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Rook placements in G_2 and F_4 and associated coadjoint orbits

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Abstract. Let $\mathfrak{n}$ be a maximal nilpotent subalgebra of a simple complex Lie algebra with root system Φ. A subset D of the set Φ^+ of positive roots is called a rook placement if it consists of roots with pairwise non-positive scalar products. To each rook placement D and each map ξ from D to the set $\mathbb{C}^\times$ of nonzero complex numbers one can naturally assign the coadjoint orbit $\Omega_{D,\xi}$ in the dual space $\mathfrak{n}^*$. By definition, $\Omega_{D,\xi}$ is the orbit of $f_{D,\xi}$, where $f_{D,\xi}$ is the sum of root covectors e_α^* multiplied by $\xi(\alpha)$, $\alpha \in D$ (in fact, almost all coadjoint orbits studied at the moment have such a form, for certain D and ξ). It follows from the results of André that if ξ_1 and ξ_2 are distinct maps from D to $\mathbb{C}^\times$ then Ω_{D,ξ_1} and Ω_{D,ξ_2} do not coincide for classical root systems Φ. We prove that this is true if Φ is of type G_2, or if Φ is of type F_4 and D is orthogonal.

1 Introduction and the main result

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, Φ be the root system of $\mathfrak{g}$, Φ^+ be the set of positive roots corresponding to $\mathfrak{b}$, $\mathfrak{n}$ be the nilradical of $\mathfrak{b}$, $N = \exp(\mathfrak{n})$ be the corresponding nilpotent algebraic group, and $\mathfrak{n}^*$ be the dual space to $\mathfrak{n}$. The group N acts on $\mathfrak{n}$ by the adjoint action; the dual action of N on the space $\mathfrak{n}^*$ is called coadjoint; we will denote the result of this action by $g.\lambda$ for $g \in N$, $\lambda \in \mathfrak{n}^*$. According to the orbit method discovered by A.A. Kirillov in 1962, coadjoint orbits play a key role in representation theory of N (see [10], [11]). We will consider a special class of coadjoint orbits defined below.

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Definition 1.1. A subset D of Φ^+ is called a *rook placement* if $(\alpha, \beta) \leq 0$ for all distinct $\alpha, \beta \in D$, where $(-,-)$ denotes the inner product.

The root vectors e_α, $\alpha \in \Phi^+$ form a basis of $\mathfrak{n}$; we denote by $\{e_\alpha^*, \alpha \in D\}$ the dual basis of $\mathfrak{n}^*$. Given a rook placement D and a map $\xi\colon D \to \mathbb{C}^\times$, we put

$$f_{D,\xi} = \sum_{\alpha \in D} \xi(\alpha) e_\alpha^* \in \mathfrak{n}^*.$$

Definition 1.2. We say that the coadjoint orbit $\Omega_{D,\xi}$ of the linear form $f_{D,\xi}$ is *associated* with the rook placement D and the map ξ.

It turns out that almost all coadjoint orbit studied to the moment are associated with certain D and ξ (see, e.g. [1], [2], [12], [13], [14], [7], [3], [4], [5]). On the other hand, C.A.M. André discovered that, for the case of A_{n-1}, rook placements themselves provide a nice splitting of $\mathfrak{n}^*$ into a disjoint union of N-stable affine subvarieties called basic subvarieties. (We will recall André's results in detail in Section 2, because we will use them for the case of F_4.) By definition, the *basic subvariety* $\mathcal{O}_{D,\xi}$ corresponding to a rook placement D and a map $\xi\colon D \to \mathbb{C}^\times$ is

$$\mathcal{O}_{D,\xi} = \sum_{\alpha \in D} \Omega_{\{\alpha\},\xi_\alpha},$$

where ξ_α is the restriction of ξ to $\{\alpha\}$. For A_{n-1}, $\mathfrak{n}^* = \bigsqcup_{D,\xi} \mathcal{O}_{D,\xi}$ and all $\mathcal{O}_{D,\xi}$'s are affine subvarieties of $\mathfrak{n}^*$ (see [1, Theorem 1]).

Even for B_n, C_n and D_n, the analogous question is still open. Nevertheless, we may formulate the following conjecture for an arbitrary root system. Non-singularity of a rook placement D means that if $\alpha, \beta \in D$ and $\alpha \neq \beta$ then $\alpha - \beta \notin \Phi^+$; for A_{n-1}, all rook placements are automatically non-singular.

