cdot, e_2)} \left(k_l u_{2x}(\cdot, e_2-) - k_a u_{2x}(\cdot, e_2+) \right) \right|_{L^3(0,T_4)} \\
&\leq \left| \frac{k_l}{l} \left(\frac{1}{w_1(\cdot, e_1)} \left(u_{1x}(\cdot, e_1-) \right) - \frac{1}{w_2(\cdot, e_2)} \left(u_{2x}(\cdot, e_2-) \right) \right) \right|_{L^3(0,T_4)} \\
&\qquad + \left| \frac{k_a}{l} \left(\frac{1}{w_1(\cdot, e_1)} \left(u_{1x}(\cdot, e_1+) \right) - \frac{1}{w_2(\cdot, e_2)} \left(u_{2x}(\cdot, e_2+) \right) \right) \right|_{L^3(0,T_4)} \\
&=: I_1 + I_2
\end{aligned}$$

First, we estimate I_1. Easily, we have

$$\begin{aligned}
I_1 &\leq \frac{k_l}{l} \left| \left(\frac{1}{w_1(\cdot, e_1)} - \frac{1}{w_2(\cdot, e_2)} \right) u_{1x}(\cdot, e_1-) \right|_{L^3(0,T_4)} \\
&\quad + \frac{k_l}{l} \left| \frac{1}{w_2(\cdot, e_2)} \left(u_{1x}(\cdot, e_1-) - u_{2x}(\cdot, e_2-) \right) \right|_{L^3(0,T_4)} \\
&= \frac{k_l}{l} (I_{11} + I_{12}).
\end{aligned}$$

Let δ_1 and C_1 be positive constants given in Lemma 6.1 and (14), respectively. Then, we have

$$\begin{aligned}
I_{11} \leq \frac{C_1}{\delta_1^2} \Bigg\{ &\left(\int_0^{T_4} |w_1(t, e_1(t)) - w_2(t, e_1(t))|^3 \, dt \right)^{\frac{1}{3}} \\
&+ \left(\int_0^{T_4} |w_2(t, e_1(t)) - w_2(t, e_2(t))|^3 \, dt \right)^{\frac{1}{3}} \Bigg\} =: I_{11}^{(1)} + I_{11}^{(2)}.
\end{aligned}$$

By applying Lemmas 3.1 and 5.4, we see that

$$\left(I_{11}^{(1)} \right)^3 \leq \left(\frac{C_1}{\delta_1^2} \right)^3 \int_0^{T_4} |w_1 - w_2|^3_{L^\infty(0,1)} \, dt$$

$$
\begin{aligned}
&\le C_2 \int_0^{T_4} \left(|w_1 - w_2|_{L^2(0,1)} + |w_1 - w_2|^{1/2}_{L^2(0,1)} |w_{1x} - w_{2x}|^{1/2}_{L^2(0,1)} \right) dt \\
&\le 8C_2 \Bigg(|w_1 - w_2|^3_{L^\infty(0,T_4;L^2(0,1))} T_4 \\
&\qquad + |w_1 - w_2|^{3/2}_{L^\infty(0,T_4;L^2(0,1))} \int_0^{T_4} |w_{1x} - w_{2x}|^{3/2}_{L^2(0,1)} dt \Bigg) \\
&\le 8C_2 \Bigg(|w_1 - w_2|^3_{L^\infty(0,T_4;L^2(0,1))} T_4 \\
&\qquad + |w_1 - w_2|^{3/2}_{L^\infty(0,T_4;L^2(0,1))} |w_{1x} - w_{2x}|^{3/2}_{L^2(0,T_4;L^2(0,1))} T_4^{1/4} \Bigg) \\
&\le C_3 T_4^{1/4} \left(\int_0^{T_4} \left(|e_1 - e_2|^2 + |e_1' - e_2'|^2 \right) dt \right)^{3/2} \\
&\le C_3 T_4^{3/4} \int_0^{T_4} \left(|e_1 - e_2|^3 + |e_1' - e_2'|^3 \right) dt,
\end{aligned}
$$

and

$$
I_{11}^{(1)} \le C_3^{1/3} T_4^{1/4} \left(|e_1 - e_2|_{L^3(0,T_4)} + |e_1' - e_2'|_{L^3(0,T_4)} \right),
$$

