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On polynomial equations over split-octonions: the arbitrary field case

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Abstract. Over the split-octonion algebra defined over an arbitrary field, we solve all polynomial equations whose coefficients are scalar except for the constant term. As an application, we determine the square and cubic roots of an octonion.

Contents

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1 Introduction

Unless stated otherwise, let $\mathbb{F}$ be a field (which may be finite) of arbitrary characteristic $\operatorname{char}\mathbb{F} \geqslant 0$. All vector spaces and algebras are assumed to be defined over $\mathbb{F}$.

1.1 Equations over octonions

The problem of solving polynomial equations has historically been one of the central problems in mathematics and has played a fundamental role in the development of algebraic geometry and other branches of mathematics. Polynomial equations have been studied not only over fields, but also over matrix algebras, quaternion algebras, octonion algebras, and other noncommutative or nonassociative structures.

In general, an *octonion algebra* $\mathbf{C}$ (also called a *Cayley algebra*) over a field $\mathbb{F}$ is a nonassociative alternative unital algebra of dimension 8, endowed with a non-singular quadratic multiplicative form

$$n\colon \mathbf{C} \to \mathbb{F},$$

called the *norm*. The norm n is said to be

- *isotropic* if $n(a) = 0$ for some non-zero $a \in \mathbf{C}$. In this case, there exists a unique octonion algebra $\mathbf{O}_{\mathbb{F}}$ over $\mathbb{F}$ with isotropic norm (see Theorem 1.8.1 in [31]). This algebra is called the *split-octonion algebra*.

- *anisotropic*, otherwise. In this case, the octonion algebra $\mathbf{C}$ is a division algebra.

Note that if the field $\mathbb{F}$ is algebraically closed, then any octonion algebra over $\mathbb{F}$ is isomorphic to the split-octonion algebra $\mathbf{O}_{\mathbb{F}}$ (see, for example, Lemma 2.2 in [28]). By Artin's theorem, in any alternative algebra every subalgebra generated by two elements is associative. Consequently, every octonion algebra is power-associative; that is, the subalgebra generated by a single element is associative. Therefore, for any $a \in \mathbf{C}$, the power a^n is well defined without the need to specify the placement of parentheses.

Polynomial equations over octonion algebras have been studied in various settings. In particular, Rodríguez-Ordóñez [30] proved that every polynomial equation of positive degree over the algebra $\mathbf{A}_{\mathbb{R}}$ of *Cayley numbers* (i.e., the division algebra of real octonions), with the only highest-degree term present, has at least one solution. An explicit algorithm for solving quadratic equations of the form $x^2+bx+c=0$ over $\mathbf{A}_{\mathbb{R}}$ was provided by Wang, Zhang, and Zhang [33], along with criteria determining whether such an equation has one, two, or infinitely many solutions.

Flaut and Shpakivskyi [16] studied the equation $x^n = a$ over real octonion division algebras. For an octonion division algebra $\mathbf{C}$ over an arbitrary field $\mathbb{F}$, Chapman [11] developed a complete method for finding the solutions of a general polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ over $\mathbf{C}$. Furthermore, Chapman and Vishkautsan [13], working over a division algebra $\mathbf{C}$, determined the solutions of the polynomial equation $(a_n c)x^n + (a_{n-1}c)x^{n-1} + \cdots + (a_1 c)x + (a_0 c) = 0$, and also discussed the solutions of the related equation $(ca_n)x^n + (ca_{n-1})x^{n-1} + \cdots + (ca_1)x + (ca_0) = 0$. Chapman and Levin [12] introduced a method for finding so-called *alternating roots* of polynomials over an arbitrary division Cayley–Dickson algebra. In a subsequent work, Chapman and Vishkautsan [14] investigated conditions under which, for a root a of a polynomial $f(x)$ over a general Cayley–Dickson algebra, there exists a factorization $f(x) = g(x)(x - a)$ for some polynomial $g(x)$. Working over the split-octonion algebra over an algebraically closed field, Lopatin and Rybalov [25] solved all polynomial equations in which all coefficients except the constant term are scalar. As a consequence, the n-th roots of a split-octonion were computed. In [28], Lopatin and Zubkov studied the linear equations given by $ax = c$, $(ax)b = c$ and $a(bx) = c$ over the split-octonion algebra $\mathbf{O}$ when the base field $\mathbb{F}$ is algebraically closed. It is worth noting that, over a division octonion algebra, these equations are easily solvable and admit a unique solution whenever $a, b \neq 0$. In contrast, over an algebraically closed field the situation is substantially more delicate. As a consequence of the main result in [28], it was shown that if a linear monomial equation over octonions with a non-zero constant term has at least two solutions, then it necessarily admits an invertible solution.

The split-octonion algebra has numerous applications in physics. For example, the Dirac equation, which describes the motion of a free spin-$\frac{1}{2}$ particle such as an electron or a proton, can be formulated in terms of split-octonions (see [17,18,22,23]). Further applications of split-octonions arise in electromagnetic theory (see [8–10]), geometrodynamics (see [7]), unified quantum theories (see [4,6,21]), and special relativity (see [19]).

Polynomial equations over arbitrary algebras have recently been investigated by Illmer and Netzer [20], who established conditions guaranteeing the existence of a common solution to n polynomial equations in n variables, with an application to polynomial equations over $\mathbf{A}_{\mathbb{R}}$. Linear equations over matrix algebras have also been studied extensively (see, for example, [2,3,5,15,24,32]). The main questions addressed in these works include

- determining conditions for the existence of solutions to linear equations;
- describing the general form of the solutions.

1.2 Results

In Section 2 we define the octonion algebra $\mathbf{O}$ and its automorphism group $\mathrm{Aut}(\mathbf{O})$. In Section 3 we extend the results of [25] from the case of an algebraically closed field to the case of an arbitrary field, which includes, in particular, the case of $\mathbb{R}$ that is important for applications in physics.