Conjecture 1.3. *Each basic subvariety $\mathcal{O}_{D,\xi}$ is an affine subvariety of $\mathfrak{n}^*$, and*

$$\mathfrak{n}^* = \bigsqcup_{D,\xi} \mathcal{O}_{D,\xi},$$

where the union is taken over all non-singular rook placements D and all maps $\xi\colon D \to \mathbb{C}^\times$.

Direct computations show that this conjecture is true for classical root systems of low rank. In the present paper, we check that this conjecture is true for the case of G_2. This is our first main result. In fact, given D and ξ, we present an explicit system of equations describing $\mathcal{O}_{D,\xi}$.

Theorem 1.4. *Let $\Phi = G_2$. Then each basic subvariety $\mathcal{O}_{D,\xi}$ is an affine subvariety of $\mathfrak{n}^*$, and*

$$\mathfrak{n}^* = \bigsqcup_{D,\xi} \mathcal{O}_{D,\xi},$$

where the union is taken over all non-singular rook placements D and all maps $\xi\colon D \to \mathbb{C}^\times$.

It turns out that, for A_{n-1}, if D is a rook placement and ξ_1, ξ_2 are distinct maps from D to $\mathbb{C}^\times$ then the associated orbits Ω_{D,ξ_1} and Ω_{D,ξ_2} do not coincide (it follows immediately from André's theory, since $\Omega_{D,\xi} \subset \mathcal{O}_{D,\xi}$, see Section 2). For other classical root systems this fact can be obtained as a corollary of the case of A_{n-1} (see also [2]). This was used by M.V. Ignatyev and I. Penkov in [8] and [6] for explicit classification of centrally generated primitive ideals in the universal enveloping algebra $U(\mathfrak{n})$ for classical root systems.

In [9], M.V. Ignatyev and A.A. Shevchenko, while classifying centrally generated primitive ideals in $U(\mathfrak{n})$ for exceptional root systems, proved that the analogous is true for certain orthogonal rook placements in F_4 and E_6, E_7, E_8. This allows us to formulate the second conjecture for an arbitrary root system.

Conjecture 1.5. *Let D be a non-singular rook placement and ξ_1, ξ_2 be distinct maps from D to $\mathbb{C}^\times$. Then the associated coadjoint orbits Ω_{D,ξ_1} and Ω_{D,ξ_2} do not coincide.*

Our second main result is that this conjecture is true for F_4 if D is orthogonal (i.e. if all roots from D are pairwise orthogonal).

Theorem 1.6. *Let $\Phi = F_4$, D be an orthogonal non-singular rook placement and ξ_1, ξ_2 be distinct maps from D to $\mathbb{C}^\times$. Then the associated coadjoint orbits Ω_{D,ξ_1} and Ω_{D,ξ_2} do not coincide.*

2 André's theory

In this section, we briefly recall André's results from [1], which will be needed in the sequel. Throughout this section, Φ will be of type A_{n-1}. As usual, we identify the set of positive roots with the following subset of the Euclidean space $\mathbb{R}^n$:

$$A_{n-1}^+ = \{\varepsilon_i - \varepsilon_j, 1 \leq i < j \leq n\},$$

with the standard inner product. Here, $\varepsilon_1, \ldots, \varepsilon_n$ denotes the standard basis of $\mathbb{R}^n$.

In this case, $\mathfrak{n}$ can be identified with the space of strictly upper-triangular $n\times n$ matrices. Given $\alpha = \varepsilon_i - \varepsilon_j \in \Phi^+$, one can pick the (i,j)-th elementary matrix $e_{i,j}$ as a root vector e_α, so that $[e_\alpha, e_\beta] = \pm e_{\alpha+\beta}$ for $\alpha, \beta \in \Phi^+$ (we put $e_{\alpha+\beta} = 0$ if $\alpha+\beta \notin \Phi^+$). We will identify the dual space $\mathfrak{n}^*$ with the space $\mathfrak{n}^-$ of strictly lower-triangular $n\times n$ matrices via the formula $\langle\lambda, x\rangle = \mathrm{tr}(\lambda x)$ for $x \in \mathfrak{n}$, $\lambda \in \mathfrak{n}^-$. The root vectors e_α, $\alpha \in \Phi^+$ form a basis of $\mathfrak{n}$; let $\{e_\alpha^*, \alpha \in \Phi^+\}$ be the dual basis of $\mathfrak{n}^*$ (in fact, $e_{i,j}^* = e_{j,i}$).