where C_2 and C_3 are positive constants. Next, by applying Lemma 3.1, again, in case $e_1(t) \le e_2(t)$ for some $t \in [0, T]$,

$$
\begin{aligned}
&|w_2(t, e_1(t)) - w_2(t, e_2(t))| \\
&\le |w_{2x}(t)|_{L^\infty(0,e_2(t))} |e_1(t) - e_2(t)| \\
&\le C \left(|w_{2x}(t)|_{L^2(0,e_2(t))} + |w_{2x}(t)|^{1/2}_{L^2(0,e_2(t))} |w_{2xx}(t)|^{1/2}_{L^2(0,e_2(t))} \right) |e_1(t) - e_2(t)| \\
&\le C \left(|w_{2x}(t)|_{L^2(0,1)} + |w_{2x}(t)|^{1/2}_{L^2(0,1)} \left(|w_{2xx}(t)|^{1/2}_{L^2(0,e_2(t))} + |w_{2xx}(t)|^{1/2}_{L^2(e_2(t),1)} \right) \right) \\
&\quad \times |e_1(t) - e_2(t)|.
\end{aligned}
$$

Even if $e_1(t) > e_2(t)$, this inequality still holds. Accordingly, it is clear that

$$\begin{aligned}\left(I_{11}^{(2)}\right)^3 \le C_4 \left|e_1(t) - e_2(t)\right|^3_{C([0,T_4])} \int_0^{T_4} \Big(& |w_{2x}(t)|_{L^2(0,1)} \\ & + |w_{2x}(t)|^{1/2}_{L^2(0,1)} \left(|w_{2xx}(t)|^{1/2}_{L^2(0,e_2(t))} + |w_{2xx}(t)|^{1/2}_{L^2(e_2(t),1)} \right) \Big)^3 dt \\ \le 8C_4 T_4^2 \left|e_1' - e_2'\right|^3_{L^3(0,T_4)} \Bigg\{ & |w_{2x}|^3_{L^\infty(0,T_4;L^2(0,1))} T_4 \\ & + |w_{2x}|^{3/2}_{L^\infty(0,T_4;L^2(0,1))} \int_0^{T_4} \left(|w_{2xx}(t)|^{3/2}_{L^2(0,e_2(t))} + |w_{2xx}(t)|^{3/2}_{L^2(e_2(t),1)} \right) dt \Bigg\} \\ \le C_5 T_4^2 \left|e_1' - e_2'\right|^3_{L^3(0,T_4)}, &\end{aligned}$$

where C_4 and C_5 are positive constants. Namely, for some positive constant C_6 it holds that

$$I_{11} \le C_6 T_4^{1/4} |e_1 - e_2|_{W^{1,3}(0,T_4)}.$$

On I_{12}, by using (18), we have

$$\begin{aligned} I_{12} &\le \frac{1}{\delta_1} \left| u_{1x}(\cdot, e_1-) - u_{2x}(\cdot, e_2-) \right|_{L^3(0,T_4)} \\ &= \frac{1}{\delta_1} \left(\int_0^{T_4} \left| u_{1x}(t, e_1(t)-) - u_{2x}(t, e_2(t)-) \right|^3 dt \right)^{\frac{1}{3}} \\ &\le C_7 |e_1 - e_2|_{W^{1,3}(0,T_4)} T_4^{\frac{1}{12}}, \end{aligned} \tag{38}$$

where C_7 is a positive constant. Hence, we have

$$I_1 \le C_8 T_4^{\frac{1}{12}} |e_1 - e_2|_{W^{1,3}(0,T_4)} \text{ for } 0 \le T_4 \le T_3,$$

where C_8 is a positive constant. Similarly to I_1, we can get same estimate for I_2. Thus, we obtain

$$\left| \frac{d}{dt}(\Gamma(e_1) - \Gamma(e_2)) \right|_{L^3(0,T_4)} \le C_9 T_4^{\frac{1}{12}} |e_1 - e_2|_{W^{1,3}(0,T_4)} \text{ for } 0 \le T_4 \le T_3,$$

where C_9 is a positive constant. By using above inequality, we have

$$\begin{aligned} |\Gamma(e_1) - \Gamma(e_2)|_{L^3(0,T_4)} &\le T_4 \left| \frac{d}{d\tau}(\Gamma(e_1) - \Gamma(e_2)) \right|_{L^3(0,T_4)} \\ &\le C_{10} T_4 |e_1 - e_2|_{W^{1,3}(0,T_4)} \text{ for } 0 \le T_4 \le T_3, \end{aligned}$$