More precisely, we solve the equation

$$\alpha_n x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_1 x = c \tag{1}$$

with scalar coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ and a possibly non-scalar constant term $c \in \mathbf{O}$, where the variable is $x \in \mathbf{O}$ (see Theorem 3.3). The solution of equation (1) is reduced to solving polynomial equations over the field $\mathbb{F}$. In Corollary 3.4 we provide further details for the case where $c \in \mathbf{O}$ is non-scalar. As applications, we consider

- the quadratic equation $x^2 = c$ in Proposition 4.2;
- the cubic equation $x^3 = c$ over the real numbers in Proposition 4.4.

In Corollary 3.5 we show that, when c is non-scalar, the number of solutions of equation (1) is finite.

Besides allowing an arbitrary base field, the main difference from the results of [25] is that we do not assume c to be in a *canonical* form. Consequently, our proofs differ substantially from those in [25]. Moreover, when we employ a canonical form of an octonion (see Proposition 2.2), it is different from the canonical forms considered in [25] (see Definition 2.3 in [25] for details).

2 Octonions

The definitions presented in this section are taken from [26]. Additional material on octonions can be found in the books [29, 34].

2.1 Split-octonions

The *split octonion algebra* $\mathbf{O} = \mathbf{O}(\mathbb{F})$, also known as the *split Cayley algebra*, is the vector space consisting of all matrices of the form

$$a = \begin{pmatrix} \alpha & \mathbf{u} \\ \mathbf{v} & \beta \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{F}^3$, endowed with the multiplication

$$aa' = \begin{pmatrix} \alpha\alpha' + \mathbf{u}\cdot\mathbf{v}' & \alpha\mathbf{u}' + \beta'\mathbf{u} - \mathbf{v}\times\mathbf{v}' \\ \alpha'\mathbf{v} + \beta\mathbf{v}' + \mathbf{u}\times\mathbf{u}' & \beta\beta' + \mathbf{v}\cdot\mathbf{u}' \end{pmatrix}, \qquad a' = \begin{pmatrix} \alpha' & \mathbf{u}' \\ \mathbf{v}' & \beta' \end{pmatrix}.$$

Here

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3, \qquad \mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1).$$

For brevity, we denote by $\mathbf{c}_1 = (1,0,0)$, $\mathbf{c}_2 = (0,1,0)$, $\mathbf{c}_3 = (0,0,1)$, and $\mathbf{0} = (0,0,0)$ the standard basis vectors of $\mathbb{F}^3$ and the zero vector, respectively. Consider the following basis of $\mathbf{O}$:

$$e_1 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix},\ \mathbf{u}_i = \begin{pmatrix} 0 & \mathbf{c}_i \\ \mathbf{0} & 0 \end{pmatrix},\ \mathbf{v}_i = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{c}_i & 0 \end{pmatrix},\ i = 1,2,3.$$

The unity element of $\mathbf{O}$ is given by $1_{\mathbf{O}} = e_1 + e_2$. Note that the multiplication in this basis satisfies

$$\mathbf{u}_i\mathbf{u}_j = (-1)^{\epsilon_{ij}}\mathbf{v}_k, \qquad \mathbf{v}_i\mathbf{v}_j = (-1)^{\epsilon_{ji}}\mathbf{u}_k,$$

where $\{i,j,k\} = \{1,2,3\}$ and ϵ_{ij} denotes the parity of the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ k & i & j \end{pmatrix}.$$

The algebra $\mathbf{O}$ is endowed with a linear involution

$$\overline{a} = \left(\begin{array}{cc} \beta & -\mathbf{u} \\ -\mathbf{v} & \alpha \end{array} \right),$$

which satisfies $\overline{aa'} = \overline{a'}\,\overline{a}$. The associated *norm* is defined by

$$n(a) = \alpha\beta - \mathbf{u}\cdot\mathbf{v},$$

and it induces a nondegenerate symmetric bilinear *form*

$$q(a,a') = n(a+a') - n(a) - n(a') = \alpha\beta' + \alpha'\beta - \mathbf{u}\cdot\mathbf{v}' - \mathbf{u}'\cdot\mathbf{v}.$$

The linear *trace* function is given by $\operatorname{tr}(a) = \alpha + \beta$. Note that

$$\operatorname{tr}(a)\,1_{\mathbf{O}} = a + \overline{a} \qquad \text{and} \qquad n(a)\,1_{\mathbf{O}} = a\,\overline{a}.$$

The subspace of traceless octonions is denoted by

$$\mathbf{O}_0 = \{a \in \mathbf{O} \mid \operatorname{tr}(a) = 0\},$$

and the affine variety of octonions of zero norm by

$$\mathbf{O}_{\#} = \{a \in \mathbf{O} \mid n(a) = 0\}.$$

The following identities hold for all $a, a' \in \mathbf{O}$:

$$\operatorname{tr}(aa') = \operatorname{tr}(a'a), \qquad n(aa') = n(a)\,n(a'), \tag{2}$$

and

$$n(a+a') = n(a) + n(a') - \operatorname{tr}(aa') + \operatorname{tr}(a)\operatorname{tr}(a'). \tag{3}$$

Moreover, every element $a \in \mathbf{O}$ satisfies the quadratic identity

$$a^2 - \operatorname{tr}(a)\,a + n(a)\,1_{\mathbf{O}} = 0. \tag{4}$$

The algebra $\mathbf{O}$ is a simple *alternative* algebra; that is, for all $a, b \in \mathbf{O}$,

$$a(ab) = (aa)b, \qquad (ba)a = b(aa). \tag{5}$$

Furthermore, the involution interacts with multiplication via

$$\overline{a}(ab) = n(a)\,b, \qquad (ba)\overline{a} = n(a)\,b. \tag{6}$$

The following remark is well known and can be proved easily.