The group N is the group of all upper-triangular $n\times n$ matrices with 1's on the diagonal. It acts on its Lie algebra $\mathfrak{n}$ via the adjoint action $\mathrm{Ad}_g(x) = gxg^{-1}$, $g \in N$, $x \in \mathfrak{n}$. The dual action of N on the space $\mathfrak{n}^*$ is called coadjoint; we will denote the result of this action by $g.\lambda$, $g \in N$, $\lambda \in \mathfrak{n}^*$. It is easy to check that, after the identification of $\mathfrak{n}^*$ with $\mathfrak{n}^-$, this action has the form $g.\lambda = (g\lambda g^{-1})_{\mathrm{low}}$. Here, given an $n\times n$ matrix a, we set

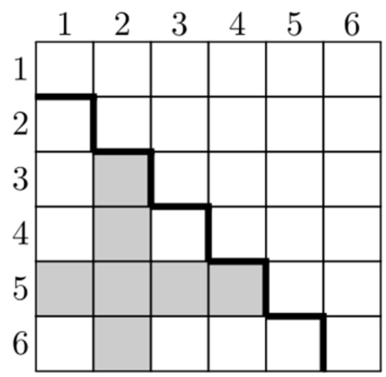
$$(a_{\mathrm{low}})_{i,j} = \begin{cases} a_{i,j}, & \text{if } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.1. Pick a number k from 1 to n. We call the sets

$$\mathcal{R}_k = \{\varepsilon_j - \varepsilon_k, 1 \leq j < k\}, \mathcal{C}_k = \{\varepsilon_k - \varepsilon_i, k < i \leq n\}$$

the *k-th row* and the *k-th column* of Φ^+, respectively. We say that the number i (respectively, the number j) is the *row* (respectively, the *column*) of a root $\alpha = \varepsilon_i - \varepsilon_j$.

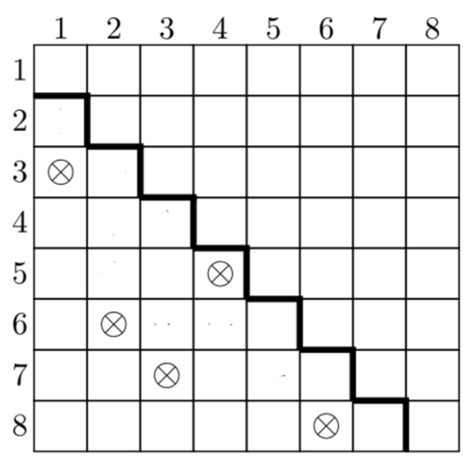
Example 2.2. Let $n = 6$. On the picture below boxes from $\mathcal{R}_5 \cup \mathcal{C}_2$ are grey. Here we identify a root $\varepsilon_i - \varepsilon_j \in \Phi^+$ with the box (j, i).



To each rook placement $D \subset \Phi^+$ and each map $\xi \colon D \to \mathbb{C}^\times$, one can assign the linear form

$$f_{D,\xi} = \sum_{\alpha \in D} \xi(\alpha) e^*_\alpha \in \mathfrak{n}^*.$$

Example 2.3. Let $n = 8$, $D = \{\varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_6, \varepsilon_3 - \varepsilon_7, \varepsilon_4 - \varepsilon_5, \varepsilon_6 - \varepsilon_8\}$. On the picture below we schematically draw the linear form $f_{D,\xi}$ putting symbols $\otimes$ in the boxes corresponding to the roots from D.



It follows immediately from the definition of a rook placement that

$$|D \cap \mathcal{R}_k| \leq 1, |D \cap \mathcal{C}_k| \leq 1$$

for all k. This explains the term "rook placement": if we identify symbols $\otimes$ from $f_{D,\xi}$ with rooks on the lower-triangular chessboard, then these rooks do not hit each other.