where C_{10} is a positive constant. Therefore, Lemma 6.4 have been proved. □

Since Lemmas 6.1 - 6.4 hold, we can choose small $T_0 \in (0, T]$ such that Γ is a function $K(\delta, e_0, M, T_0) \to K(\delta, e_0, M, T_0)$ and a contraction mapping on $W^{1,3}(0, T_0)$. Hence, there exists one and only one $e \in K(\delta, e_0, M, T_0)$ with $\Gamma(e) = e$ by applying Banach's fixed-point theorem to Γ. Obviously, (6) holds on $[0, T_0]$. Consequently, P has at least one solution (e, u, w) on $[0, T_0]$ for some $T_0 \in (0, T]$.

Next, we will prove the uniqueness of solutions to P on $[0, T_0]$ for any $T_0 \in (0, T]$. For $0 < T_0 \leq T$ and $i = 1, 2$ let (e_i, u_i, w_i) be solutions of P on $[0, T_0]$. Namely, by Definition 2.2, it holds

$$
\begin{aligned}
& e_i \in W^{1,\infty}(0, T_0), \quad \delta \leq e_i \leq 1 - \delta \text{ on } [0, T_0], \\
& u_i, w_i \in W^{1,2}(0, T_0; L^2(0, 1)) \cap L^\infty(0, T_0; W^{1,2}(0, 1)), \\
& u_{ixx}, w_{ixx} \in L^2(Q_l(T_0, e_i)) \cap L^2(Q_a(T_0, e_i)), \\
& u_{ix}(\cdot, e_i \pm) \in L^\infty(0, T_0), \quad w_i(\cdot, e_i) \geq \delta_1 \text{ on } [0, T_0].
\end{aligned}
$$

Clearly, $\Gamma(e_i) = e_i$ on $[0, T_0]$ for $i = 1, 2$. On account of the proof of the estimates, we see that $e_1 = e_2$ on $[0, T_*]$ for some $T_* \in (0, T]$, and the uniqueness of AP(e) guaranteed by Lemma 5.1 implies that $u_1 = u_2$ and $w_1 = w_2$ on $(0, T_*) \times (0, 1)$. By repeating the argument above finite times, we have proved the uniqueness of solutions to P.

7 Proof of Theorem 2.6

Throughout this section, we suppose that all the assumptions of Theorem 2.6 hold.

Lemma 7.1. *Let (e, u, w) be a solution of P on $[0, T]$. Then, it holds that $u + \theta \leq u_b$ on $(0, T) \times (0, 1)$.*

Proof. First, (A2) and the assumption $u_b \geq u_0 + \theta_c$ on $[0, 1]$ suggests that $u_0(e_0) = 0$ and $u_b \geq \theta_c$. Accordingly, for any $t \in [0, T]$, by multiplying $[u(t) + \theta_c - u_b]^+$ on both sides of (1) and integrating it with x on $[0, e(t)]$, $[e(t), 1]$, respectively, we have

$$
\begin{aligned}
\int_0^e c_l u_t [u + \theta_c - u_b]^+ dx &= \int_0^e k_l u_{xx} [u + \theta_c - u_b]^+ dx \\
&= -\int_0^e k_l u_x ([u + \theta_c - u_b]^+)_x dx \\
&= -\int_0^e k_l \left| ([u + \theta_c - u_b]^+)_x \right|^2 dx \\
&\leq 0 \text{ a.e. on } [0, T],
\end{aligned}
$$

and from the monotonicity of the boundary condition it follows

$$
\int_e^1 c_a u_t [u + \theta_c - u_b]^+ dx
$$

$$
\begin{aligned}
&= \int_e^1 k_a u_{xx}[u+\theta_c-u_b]^+ dx \\
&= -(h(u(\cdot,1)+\theta_c-u_b)+\sigma((u(\cdot,1)+\theta_c)^4-u_b^4))[u(\cdot,1)+\theta_c-u_b]^+ \\
&\quad - \int_e^1 k_a u_x([u+\theta_c-u_b]^+)_x dx \\
&\le - \int_e^1 k_a \left|([u+\theta_c-u_b]^+)_x\right|^2 dx \\
&\le 0 \text{ a.e. on } [0,T].
\end{aligned}
$$