Remark 2.1. Let $a \in \mathbf{O}$. Then exactly one of the following cases occurs:

- If $n(a) \neq 0$, then there exist unique elements $b, c \in \mathbf{O}$ such that $ba = 1_{\mathbf{O}}$ and $ac = 1_{\mathbf{O}}$. In this case, $b = c = \overline{a}/n(a)$, and we denote this element by a^{-1}.
- If $n(a) = 0$, then a admits neither a left inverse nor a right inverse in $\mathbf{O}$.

As a direct consequence of identities (6), for every $a \in \mathbf{O}\backslash\mathbf{O}_{\#}$ we have

$$a^{-1}(ab) = b, \qquad (ba)a^{-1} = b, \tag{7}$$

for all $b \in \mathbf{O}$.

2.2 The group $\mathrm{Aut}(\mathbf{O})$

The group $\mathrm{Aut}(\mathbf{O})$ of all automorphisms of the algebra $\mathbf{O}$ is the exceptional simple algebraic group $\mathrm{G}_2(\mathbb{F})$ when the field $\mathbb{F}$ is algebraically closed. In the general case, the group $\mathrm{Aut}(\mathbf{O})$ contains a subgroup isomorphic to $\mathrm{SL}_3(\mathbb{F})$. More precisely, every element $g \in \mathrm{SL}_3(\mathbb{F})$ defines an automorphism of $\mathbf{O}$ by

$$a \longmapsto \begin{pmatrix} \alpha & \mathbf{u}g \\ \mathbf{v}g^{-T} & \beta \end{pmatrix},$$

where g^{-T} denotes $(g^{-1})^T$ and the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}^3$ are regarded as row vectors. For every $\mathbf{u}, \mathbf{v} \in \mathbb{F}^3$, define automorphisms $\delta_1(\mathbf{u}), \delta_2(\mathbf{v}) \in \mathrm{Aut}(\mathbf{O})$ by

$$\delta_1(\mathbf{u})(a') = \begin{pmatrix} \alpha' - \mathbf{u}\cdot\mathbf{v}' & (\alpha' - \beta' - \mathbf{u}\cdot\mathbf{v}')\mathbf{u} + \mathbf{u}' \\ \mathbf{v}' - \mathbf{u}' \times \mathbf{u} & \beta' + \mathbf{u}\cdot\mathbf{v}' \end{pmatrix},$$

$$\delta_2(\mathbf{v})(a') = \begin{pmatrix} \alpha' + \mathbf{u}'\cdot\mathbf{v} & \mathbf{u}' + \mathbf{v}' \times \mathbf{v} \\ (-\alpha' + \beta' - \mathbf{u}'\cdot\mathbf{v})\mathbf{v} + \mathbf{v}' & \beta' - \mathbf{u}'\cdot\mathbf{v} \end{pmatrix},$$

where $a' = \begin{pmatrix} \alpha' & \mathbf{u}' \\ \mathbf{v}' & \beta' \end{pmatrix}$.

A direct computation shows that the map

$$\hbar\colon \mathbf{O} \to \mathbf{O}, \qquad a \longmapsto \begin{pmatrix} \beta & -\mathbf{v} \\ -\mathbf{u} & \alpha \end{pmatrix}, \tag{8}$$

also belongs to $\mathrm{Aut}(\mathbf{O})$.

The action of $\mathrm{Aut}(\mathbf{O})$ on $\mathbf{O}$ satisfies the following properties:

$$\overline{ga} = g\overline{a}, \qquad \mathrm{tr}(ga) = \mathrm{tr}(a), \qquad n(ga) = n(a), \qquad q(ga, ga') = q(a, a'), \tag{9}$$

for all $g \in \mathrm{Aut}(\mathbf{O})$ and $a, a' \in \mathbf{O}$ (see, for example, equation (2.2) in [1]). Consequently, the group $\mathrm{Aut}(\mathbf{O})$ also acts naturally on the subspaces $\mathbf{O}_0$ and $\mathbf{O}_{\#}$.

2.3 Canonical octonions

A minimal set of representatives for the $\mathrm{Aut}(\mathbf{O})$-orbits in $\mathbf{O}$ was described in Proposition 3.3 of [27] in the case where the base field is algebraically closed. We extend this result to an arbitrary field in the following proposition. The elements appearing in Proposition 2.2 will be referred to as *canonical octonions*.

Proposition 2.2. *A minimal set of representatives for the* $\mathrm{Aut}(\mathbf{O})$*-orbits in* $\mathbf{O}$ *consists of the following elements:*

1. $\alpha\, 1_{\mathbf{O}}$,
2. $\begin{pmatrix} \alpha & (\beta,0,0) \\ (1,0,0) & 0 \end{pmatrix} \in \mathbf{O}$,

where $\alpha, \beta \in \mathbb{F}$*. In other words,* $\mathbf{O}$ *is the disjoint union of the following* $\mathrm{Aut}(\mathbf{O})$*-orbits:*

$$\alpha\, 1_{\mathbf{O}} \qquad \text{and} \qquad O(\alpha,\beta) := \mathrm{Aut}(\mathbf{O}) \cdot \begin{pmatrix} \alpha & (\beta,0,0) \\ (1,0,0) & 0 \end{pmatrix},$$

with $\alpha, \beta \in \mathbb{F}$*.*

Proof. By Lemma 3.3 of [1], two elements $a, b \in \mathbf{O}\backslash\mathbb{F}1_{\mathbf{O}}$ belong to the same $\mathrm{Aut}(\mathbf{O})$-orbit if and only if $\mathrm{tr}(a) = \mathrm{tr}(b)$ and $n(a) = n(b)$. The proof is completed by observing that

$$\mathrm{tr}\left(\begin{pmatrix} \alpha & (\beta,0,0) \\ (1,0,0) & 0 \end{pmatrix}\right) = \alpha, \qquad n\left(\begin{pmatrix} \alpha & (\beta,0,0) \\ (1,0,0) & 0 \end{pmatrix}\right) = -\beta.$$

□

The above proof of Proposition 2.2 is non-constructive and relies on Lemma 3.3 of [1]. For the sake of completeness, we present an alternative proof that does not depend on Lemma 3.3 of [1] and that, for every $a \in \mathbf{O}$, explicitly constructs an element $g \in \mathrm{Aut}(\mathbf{O})$ such that ga is in canonical form. This constructive approach may be useful for practical computations involving octonions.