Now, given a root $\alpha \in D$, we denote by ξ_α the restriction of the map ξ to the subset $\{\alpha\}$, and put

$$\mathcal{O}_{D,\xi} = \sum_{\alpha \in D} \mathcal{O}_{\{\alpha\},\xi_\alpha}.$$

Clearly, $\Omega_{D,\xi} \subset \mathcal{O}_{D,\xi}$.

Definition 2.4. The set $\mathcal{O}_{D,\xi}$ is called a *basic subvariety* of $\mathfrak{n}^*$ corresponding to the rook placement D and the map ξ.

Accordingly to [1, Theorem 1], $\mathfrak{n}^*$ is a disjoint unions of basic subvarieties:

$$\mathfrak{n}^* = \bigsqcup_{D,\xi} \mathcal{O}_{D,\xi},$$

where the union is taken over all rook placements in A_{n-1}^+ and all maps $\xi\colon D \to \mathbb{C}^\times$. Formally, André considered the case of finite ground field, but his proofs are valid over $\mathbb{C}$, too. Furthermore, each basic subvariety $\mathcal{O}_{D,\xi}$ is in fact an affine subvariety of $\mathfrak{n}^*$, and André presented an explicit set of defining equations for it. To describe this set, we need some more notation.

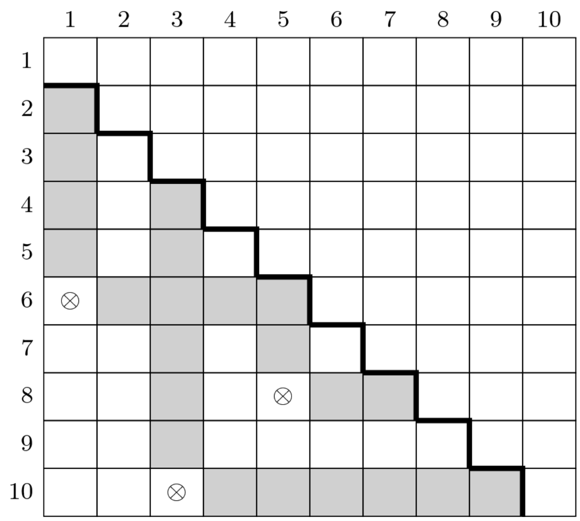
Definition 2.5. A root $\alpha \in \Phi^+$ is called β*-singular* for a root $\beta \in \Phi^+$ if $\beta - \alpha \in \Phi^+$. The set of all β-singular roots is denoted by $S(\beta)$.

Let D be a rook placement and $\xi\colon D \to \mathbb{C}^\times$ be a map. We denote

$$S(D) = \bigcup_{\beta \in D} S(\beta)$$

and $R(D) = \Phi^+ \setminus S(D)$. Obviously, $D \subset R(D)$. Roots from $S(D)$ (respectively, from $R(D)$) are called D*-singular* (respectively, D*-regular*).

Example 2.6. Let $n = 10$, $D = \{\varepsilon_1 - \varepsilon_6, \varepsilon_3 - \varepsilon_{10}, \varepsilon_5 - \varepsilon_8\}$. On the picture below, boxes corresponding to the roots from D are filled by $\otimes$'s, as above, while the boxes corresponding by the D-singular roots are marked gray.



It turns out that to each D-regular root α one can assign the defining equation of $\mathcal{O}_{D,\xi}$. Namely, there is a natural partial order on Φ^+: we write $\alpha \leq \beta$ if $\beta-\alpha$ is a sum of positive roots. Evidently, $\varepsilon_i - \varepsilon_j \leq \varepsilon_r - \varepsilon_s$ if and only if $s \leq j$ and $r \geq i$ (in other words, on our pictures $\varepsilon_r - \varepsilon_s$ is located non-strictly to the South-West from $\varepsilon_i - \varepsilon_j$). Given $\alpha \in R(D)$, we set $D(\alpha) = \{\alpha\} \cup \{\beta \in D \mid \beta \geq \alpha\}$. Now, let $R_D(\alpha)$ (respectively $C_D(\alpha)$) be the set of all rows (respectively, of all columns) of the roots from $D(\alpha)$. Finally, for a matrix $\lambda \in \mathfrak{n}^*$, we denote by $\Delta_\alpha^D(\lambda)$ the minor of the matrix λ with the set of rows $R_D(\alpha)$ and the set of columns $C_D(\alpha)$. We assume that the numbers of rows and columns are taken in the increasing order.