Thus, we obtain

$$\int_0^1 u_t[u+\theta-u_b]^+ dx \le 0 \text{ a.e. on } [0,T].$$

Since u_b is a constant, we see that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left|[u+\theta_c-u_b]^+\right|^2 dx \le 0 \text{ a.e. on } [0,T].$$

The assumption $u_b \ge u_0+\theta$ on $[0,1]$ implies

$$
\begin{aligned}
\int_0^1 \left|[u(t)+\theta_c-u_b]^+\right|^2 dx &\le \int_0^1 \left|[u_0+\theta_c-u_b]^+\right|^2 dx \\
&= 0 \text{ on } [0,T].
\end{aligned}
$$

This shows that Lemma 7.1 holds. □

Lemma 7.2. *Let (e,u,w) be a solution of P on $[0,T]$. Then there exists a positive constant δ_w such that $w \ge \delta_w$ on $(0,T)\times(0,1)$.*

Proof. By the assumption $w_0 > 0$ on $[0,1]$, we can choose δ_w such that $w_0(x) \ge \delta_w$ for $x \in [0,1]$. For any $t \in [0,T]$ we multiply $[-w(t)+\delta_w]^+$ on both sides of (2) and integrate it with x on $[0,e(t)]$, $[e(t),1]$, respectively. By integration by parts, we have

$$
\begin{aligned}
\int_0^1 w_t[-w+\delta_w]^+ dx &= -\int_0^e d_l w_x([-w+\delta_w]^+)_x dx - \int_e^1 d_a w_x([-w+\delta_w]^+)_x dx \\
&\quad + d_a w_x(\cdot,1)[-w(\cdot,1)+\delta_w]^+ \\
&= \int_0^e d_l \left|([-w+\delta_w]^+)_x\right|^2 dx + \int_e^1 d_a \left|([-w+\delta_w]^+)_x\right|^2 dx \\
&\quad - \{b_1 p(u(\cdot,1)+\theta_c) - b_2 p(u_b)\}[-w(\cdot,1)+\delta_w]^+ \text{ a.e. on } [0,T].
\end{aligned}
$$

Moreover, by the assumption $b_1 \le b_2$, Lemma 7.1 and the monotonicity of p, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 \left|[-w+\delta_w]^+\right|^2 dx &\le \{b_1 p(u(\cdot,1)+\theta_c) - b_2 p(u_b)\}[-w(\cdot,1)+\delta_w]^+ \\ &\le 0 \text{ a.e. on } [0,T].\end{aligned}$$

Thanks to $w_0(x) \ge \delta_w$ for $x \in [0,1]$, we have

$$\begin{aligned}\int_0^1 \left|[-w+\delta_w]^+\right|^2 dx &\le \int_0^1 \left|[-w_0+\delta_w]^+\right|^2 dx \\ &\le 0 \quad \text{on } [0,T].\end{aligned}$$

Thus, Lemma 7.2 has been proved. □

Lemma 7.3. *Let $[0,T^*)$ be the maximal interval of existence of the solution (u,w,e) to P. Then, there exists a positive constant C_1 such that*

$$\int_0^t |u_t|^2_{L^2(0,1)}\, d\tau + |u(t)|^2_{W^{1,2}(0,1)} + \int_0^t |e'|^3\, d\tau \le C_1 \textit{ for } t \in [0,T^*). \tag{39}$$

Proof. Multiply $k_l u_t$, $k_a u_t$ on both sides of the first equation of (1), the second one and integrate it with x on $[0,e]$, $[e,1]$, respectively. Then, similarly to the proof of Lemma 4.4 in [2], we can obtain

$$\begin{aligned}&C_* \int_0^1 |u_t|^2 dx + \frac{k_l^2}{2}\frac{d}{dt}\int_0^e |u_x|^2 dx + \frac{k_a^2}{2}\frac{d}{dt}\int_e^1 |u_x|^2 dx \\ &\quad + \frac{k_l^2}{2} u_x(\cdot,e-)^2 e' - \frac{k_a^2}{2} u_x(\cdot,e+)^2 e' \\ &\quad + k_a(h(u(\cdot,1)+\theta_c-u_b) + \sigma((u(\cdot,1)+\theta_c)^4 - u_b^4))u_t(\cdot,1) \\ &\le 0 \text{ a.e. on } [0,T^*),\end{aligned}$$

where $C_* = \min\{c_l k_l, c_a k_a\}$. We note that $u_t(\cdot,1)$ is well-defined by Proposition 2.7. Easily, we get

$$\begin{aligned}\frac{k_l^2}{2} u_x(\cdot,e-)^2 e' - \frac{k_a^2}{2} u_x(\cdot,e+)^2 e' &= \frac{|e'|^2}{2}(k_l u_x(\cdot,e-) + k_a u_x(\cdot,e+)) l w(\cdot,e) \\ &=: I_1 \text{ a.e. on } [0,T^*).\end{aligned}$$