Constructive proof of Proposition 2.2. In this proof, we use the symbol $*$ to denote an arbitrary element of $\mathbb{F}$. Let $a = \begin{pmatrix} \alpha_1 & \mathbf{u} \\ \mathbf{v} & \alpha_8 \end{pmatrix}$. Then one of the following two cases occurs.

1. Assume that $\mathbf{u} = \mathbf{v} = \mathbf{0}$. If $\alpha_1 = \alpha_8$, then $a \in \mathbb{F}1_{\mathbf{O}}$ is canonical.

Now assume that $\alpha_1 \neq \alpha_8$. By acting on a with $\delta_2(1,0,0)$, we may assume that $a = \begin{pmatrix} \alpha_1 & \mathbf{0} \\ (\alpha_8 - \alpha_1, 0, 0) & \alpha_8 \end{pmatrix}$. Applying part (b) of Lemma 3.2 from [27], we may further assume that $a = \begin{pmatrix} \alpha_1 & \mathbf{0} \\ (1,0,0) & \alpha_8 \end{pmatrix}$.

If $\alpha_8 = 0$, then a is canonical. Otherwise, acting on a with $\delta_1(-\alpha_8, 0, 0)$, we obtain the canonical octonion $\begin{pmatrix} \alpha_1 + \alpha_8 & (-\alpha_1\alpha_8, 0, 0) \\ (1, 0, 0) & 0 \end{pmatrix}$.

2. Assume that $\mathbf{u}$ or $\mathbf{v}$ is non-zero. Applying the automorphism $\hbar$, we may assume that $\mathbf{v}$ is non-zero. Using part (a) of Lemma 3.2 from [27] together with the automorphism $\hbar$, we reduce to the case $\mathbf{v} = (1, 0, 0)$. Acting by $\delta_1(-\alpha_8, 0, 0)$, we then obtain the octonion

$$\begin{pmatrix} \alpha_1 + \alpha_8 & \mathbf{u}' \\ (1, *, *) & 0 \end{pmatrix}.$$

As above, similarly to part (a) of Lemma 3.2 from [27], we further reduce to the octonion

$$\begin{pmatrix} \alpha & \mathbf{u}'' \\ (1, 0, 0) & 0 \end{pmatrix},$$

where $\alpha = \alpha_1 + \alpha_8$ and $\mathbf{u}''$ is one of the following two vectors:

- $\mathbf{u}'' = (*, 0, 0)$. In this case, we have obtained a canonical octonion.
- $\mathbf{u}'' = (0, 1, 0)$. If $\alpha \neq 0$, we apply $\delta_1\big(0, -\frac{1}{\alpha}, 0\big)$ to obtain the canonical octonion $\begin{pmatrix} \alpha & \mathbf{0} \\ (1, 0, 0) & 0 \end{pmatrix}$. If $\alpha = 0$, we apply $\delta_2(0, 0, 1)$ to obtain the canonical octonion $\begin{pmatrix} 0 & \mathbf{0} \\ (1, 0, 0) & 0 \end{pmatrix}$,

and the proof is finished. $\square$

3 Polynomial equations

Recursively, we define the (commutative and associative) Generalized Fibonacci polynomials $p_n = p_n(y, z) \in \mathbb{F}[y, z]$ for all $n \geqslant -1$ by

$$p_{-1} = 0, \qquad p_0 = 1, \qquad p_{k+1} = yp_k + zp_{k-1} \quad \text{for all } k \geqslant 0.$$

In particular,

$$\begin{array}{rcl} p_1 & = & y, \\ p_2 & = & y^2 + z, \\ p_3 & = & y^3 + 2yz, \\ p_4 & = & y^4 + 3y^2z + z^2, \\ p_5 & = & y^5 + 4y^3z + 3yz^2, \\ p_6 & = & y^6 + 5y^4z + 6y^2z^2 + z^3. \end{array}$$

Let $f(y) = \alpha_n y^n + \cdots + \alpha_1 y \in \mathbb{F}[y]$ be a non-zero polynomial without constant term, where $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, $\alpha_n \neq 0$, and $n \geqslant 1$. We define the polynomials $\widehat{f}(y, z)$ and $\check{f}(y, z)$ in $\mathbb{F}[y, z]$ by

$$\begin{array}{rcl} \widehat{f}(y, z) & = & \alpha_n p_{n-1}(y, z) + \cdots + \alpha_2 p_1(y, z) + \alpha_1 p_0(y, z), \\ \check{f}(y, z) & = & \alpha_n p_{n-2}(y, z) + \cdots + \alpha_2 p_0(y, z) + \alpha_1 p_{-1}(y, z). \end{array}$$

For each $x \in \mathbf{O}$, we naturally define the substitution

$$f(x) = \alpha_n x^n + \cdots + \alpha_1 x \in \mathbf{O}.$$

Proposition 3.1. *Given $a \in \mathbf{O}$, we denote $\operatorname{tr}(a) = \alpha$ and $n(a) = -\beta$, where $\alpha, \beta \in \mathbb{F}$.*

1. *For $n \geqslant 1$, we have*
$$a^n = p_{n-1}(\alpha,\beta)\, a + p_{n-2}(\alpha,\beta)\,\beta 1_{\mathbf{O}}. \tag{10}$$

2. *For a non-zero polynomial $f(y) \in \mathbb{F}[y]$ without constant term, we have*
$$f(a) = \widehat{f}(\alpha,\beta)\, a + \check{f}(\alpha,\beta)\,\beta 1_{\mathbf{O}}. \tag{11}$$

Proof. **1.** For brevity, set $q_n = p_n(\alpha,\beta)$. The quadratic equation (4) can be rewritten in the form
$$a^2 = \alpha a + \beta 1_{\mathbf{O}}. \tag{12}$$

We prove formula (10) by induction on $n \geqslant 1$. The case $n = 1$ is trivial.