For instance, in the previous example, if $\alpha = \varepsilon_4 - \varepsilon_5$ then

$$D(\alpha) = \{\varepsilon_1 - \varepsilon_6, \varepsilon_3 - \varepsilon_{10}, \varepsilon_4 - \varepsilon_5\},$$

hence $R_D(\alpha) = \{5, 6, 10\}$, $C_D(\alpha) = \{1, 3, 4\}$ and

$$\Delta_\alpha^D(\lambda) = \begin{vmatrix} \lambda_{5,1} & \lambda_{5,3} & \lambda_{5,4} \\ \lambda_{6,1} & \lambda_{6,3} & \lambda_{6,4} \\ \lambda_{10,1} & \lambda_{10,3} & \lambda_{10,4} \end{vmatrix}.$$

On the other hand, for $\alpha = \varepsilon_5 - \varepsilon_8 \in D$, one has $D(\alpha) = \{\varepsilon_3 - \varepsilon_{10}, \varepsilon_5 - \varepsilon_8\}$, $R_D(\alpha) = \{8, 10\}$, $C_D(\alpha) = \{3, 5\}$, and, consequently,

$$\Delta_\alpha^D(\lambda) = \begin{vmatrix} \lambda_{8,3} & \lambda_{8,5} \\ \lambda_{10,3} & \lambda_{10,5} \end{vmatrix}.$$

Thanks to [1, Proposition 2], a matrix $\lambda \in \mathfrak{n}^*$ belongs to $\mathcal{O}_{D,\xi}$ if and only if

$$\Delta_\alpha^D(\lambda) = \Delta_\alpha^D(f_{D,\xi}) \text{ for all } \lambda \in R(D).$$

Precisely, $\Delta_\alpha^D(\lambda) = 0$ for all $\alpha \in R(D) \setminus D$ and $\Delta_\alpha^D(\lambda) = \pm \prod_{\beta \in D(\alpha)} \xi(\beta)$ for $\alpha \in D$. It follows immediately that

$$\dim \mathcal{O}_{D,\xi} = |S(D)|.$$

Remark 2.7. Actually, André's proof of the fact that each $\lambda \in \mathfrak{n}^*$ belongs to a certain basic subvariety $\mathcal{O}_{D,\xi}$ is very straightforward. Namely, there is a total order $\leq_t$ on Φ^+ refining the partial order $\leq$ defined above. By definition, $\varepsilon_i - \varepsilon_j <_t \varepsilon_r - \varepsilon_s$ if $s < j$ or $s = j$, $i < r$. Now, given $\lambda \in \mathfrak{n}^*$, we inductively construct $\mathcal{O}_{D,\xi}$ containing λ. If $\lambda = 0$, then $D = \varnothing$ and ξ is the unique empty map from $\varnothing$ to $\mathbb{C}^\times$. If $\lambda \neq 0$ then we find the smallest (with respect to $\leq_t$) root α_1 from Φ^+ such that $\lambda(e_{\alpha_1}) \neq 0$, and put $D = \{\alpha_1\}$, $\xi(\alpha_1) = \lambda(\alpha_1)$. If $\lambda \in \mathcal{O}_{D,\xi}$, then we are done. Otherwise, let α_2 be the smallest root such that $\Delta^D_{\alpha_2}(\lambda) \neq \Delta^D_{\alpha_2}(f_{D,\xi})$. Then we add α_2 to D and define $\xi(\alpha_2)$ in the obvious way. Now, one can repeat this procedure to obtain the required basic subvariety $\mathcal{O}_{D,\xi}$.

3 Case $\Phi = G_2$

In this section, we prove our first main result, Theorem 1.4. First, we briefly recall some basic facts about the simple Lie algebra $\mathfrak{g}$ of type G_2 and its maximal nilpotent subalgebra $\mathfrak{n}$. By definition, the root system $\Phi = G_2$ has the form $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$, $\Phi^- = -\Phi^+$, and α, β are vectors from $\mathbb{R}^2$ such that $||\alpha||^2 = 1, ||\beta||^2 = 3$ and the angle between α and β equals $5\pi/6$. There is a Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{n}$ has a basis consisting of the root vectors e_γ, $\gamma \in \Phi^+$.