If $e'(t) > 0$ for some $t \in [0,T^*)$, then $k_l u_x(t,e(t)) > k_a u_x(t,e(t))$ implies that

$$|e'(t)| = k_l u_x(t,e(t)-) - k_a u_x(t,e(t)+).$$

Hence, we have

$$\begin{aligned} I_1(t) &\geq \frac{|e'(t)|^2}{2}(k_l u_x(t,e(t)-) + k_a u_x(t,e(t)+))lw(t,e(t)) \\ &\geq \frac{|e'(t)|^2}{2}(k_l u_x(t,e(t)-) - k_a u_x(t,e(t)+))lw(t,e(t)) \\ &= \frac{|e'(t)|^3}{2} l^2 w(t,e(t))^2. \end{aligned}$$

Here, we note that $u_x(t,e(t)) \geq 0$ because of $u(t,e(t)) = 0$ and $u(t,x) \geq 0$ for all $x \in [e(t),1]$. In case $e'(t) \leq 0$ for some $t \in [0,T^*)$, we can get the same inequality. Also, we obtain

$$\begin{aligned} &(h(u(\cdot,1) + \theta_c - u_b) + \sigma((u(\cdot,1)+\theta_c)^4 - u_b^4))u_t(\cdot,1) \\ &= \frac{d}{dt}G + hu_b' u(\cdot,1) + 4\sigma u_b^3 u_b' u(\cdot,1) \text{ a.e. on } [0,T^*), \end{aligned}$$

where

$$G = \frac{h}{2}(u(\cdot,1)+\theta_c)^2 - hu_b u(\cdot,1) + \frac{\sigma}{5}(u(\cdot,1)+\theta_c)^5 - \sigma u_b^4 u(\cdot,1) \text{ on } [0,T^*).$$

As given in Lemma 3.2, $u(\cdot,1) \geq 0$ on $[0,T^*)$. Accordingly, by applying Young's inequality, we have

$$\begin{aligned} G &= \frac{h}{2}(u(\cdot,1)+\theta_c)^2 - hu_b(u(\cdot,1)+\theta_c) + hu_b\theta_c \\ &\quad + \frac{\sigma}{5}(u(\cdot,1)+\theta_c)^5 - \sigma u_b^4(u(\cdot,1)+\theta_c) + \sigma u_b^4\theta_c \\ &\geq \frac{h}{2}(u(\cdot,1)+\theta_c)^2 - hu_b(u(\cdot,1)+\theta_c) - \sigma u_b^4(u(\cdot,1)+\theta_c) \\ &\geq \frac{h}{4}(u(\cdot,1)+\theta_c)^2 - 2hu_b^2 - \frac{2}{h}\sigma^2 u_b^8 \\ &\geq -C_b \text{ a.e. on } [0,T^*), \end{aligned}$$

where $C_b = 2hu_b^2 - \frac{2}{h}\sigma^2 u_b^8$. By putting

$$E_1 = \frac{k_l^2}{2}\int_0^e |u_x|^2\,dx + \frac{k_a^2}{2}\int_e^1 |u_x|^2\,dx + k_a(G + C_b) \text{ on } [0,T^*),$$

we have

$$C_*\int_0^1 |u_t|^2\,dx + \frac{|e'|^3}{2}l^2 w(\cdot,e)^2 + \frac{d}{dt}E_1 \leq 0 \text{ a.e. on } [0,T^*).$$

Therefore, we have

$$C_*\int_0^t\int_0^1 |u_t|^2\,dxd\tau + \frac{l^2}{2}\int_0^t |e'|^3 w(\cdot,e)^2 d\tau + E_1(t) \leq E_1(0) \text{ for } t \in [0,T^*).$$