Assume that (10) holds for $n = k$. Then, by the induction hypothesis and (12), we obtain
$$a^{k+1} = q_{k-1}a^2 + q_{k-2}\beta a = \big(q_{k-1}\alpha + q_{k-2}\beta\big)a + \beta q_{k-1} 1_{\mathbf{O}}.$$
By the recursive definition of $p_n(y,z)$, this proves part 1.

2. Let $f(y) = \sum_{k=1}^{n} \alpha_k y^k$, where $\alpha_k \in \mathbb{F}$. Applying part 1, we obtain

$$\begin{aligned} f(a) &= \sum_{k=1}^{n} \alpha_k p_{k-1}(\alpha,\beta)\, a + \sum_{k=1}^{n} \alpha_k p_{k-2}(\alpha,\beta)\,\beta 1_{\mathbf{O}} \\ &= \widehat{f}(\alpha,\beta)\, a + \check{f}(\alpha,\beta)\,\beta 1_{\mathbf{O}}, \end{aligned}$$

which proves (11). ∎

Proposition 3.1 implies the following remark.

Remark 3.2. Let $a = \begin{pmatrix} \alpha & (\beta,0,0) \\ (1,0,0) & 0 \end{pmatrix} \in \mathbf{O}$ for some $\alpha, \beta \in \mathbb{F}$, and let $n \geqslant 2$. Then

$$a^n = \begin{pmatrix} p_n(\alpha,\beta) & (\beta\, p_{n-1}(\alpha,\beta),0,0) \\ (p_{n-1}(\alpha,\beta),0,0) & \beta\, p_{n-2}(\alpha,\beta) \end{pmatrix}.$$

Recall that $O(\alpha,\beta)$ is defined in Proposition 2.2 as the $\operatorname{Aut}(\mathbf{O})$-orbit of the canonical octonion $\begin{pmatrix} \alpha & (\beta,0,0) \\ (1,0,0) & 0 \end{pmatrix}$.

Theorem 3.3. *Assume that $f(y) \in \mathbb{F}[y]$ is a non-zero polynomial without constant term, and let $c \in \mathbf{O}$. Let $X \subseteq \mathbf{O}$ be the set of all solutions of the equation $f(x) = c$, where $x \in \mathbf{O}$ is a variable.*

1. *Assume* $c = \gamma 1_{\mathbf{O}}$ *for some* $\gamma \in \mathbb{F}$. *Then*

$$X = \left\{\nu 1_{\mathbf{O}} \,\middle|\, \nu \in \mathbb{F} \text{ satisfies } f(\nu) = \gamma\right\} \cup \bigcup_{\substack{\lambda,\mu\in\mathbb{F} \\ \widehat{f}(\lambda,\mu)=0 \\ \mu\,\check{f}(\lambda,\mu)=\gamma}} O(\lambda,\mu).$$

2. *Assume* $c \notin \mathbb{F}1_{\mathbf{O}}$. *Then* $x \in X$ *if and only if there exist* $\lambda, \mu \in \mathbb{F}$ *satisfying*

$$\left\{\begin{array}{rcl} \lambda\,\widehat{f}(\lambda,\mu) + 2\mu\,\check{f}(\lambda,\mu) & = & \mathrm{tr}(c), \\ -\mu\,\widehat{f}(\lambda,\mu)^2 + \lambda\mu\,\widehat{f}(\lambda,\mu)\check{f}(\lambda,\mu) + \mu^2\,\check{f}(\lambda,\mu)^2 & = & n(c), \\ \widehat{f}(\lambda,\mu) & \neq & 0, \end{array}\right. \tag{13}$$

such that

$$x = \frac{1}{\widehat{f}(\lambda,\mu)}\Big(c - \mu\,\check{f}(\lambda,\mu)1_{\mathbf{O}}\Big). \tag{14}$$

In this case, we have $\mathrm{tr}(x) = \lambda$ *and* $n(x) = -\mu$.

Proof. **1.** Let $c = \gamma 1_{\mathbf{O}}$ for some $\gamma \in \mathbb{F}$, and consider an arbitrary $x \in \mathbf{O}$. By Proposition 2.2, there exists $g \in \mathrm{Aut}(\mathbf{O})$ such that gx is canonical. Note that the equality $f(x) = c$ is equivalent to

$$f(gx) = \gamma 1_{\mathbf{O}}. \tag{15}$$

One of the following possibilities occurs:

(a) $gx = \nu 1_{\mathbf{O}}$ for some $\nu \in \mathbb{F}$. In this case, equality (15) is equivalent to $f(\nu) = \gamma$. Hence,

$$x = g^{-1}(\nu 1_{\mathbf{O}}) = \nu 1_{\mathbf{O}}.$$

(b) $gx = \begin{pmatrix} \lambda & (\mu,0,0) \\ (1,0,0) & 0 \end{pmatrix}$ for some $\lambda, \mu \in \mathbb{F}$. By part 2 of Proposition 3.1, equality (15) is equivalent to

$$\left\{\begin{array}{rcl} \widehat{f}(\lambda,\mu) & = & 0, \\ \mu\,\check{f}(\lambda,\mu) & = & \gamma. \end{array}\right.$$

Since $x \in O(\lambda,\mu)$, we have that $x \in X$ if and only if $O(\lambda,\mu) \subseteq X$.

The claim is thus proven.

2. Assume $c \notin \mathbb{F}1_{\mathbf{O}}$.

(a) Assume that $x \in X$. By part 2 of Proposition 3.1, the equality $f(x) = c$ is equivalent to

$$\widehat{f}(\lambda,\mu)\,x + \mu\,\check{f}(\lambda,\mu)\,1_{\mathbf{O}} = c, \tag{16}$$

where $\lambda = \mathrm{tr}(x)$ and $\mu = -n(x)$. If $\widehat{f}(\lambda, \mu) = 0$, then equality (16) would imply $c \in \mathbb{F}1_{\mathbf{O}}$, a contradiction. Hence, x is given by equality (14).