It is well known that there exists nonzero scalars c_i, $1 \leq i \leq 5$, such that

$$\begin{aligned}
&[e_\alpha, e_\beta] = c_1 \cdot e_{\alpha+\beta},\\
&[e_\alpha, e_{\alpha+\beta}] = c_2 \cdot e_{2\alpha+\beta}\\
&[e_\alpha, e_{2\alpha+\beta}] = c_3 \cdot e_{3\alpha+\beta}\\
&[e_{3\alpha+\beta}, e_\beta] = c_4 \cdot e_{3\alpha+2\beta}\\
&[e_{\alpha+\beta}, e_{2\alpha+\beta}] = c_5 \cdot e_{3\alpha+2\beta}.
\end{aligned}$$

In fact, one can choose the root vectors so that $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $c_4 = 1$, $c_5 = 3$, but we will not use these explicit values in the sequel. One can immediately check that $c_1c_5 = c_3c_4$ for an arbitrary choice of the root vectors.

Recall the definition of the group $N = \exp(\mathfrak{n})$ and the coadjoint action of N on the dual space $\mathfrak{n}^*$. It is straightforward to check that this action has the form

$$\begin{aligned}
(\exp(x).\lambda)(y) &= \lambda(\exp(-\mathrm{ad}_x)(y))\\
&= \lambda(y) - \lambda([x,y]) + \frac{1}{2!}\lambda([x,[x,y]]) - \ldots,
\end{aligned}$$

for $x, y \in \mathfrak{n}, \lambda \in \mathfrak{n}^*$.

Now, let D be a non-singular rook placement in Φ^+. Recall that non-singularity means that $\gamma \notin S(\delta)$ for all distinct $\gamma, \delta \in D$, where $S(\delta)$ denotes the set of δ-singular roots in Φ^+. Fix a map $\xi \colon D \to \mathbb{C}^\times$, and recall that, by definition, $\Omega_{D,\xi}$ is the coadjoint orbit of the linear form $f_{D,\xi}$. It follows immediately that if γ is a maximal (with respect to the partial order $\leq$ on Φ^+) among all roots from D then $\lambda(e_\gamma) = f_{D,\xi}(e_\gamma) = \xi(\gamma)$ for all $\lambda \in \Omega_{D,\xi}$. Similarly, $\lambda(e_\gamma) = 0$ for all $\lambda \in \Omega_{D,\xi}$, if there are no $\delta \in D$ such that $\delta \geq \gamma$. Recall also the definition of $\mathcal{O}_{D,\xi}$.

Given $\gamma \in \Phi^+$, we write $\lambda_\gamma = \lambda(e_\gamma)$, so that $\lambda = \sum_{\gamma\in\Phi^+} \lambda_\gamma e_\gamma^*$. We will prove Theorem1.4 as an immediate corollary of the following key proposition:

Proposition 3.1. *Let D be a non-singular rook placement in Φ^+, $\xi\colon D \to \mathbb{C}^\times$ be a map. Pick a linear form $\lambda \in \mathfrak{n}^*$. Then $\lambda \in \mathcal{O}_{D,\xi}$ if and only if λ satisfy the following system of equations.*