Thus, we conclude that Lemma 7.3 holds. □

Lemma 7.4. *Let $T^* < \infty$ and $[0, T^*)$ be the maximal interval of existence of the solution (u, w, e) to P. Suppose that neither (b) nor (c) does not hold. Then, there exists $\delta_e > 0$ such that $\delta_e \le e \le 1 - \delta_e$ on $[0, T^*)$.*

Proof. From the assumption, there exist $\varepsilon_0 > 0$ and sequences $\{t_{1n}\}$ and $\{t_{2n}\}$ such that $t_{1n} > T_* - \frac{1}{n}$, $t_{2n} > T_* - \frac{1}{n}$, $e(t_{1n}) \ge \varepsilon_0$ and $e(t_{2n}) < 1 - \varepsilon_0$ for any $n \in \mathbb{N}$. Because of $e \in C([0, T^*))$, we can choose $t_n \in (t_{1n}, t_{2n})$ or (t_{2n}, t_{1n}) such that $\varepsilon_0^* \le e(t_n) \le 1 - \varepsilon_0^*$, where $\varepsilon_0^* = \min\{\frac{1}{4}, \varepsilon_0\}$. By Lemma 7.3, for $t_n < t < T^*$, we have

$$|e(t) - e(t_n)| \le \int_{t_n}^{t} |e'| dt \le C_1 (t - t_n)^{\frac{2}{3}}, \tag{40}$$

where C_1 is a positive constant defined by Lemma 7.3.

Thus, there exists $\delta_1 > 0$ such that for any $n \in \mathbb{N}$ and t with $0 < t - t_n < \delta_1$, it holds $|e(t) - e(t_n)| < \frac{\varepsilon_0^*}{2}$. Hence, we have $e(t) \ge \frac{\varepsilon_0^*}{2}$ for $t_n < t < \min\{T^*, t_n + \delta_1\}$ and any $n \in \mathbb{N}$. Moreover, we can choose $n_0 \in \mathbb{N}$ such that $|t_n - T^*| < \delta_1$ for $n \ge n_0$. Accordingly, for $T^* > t \ge t_{n0}$, $e(t) \ge \frac{\varepsilon_0^*}{2}$ holds. Since $e(t) > 0$ for $0 \le t \le t_{n0}$, there exists $\varepsilon_1 > 0$ such that $e \ge \varepsilon_1$ on $[0, t_{n0}]$. Let $\delta_e^{(1)} = \min\{\frac{\varepsilon_0^*}{2}, \varepsilon_1\}$. Then, it follows that

$$(t) \ge \delta_e^{(1)} \text{ for } 0 \le t < T^*.$$

Similarly, we obtain

$$e \le 1 - \delta_e \text{ on } [0, T^*) \text{ for some } \delta_e^{(2)} > 0.$$

Thus, Lemma 7.4 has been proved. □

Lemma 7.5. *Under the same assumption as in Lemma 7.4, there exist*

$$u_* \in W^{1,2}(0,1), \quad w_* \in W^{1,2}(0,1), \quad e_* \in (0,1), \quad \varepsilon' > 0,$$

such that

$$\begin{aligned}
\varepsilon' \le e_* \le 1 - \varepsilon', & \\
u_* \in W^{2,2}(1-\varepsilon', 1), & \qquad u_* \ge 0 \text{ on } [e_*, 1], \quad u_* \le 0 \text{ on } [0, e_*], \\
-k_a u_{*x}(1) = g(T^*, u_*(1)), & \qquad w_* \ge \delta_w \text{ on } [0,1], \\
u(t) \to u_* \text{ in } L^2(0,1) & \quad \text{weakly in } W^{2,2}(1-\varepsilon', 1), \\
w(t) \to w_* \text{ in } L^2(0,1), & \quad e(t) \to e_* \text{ in } \mathbb{R} \text{ as } t \uparrow T^*.
\end{aligned}$$