Applying the trace and norm to equality (16) and using the linearity of the trace together with formula (3), we obtain that system (13) holds.

(b) Conversely, assume that $x \in \mathbf{O}$ is defined by equality (14) for some $\lambda, \mu \in \mathbb{F}$ satisfying system (13). System (13) implies

$$\mathrm{tr}(x) = \frac{1}{\widehat{f}(\lambda, \mu)}\Big(\mathrm{tr}(c) - 2\mu\,\check{f}(\lambda, \mu)\Big) = \lambda.$$

Similarly, formula (3) together with system (13) gives

$$\begin{aligned} n(x) &= \frac{1}{\widehat{f}(\lambda, \mu)^2}\Big(n(c) + \mu^2\check{f}(\lambda, \mu)^2 - \mu\ \mathrm{tr}(c)\check{f}(\lambda, \mu)\Big) \\ &= \frac{1}{\widehat{f}(\lambda, \mu)^2}\Big(n(c) - \lambda\mu\,\widehat{f}(\lambda, \mu)\check{f}(\lambda, \mu) - \mu^2\check{f}(\lambda, \mu)^2\Big) = -\mu. \end{aligned}$$

Therefore, by part 2 of Proposition 3.1 and equality (14),

$$f(x) = \widehat{f}(\lambda, \mu)\,x + \mu\,\check{f}(\lambda, \mu)\,1_{\mathbf{O}} = c.$$

Hence, x belongs to X. ☐

The following corollary is a straightforward consequence of Theorem 3.3. In its formulation and hereafter, we use the standard notation $\mathbb{F}^{\times}$ to denote the set of all non-zero elements of $\mathbb{F}$.

Corollary 3.4. *Assume that $f(y) \in \mathbb{F}[y]$ is a non-zero polynomial without constant term, and let $c \in \mathbf{O}\backslash\mathbb{F}1_{\mathbf{O}}$. Let $X \subseteq \mathbf{O}$ be the set of all solutions of the equation $f(x) = c$, where $x \in \mathbf{O}$ is a variable.*

1. *Assume* $\mathrm{char}\,\mathbb{F} \neq 2$. *Then $x \in X$ if and only if there exist $\lambda, \mu \in \mathbb{F}$ satisfying*

$$\begin{cases} (\lambda^2 + 4\mu)\widehat{f}(\lambda, \mu)^2 &= \mathrm{tr}(c)^2 - 4\,n(c), \\ \lambda\widehat{f}(\lambda, \mu) + 2\mu\check{f}(\lambda, \mu) &= \mathrm{tr}(c), \\ \widehat{f}(\lambda, \mu) &\neq 0, \end{cases} \tag{17}$$

   such that

$$x = \frac{1}{\widehat{f}(\lambda, \mu)}\Big(c - \frac{\mathrm{tr}(c)}{2}1_{\mathbf{O}}\Big) + \frac{\lambda}{2}\,1_{\mathbf{O}}. \tag{18}$$

2. *Assume* $\mathrm{char}\,\mathbb{F} = 2$.

(a) If $\mathrm{tr}(c) \neq 0$*, then* $x \in X$ *if and only if there exist* $\lambda \in \mathbb{F}^{\times}$ *and* $\mu \in \mathbb{F}$ *satisfying*

$$\left\{ \begin{array}{rcl} \mu^2 \check{f}(\lambda,\mu)^2 + \mu\, \mathrm{tr}(c)\, \check{f}(\lambda,\mu) & = & n(c) + \frac{\mu}{\lambda^2}\,\mathrm{tr}(c)^2, \\ \lambda \widehat{f}(\lambda,\mu) & = & \mathrm{tr}(c), \end{array} \right. \tag{19}$$

such that

$$x = \frac{\lambda}{\mathrm{tr}(c)}\Big(c + \mu\, \check{f}(\lambda,\mu) 1_{\mathbf{O}}\Big). \tag{20}$$

(b) If $\mathrm{tr}(c) = 0$*, then* $x \in X$ *if and only if there exists* $\mu \in \mathbb{F}$ *satisfying*

$$\left\{ \begin{array}{rcl} \mu \widehat{f}(0,\mu)^2 + \mu^2 \check{f}(0,\mu)^2 & = & n(c), \\ \widehat{f}(0,\mu) & \neq & 0, \end{array} \right. \tag{21}$$

such that

$$x = \frac{1}{\widehat{f}(0,\mu)}\Big(c + \mu\, \check{f}(0,\mu) 1_{\mathbf{O}}\Big). \tag{22}$$

Note that in each case we have $\mathrm{tr}(x) = \lambda$ *and* $n(x) = -\mu$*, where in case 2(b) we have* $\lambda = 0$*.*

Corollary 3.5. *Assume that* $f(y) \in \mathbb{F}[y]$ *is a non-zero polynomial of degree* $n \geqslant 1$ *without constant term, and let* $c \in \mathbf{O} \backslash \mathbb{F}1_{\mathbf{O}}$*. Denote by* $X \subseteq \mathbf{O}$ *the set of all solutions of the equation* $f(x) = c$*. Then*

$$|X| \leqslant n^2.$$

Proof. Consider the inclusion $\mathbb{F} \subseteq \overline{\mathbb{F}}$, where $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$. Let $X' \subseteq \mathbf{O}(\overline{\mathbb{F}})$ denote the set of all solutions of $f(x) = c$ in $\mathbf{O}(\overline{\mathbb{F}})$.

Since $c \notin \mathbb{F}1_{\mathbf{O}}$, Corollary 3.3 of [25] implies that $|X'| \leqslant n^2$.