	D	System of equations for $\mathcal{O}_{D,\xi}$
1	$\varnothing$	$\lambda_\gamma = 0$ *for all* $\gamma \in \Phi^+$
2	α	$\lambda_\alpha = \xi(\alpha)$, $\lambda_\gamma = 0$ *for* $\gamma \neq \alpha$
3	β	$\lambda_\beta = \xi(\beta)$, $\lambda_\gamma = 0$ *for* $\gamma \neq \beta$
4	$\alpha+\beta$	$\lambda_{\alpha+\beta} = \xi(\alpha+\beta)$, $\lambda_{2\alpha+\beta} = \lambda_{3\alpha+\beta} = \lambda_{3\alpha+2\beta} = 0$
5	$2\alpha+\beta$	$\lambda_{2\alpha+\beta} = \xi(2\alpha+\beta)$, $2c_2\lambda_\beta\lambda_{2\alpha+\beta} - c_1\lambda_{\alpha+\beta}^2 = 0$, $\lambda_{3\alpha+\beta} = \lambda_{3\alpha+2\beta} = 0$
6	$3\alpha+\beta$	$6c_3^2\lambda_\beta\lambda_{3\alpha+\beta}^2 - c_1c_2\lambda_{2\alpha+\beta}^3 = 0$, $2c_3\lambda_{\alpha+\beta}\lambda_{3\alpha+\beta} - c_2\lambda_{2\alpha+\beta}^2 = 0$, $\lambda_{3\alpha+\beta} = \xi(3\alpha+\beta)$, $\lambda_{3\alpha+2\beta} = 0$
7	$3\alpha+2\beta$	$2c_5\lambda_\alpha\lambda_{3\alpha+2\beta} - 2c_3\lambda_{\alpha+\beta}\lambda_{3\alpha+\beta} + c_2\lambda_{2\alpha+\beta}^2 = 0$, $\lambda_{3\alpha+2\beta} = \xi(3\alpha+2\beta)$
8	α, β	$\lambda_\alpha = \xi(\alpha)$, $\lambda_\beta = \xi(\beta)$, $\lambda_\gamma = 0$ *for* $\gamma \neq \alpha, \beta$
9	$\beta, 2\alpha+\beta$	$\lambda_{2\alpha+\beta} = \xi(2\alpha+\beta)$, $2c_2\lambda_\beta\lambda_{2\alpha+\beta} - c_1\lambda_{\alpha+\beta}^2 =$ $2c_2\xi(\beta)\xi(2\alpha+\beta)$, $\lambda_{3\alpha+\beta} = \lambda_{3\alpha+2\beta} = 0$
10	$\beta, 3\alpha+\beta$	$6c_3^2\lambda_\beta\lambda_{3\alpha+\beta}^2 - c_1c_2\lambda_{2\alpha+\beta}^3 = 6c_3^2\xi(\beta)\xi(3\alpha+\beta)^2$, $2c_3\lambda_{\alpha+\beta}\lambda_{3\alpha+\beta} - c_2\lambda_{2\alpha+\beta}^2 = 0$, $\lambda_{3\alpha+\beta} = \xi(3\alpha+\beta)$, $\lambda_{3\alpha+2\beta} = 0$
11	$\alpha+\beta, 3\alpha+\beta$	$2c_3\lambda_{\alpha+\beta}\lambda_{3\alpha+\beta} - c_2\lambda_{2\alpha+\beta}^2 = 2c_3\xi(\alpha+\beta)\xi(3\alpha+\beta)$, $\lambda_{3\alpha+\beta} = \xi(3\alpha+\beta)$, $\lambda_{3\alpha+2\beta} = 0$

12	$\alpha, 3\alpha+2\beta$	$2c_5\lambda_\alpha\lambda_{3\alpha+2\beta} - 2c_3\lambda_{\alpha+\beta}\lambda_{3\alpha+\beta} + c_2\lambda^2_{2\alpha+\beta} = c_5\xi(\alpha)\xi(3\alpha+2\beta)$, $\lambda_{3\alpha+2\beta} = \xi(3\alpha+2\beta)$

Proof. The proof will be performed for all rook placements in Φ^+ subsequently. First, assume that $|D| = 1$, and in that case, $\Omega_{D,\xi} = \mathcal{O}_{D,\xi}$. Pick a linear form $\lambda \in \mathcal{O}_{D,\xi}$. Then there exists $x = \sum_{\gamma\in\Phi^+} x_\gamma e_\gamma \in \mathfrak{n}$ such that $\lambda = \exp(x).f_{D,\xi}$. Let us proceed case-by-case. Cases 1, 2, 3 from the table above are evident, so we start from case 4.

Case 4: $D = \{\alpha+\beta\}$.

It follows immediately from the paragraph before Proposition 3.1 that $\lambda_{\alpha+\beta} = \xi(\alpha+\beta)$ and $\lambda_{2\alpha+\beta} = \lambda_{3\alpha+\beta} = \lambda_{3\alpha+2\beta} = 0$. To compute λ_α, we note that, obviously, $(\alpha+\beta)-\alpha$ can be uniquely represented as a sum of positive roots: $(\alpha+\beta)-\alpha = \beta$. Therefore,

$$\lambda_\alpha = \lambda(e_\alpha) = \xi(\alpha+\beta)e^*_{\alpha+\beta}(e_\alpha - [x, e_\alpha]) = \xi(\alpha+\beta)e^*_{\alpha+\beta}(-[x_\beta e_\beta, e_\alpha]) = \xi(\alpha+\beta)c_1x_\beta.$$

Similarly, $\lambda_\beta = -\xi(\alpha+\beta)c_1x_\alpha$. Since x_α and x_β can be arbitrary, we obtain the required system of equations.