Proof. First, by Lemma 7.3, we see that

$$\begin{aligned}
|u(t) - u(t')|_{L^2(0,1)} &\le \int_{t'}^{t} |u_\tau|_{L^2(0,1)} \, d\tau \\
&\le C'(t - t')^{1/2} \text{ for } 0 < t' < t < T^*,
\end{aligned}$$

where C' is a positive constant. This shows that $\{u(t)\}_{t\uparrow T^*}$ is a Cauchy sequence in $L^2(0,1)$. Similarly, (40) implies that $\{e(t)\}_{t\uparrow T^*}$ is also a Cauchy sequence in $\mathbb{R}$. Hence, there exist $e_* \in \mathbb{R}$, $\varepsilon > 0$ and $u_* \in L^2(0,1)$ such that

$$e(t) \to e_* \text{ in } \mathbb{R} \text{ as } t \uparrow T^*, \quad \varepsilon \le e_* \le 1-\varepsilon,$$

and

$$u(t) \to u_* \text{ in } L^2(0,1) \text{ as } t \uparrow T^*.$$

Here, we note that $\varepsilon > 0$, since neither (b) nor (c) does not occur. Moreover, thanks to Lemmas 4.3 and 4.4, we observe that $\{u(t) | 0 \le t < T^*\}$ is bounded in $W^{1,2}(0,1)$ and $\varepsilon' \le e \le 1-\varepsilon'$ on $[0,T^*)$ for some $\varepsilon' > 0$. Hence, by applying Lemma 4.4 and $u_{xx} = \frac{c_a}{k_a} u_t$, we observe that $\{u(t) | 0 \le t < T^*\}$ is also bounded in $W^{2,2}(1-\varepsilon',1)$. Immediately, it holds that

$$u(t) \to u_* \text{ in } C([0,1]), \text{ and weakly in } W^{1,2}(0,1) \text{ and } W^{2,2}(1-\varepsilon',1) \text{ as } t \uparrow T^*.$$

Next, by Lemma 3.2, we see that $u(t) \ge 0$ on $[e(t),1]$ for $t \in [0,T^*)$. Hence, it is easy to obtain $u_* \ge 0$ on $[e_*,1]$. Similarly, we can prove $u_* \le 0$ on $[0,e_*]$.

From now on, we show that $w_* \ge \delta_w$ on $[0,1]$. First, Lemma 5.2 guarantees that $\{w(t)\}_{t\uparrow T^*}$ is a Cauchy sequence in $L^2(0,1)$. Accordingly, there exists $w_* \in L^2(0,1)$ such that

$$w(t) \to w_* \text{ in } L^2(0,1) \text{ as } t \uparrow T^*.$$

Moreover, the boundedness of $\{w(t) | 0 \le t < T^*\}$ in $W^{1,2}(0,1)$ implies that

$$w(t) \to w_* \text{ in } C([0,1]) \text{ and weakly in } W^{1,2}(0,1) \text{ as } t \uparrow T^*.$$

Since $w(t) \ge \delta_w$ on $(0,T^*) \times (0,1)$ by Lemma 7.2, we can show $w_* \ge \delta_w$ on $[0,1]$.

Finally, we show $-k_a u_{*x}(1) = g(T^*, u_*(1))$. By Lemma 4.4, $\{u_x(t) | 0 \le t < T^*\}$ is bounded in $W^{1,2}(1-\varepsilon',1)$, $\{u_x(t_n)\}$ is bounded in $W^{1,2}(1-\varepsilon',1)$. Hence,

$$u_x(t) \to u_{*x} \text{ in } C([1-\varepsilon',1]) \text{ as } t \uparrow T^*.$$

It is also clear that

$$u(t) \to u_* \text{ in } C([1-\varepsilon',1]) \text{ as } t \uparrow T^*. \tag{41}$$

By using (41), we obtain $-k_a u_{*x}(1) = g(T^*, u_*(1))$. □

Proof of Theorem 2.6. Suppose that $T^* < \infty$ and either (b) or (c) does not hold. Then, by Lemma 7.5, there exists a triplet (e_*, u_*, w_*) satisfying (A1)–(A5) as the initial time T^*. Consequently, Theorem 2.2 implies existence of a unique solution $(\hat{e}, \hat{u}, \hat{w})$ on $[T^*, \hat{T}]$ for some $\hat{T} > T^*$. Immediately, we can extend the solution (e,u,w) beyond T^* by using $(\hat{e}, \hat{u}, \hat{w})$. This contradicts the definition of $T^* < \infty$. Hence, $T^* = \infty$. This completes the proof of Theorem 2.6. □

8 Conclusion

In this paper, we have established existence and uniqueness of a strong solution under high regularity for the initial data and shown the behavior of the free boundary. For future works, we consider the behavior of the free boundary. As a first step, we will prove that (c) in Theorem 2.6 does not occur.

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