Finally, since $\mathbf{O} \subseteq \mathbf{O}(\overline{\mathbb{F}})$, we have $X \subseteq X'$, which proves the claim. □

4 Roots of octonions

Definition 4.1.

- We denote by $\mathbb{F}^2$ the set $\{\alpha^2 \mid \alpha \in \mathbb{F}\}$.
- Let $\beta \in \mathbb{F}^2$. We fix an element $\sqrt{\beta}$ from the non-empty set $\{\alpha \in \mathbb{F} \mid \alpha^2 = \beta\}$. Observe that the set of all solutions to the equation $\nu^2 = \beta$, where $\nu \in \mathbb{F}$ is a variable, is $\{\pm\sqrt{\beta}\}$.

Proposition 4.2. *Let* $X \subseteq \mathbf{O}$ *be the set of all solutions of the equation* $x^2 = c$*, where* $x \in \mathbf{O}$ *is a variable and* $c \in \mathbf{O}$*.*

1. *Assume* $c = \gamma 1_{\mathbf{O}}$ *for some* $\gamma \in \mathbb{F}$*. Then*

$$X = \pm\sqrt{\gamma} 1_{\mathbf{O}} \ \cup\ O(0,\gamma).$$

2. Assume $c \notin \mathbb{F}1_{\mathbf{O}}$ *and* $\operatorname{char}\mathbb{F} \neq 2$.

(a) If $n(c) \in \mathbb{F}^2$, *denote*

$$\alpha = \operatorname{tr}(c) + 2\sqrt{n(c)}, \qquad \beta = \operatorname{tr}(c) - 2\sqrt{n(c)}.$$

- *If* $\alpha, \beta \in (\mathbb{F}^\times)^2$, *then*

$$X = \Big\{ \pm \frac{1}{\sqrt{\alpha}}\big(c + \sqrt{n(c)}1_{\mathbf{O}}\big),\ \pm\frac{1}{\sqrt{\beta}}\big(c - \sqrt{n(c)}1_{\mathbf{O}}\big)\Big\}.$$

- *If* $\alpha \in (\mathbb{F}^\times)^2$ *and* $\beta \notin (\mathbb{F}^\times)^2$, *then*

$$X = \Big\{ \pm \frac{1}{\sqrt{\alpha}}\big(c + \sqrt{n(c)}1_{\mathbf{O}}\big)\Big\}.$$

- *If* $\alpha \notin (\mathbb{F}^\times)^2$ *and* $\beta \in (\mathbb{F}^\times)^2$, *then*

$$X = \Big\{ \pm \frac{1}{\sqrt{\beta}}\big(c - \sqrt{n(c)}1_{\mathbf{O}}\big)\Big\}.$$

- *If* $\alpha, \beta \notin (\mathbb{F}^\times)^2$, *then* $X = \varnothing$.

(b) If $n(c) \notin \mathbb{F}^2$, *then* $X = \varnothing$.

3. Assume $c \notin \mathbb{F}1_{\mathbf{O}}$ *and* $\operatorname{char}\mathbb{F} = 2$.

(a) If $\operatorname{tr}(c) \neq 0$, *then*

$$X = \begin{cases} \frac{1}{\sqrt{\operatorname{tr}(c)}}\Big(c + \sqrt{n(c)}1_{\mathbf{O}}\Big), & \text{if } \operatorname{tr}(c), n(c) \in \mathbb{F}^2, \\ \varnothing, & \text{otherwise.} \end{cases}$$

(b) If $\operatorname{tr}(c) = 0$, *then* $X = \varnothing$.

Proof. Note that for $f(y) = y^2 \in \mathbb{F}[y]$ and every $\lambda, \mu \in \mathbb{F}$, we have

$$\widehat{f}(\lambda, \mu) = p_1(\lambda, \mu) = \lambda, \qquad \check{f}(\lambda, \mu) = p_0(\lambda, \mu) = 1.$$

1. Assume $c = \gamma 1_{\mathbf{O}}$ for some $\gamma \in \mathbb{F}$. Then the required statement follows immediately from Theorem 3.3.

2. Assume $c \notin \mathbb{F}1_{\mathbf{O}}$ and $\operatorname{char}\mathbb{F} \neq 2$. By Corollary 3.4, $x \in X$ if and only if

$$x = \frac{1}{\lambda}\Big(c - \frac{\operatorname{tr}(c)}{2}1_{\mathbf{O}}\Big) + \frac{\lambda}{2}1_{\mathbf{O}}$$

for some $\lambda \in \mathbb{F}^{\times}$ and $\mu \in \mathbb{F}$ satisfying

$$\left\{ \begin{array}{rcl} (\lambda^2 + 4\mu)\,\lambda^2 & = & \mathrm{tr}(c)^2 - 4\,n(c), \\ \lambda^2 + 2\mu & = & \mathrm{tr}(c). \end{array} \right. \tag{23}$$

System (23) is equivalent to

$$\left\{ \begin{array}{rcl} (\lambda^2 - \mathrm{tr}(c))^2 & = & 4\,n(c), \\ \mu & = & \frac{1}{2}\big(\,\mathrm{tr}(c) - \lambda^2\big), \end{array} \right.$$

and the required statement follows.

3. Assume $c \notin \mathbb{F}1_{\mathbf{O}}$ and $\mathrm{char}\,\mathbb{F} = 2$.
(a) Let $\mathrm{tr}(c) \neq 0$. By Corollary 3.4, $x \in X$ if and only if

$$x = \frac{\lambda}{\mathrm{tr}(c)}\Big(c + \mu 1_{\mathbf{O}}\Big)$$

for some $\lambda \in \mathbb{F}^{\times}$ and $\mu \in \mathbb{F}$ satisfying

$$\left\{ \begin{array}{rcl} \mu^2 + \mu\;\mathrm{tr}(c) & = & n(c) + \frac{\mu}{\lambda^2}\;\mathrm{tr}(c)^2, \\ \lambda^2 & = & \mathrm{tr}(c). \end{array} \right. \tag{24}$$

System (24) is equivalent to

$$\lambda^2 = \mathrm{tr}(c), \qquad \mu^2 = n(c),$$

and the required statement follows.

(b) Let $\mathrm{tr}(c) = 0$. Assume that X is non-empty. Then, for any $x \in X$, Corollary 3.4 implies the existence of $\mu \in \mathbb{F}$ such that $\widehat{f}(0,\mu) \neq 0$. But $\widehat{f}(0,\mu) = 0$, which is a contradiction. Hence $X = \varnothing$. □

Proposition 4.2 implies the following corollary.

Corollary 4.3. *Assume $\mathbb{F} = \mathbb{R}$ and let $c \in \mathbf{O}$. Then the equation $x^2 = c$ has no solutions in $\mathbf{O}$ if and only if $c \notin \mathbb{F}1_{\mathbf{O}}$ and one of the following conditions holds:*

- $0 \leqslant 4n(c) \leqslant \mathrm{tr}(c)^2$ *and* $\mathrm{tr}(c) \leqslant 0$*;*
- $n(c) < 0$.

Proposition 4.4. *Assume $\mathbb{F} = \mathbb{R}$ and let $c \in \mathbf{O}$. Let $X \subseteq \mathbf{O}$ be the set of all solutions of the equation $x^3 = c$, where $x \in \mathbf{O}$ is a variable.*

1. *Assume $c = \gamma 1_{\mathbf{O}}$ for some $\gamma \in \mathbb{R}$. Then*

$$X = \big\{\sqrt[3]{\gamma}\,1_{\mathbf{O}}\big\} \,\cup\, O\big(-\sqrt[3]{\gamma}, -\sqrt[3]{\gamma^2}\big).$$

2. *Assume* $c \notin \mathbb{R}1_{\mathbf{O}}$ *and* $\operatorname{tr}(c) \neq 0$. *Then* $x \in X$ *if and only if there exists* $\lambda \in \mathbb{R}$ *satisfying*

$$\begin{cases} \left(2\lambda^3 + \operatorname{tr}(c)\right)^2\left(\lambda^3 - 4\operatorname{tr}(c)\right) &= 27\lambda^3\left(4n(c) - \operatorname{tr}(c)^2\right), \\ 2\lambda^3 + \operatorname{tr}(c) &\neq 0, \end{cases} \tag{25}$$

such that

$$x = \frac{3\lambda}{2\lambda^3 + \operatorname{tr}(c)}\left(c - \frac{\operatorname{tr}(c)}{2}1_{\mathbf{O}}\right) + \frac{\lambda}{2}1_{\mathbf{O}}. \tag{26}$$

3. *Assume* $c \notin \mathbb{R}1_{\mathbf{O}}$ *and* $\operatorname{tr}(c) = 0$.

   (a) *If* $n(c) > 0$, *then* $X = \{x_1, x_2\}$.

   (b) *If* $n(c) = 0$, *then* $X = \varnothing$.

   (c) *If* $n(c) < 0$, *then* $X = \{x_1\}$,

   where

$$x_1 = -\frac{c}{\sqrt[3]{n(c)}}, \qquad x_2 = \frac{1}{2}\left(\frac{c}{\sqrt[3]{n(c)}} + \sqrt{3}\,\sqrt[6]{n(c)}\,1_{\mathbf{O}}\right).$$

Proof. Note that for $f(y) = y^3 \in \mathbb{R}[y]$ and every $\lambda, \mu \in \mathbb{R}$, we have

$$\widehat{f}(\lambda, \mu) = p_2(\lambda, \mu) = \lambda^2 + \mu, \qquad \check{f}(\lambda, \mu) = p_1(\lambda, \mu) = \lambda.$$

I. Assume $c = \gamma 1_{\mathbf{O}}$ for some $\gamma \in \mathbb{R}$. By Theorem 3.3,

$$X = \left\{\sqrt[3]{\gamma}\,1_{\mathbf{O}}\right\} \cup \left\{O(\lambda, \mu) \mid \lambda, \mu \in \mathbb{R} \text{ satisfy } \lambda^2 + \mu = 0 \text{ and } \lambda\mu = \gamma\right\}.$$

Considering separately the cases $\gamma = 0$ and $\gamma \neq 0$, we obtain the statement of part 1.

II. Assume $c \notin \mathbb{R}1_{\mathbf{O}}$. By Corollary 3.4, an element $x \in \mathbf{O}$ belongs to X if and only if

$$x = \frac{1}{\lambda^2 + \mu}\left(c - \frac{\operatorname{tr}(c)}{2}1_{\mathbf{O}}\right) + \frac{\lambda}{2}1_{\mathbf{O}}, \tag{27}$$

for some $\lambda, \mu \in \mathbb{R}$ satisfying

$$\begin{cases} (\lambda^2 + 4\mu)(\lambda^2 + \mu)^2 &= \operatorname{tr}(c)^2 - 4n(c), \\ 2\lambda\mu + \lambda(\lambda^2 + \mu) &= \operatorname{tr}(c), \\ \lambda^2 + \mu &\neq 0. \end{cases} \tag{28}$$

The second equation of system (28) is equivalent to

$$3\lambda\mu = \operatorname{tr}(c) - \lambda^3. \tag{29}$$

(a) Assume $\mathrm{tr}(c) \neq 0$. Then $\lambda \neq 0$, since otherwise (29) would imply $\mathrm{tr}(c) = 0$, a contradiction. Hence

$$\mu = \frac{\mathrm{tr}(c) - \lambda^3}{3\lambda}.$$

Substituting this expression into system (28) and equality (27) yields system (25) and equality (26), respectively. Note that the first equation of system (25) implies that $\lambda \neq 0$. This completes the proof of part 2.

(b) Assume $\mathrm{tr}(c) = 0$. Considering separately the cases $\lambda = 0$ and $\lambda \neq 0$, we obtain the statements of part 3. □

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