Case 5: $D = \{2\alpha+\beta\}$.

Here $\lambda_{2\alpha+\beta} = \xi(2\alpha+\beta)$ and $\lambda_{3\alpha+\beta} = \lambda_{3\alpha+2\beta} = 0$. Now, $(2\alpha+\beta)-(\alpha+\beta) = \alpha$ is the unique representation of $(2\alpha+\beta)-(\alpha+\beta)$ as a sum of positive roots, hence

$$\lambda_{\alpha+\beta} = \lambda(e_{\alpha+\beta}) = \xi(2\alpha+\beta)e^*_{2\alpha+\beta}(e_{\alpha+\beta} - [x, e_{\alpha+\beta}]) = -\xi(2\alpha+\beta)c_2x_\alpha.$$

Next, since $(2\alpha+\beta)-\beta = 2\alpha$ is the unique representation of $(2\alpha+\beta)-\beta$ as a sum of positive roots, we obtain

$$\begin{aligned}\lambda_\beta &= \lambda(e_\beta)\\ &= \xi(2\alpha+\beta)e^*_{2\alpha+\beta}(e_\beta - [x, e_\beta] + \frac{1}{2!}[x,[x,e_\beta]])\xi(2\alpha+\beta)e^*_{2\alpha+\beta}(\frac{1}{2}[x_\alpha e_\alpha,[x_\alpha e_\alpha, e_\beta]])\\ &= \frac{1}{2}\xi(2\alpha+\beta)c_1x_\alpha^2e^*_{2\alpha+\beta}([e_\alpha, e_{\alpha+\beta}]) = \frac{1}{2}\xi(2\alpha+\beta)c_1c_2x_\alpha^2.\end{aligned}$$

Finally, since $2\alpha \notin \Phi^+$, we can obtain $2\alpha+\beta$ either by adding to α the roots β and α subsequently, or by adding to α the root $\alpha+\beta$. So,

$$\lambda_\alpha = \lambda(e_\alpha) = \xi(2\alpha+\beta)e^*_{2\alpha+\beta}(e_\alpha - [x, e_\alpha] + \frac{1}{2!}[x,[x,e_\alpha]]) = \xi(2\alpha+\beta)(c_2x_{\alpha+\beta} - \frac{1}{2}c_1c_2x_\alpha x_\beta).$$

Thus, λ_α can be arbitrary, while $2c_2\lambda_{2\alpha+\beta}\lambda_\beta = c_1\lambda^2_{\alpha+\beta}$, as required.

Case 6: $D = \{3\alpha+\beta\}$.

Arguing as above, we see that $\lambda_{3\alpha+\beta} = \xi(3\alpha+\beta)$, $\lambda_{3\alpha+2\beta} = 0$,

$$\begin{aligned}&\lambda_{2\alpha+\beta} = -\xi(3\alpha+\beta)c_3x_\alpha, \lambda_{\alpha+\beta} = \frac{1}{2}\xi(3\alpha+\beta)c_2c_3x_\alpha^2, \lambda_\beta = -\frac{1}{6}\xi(3\alpha+\beta)c_1c_2c_3x_\alpha^3,\\ &\lambda_\alpha = \xi(3\alpha+\beta)(c_3x_{2\alpha+\beta} - \frac{1}{2}c_2c_3x_\alpha x_{\alpha+\beta} + \frac{1}{6}c_1c_2c_3x_\alpha^2x_\beta).\end{aligned}$$

Now, it is clear that the equations from the table above define $\mathcal{O}_{D,\xi}$.

Case 7: $D = \{3\alpha + 2\beta\}$.

Here $\lambda_{3\alpha+2\beta} = \xi(3\alpha + 2\beta)$, $\lambda_{3\alpha+\beta} = \xi(3\alpha + 2\beta)c_4x_\beta$,

$$\begin{aligned}
&\lambda_{2\alpha+\beta} = \xi(3\alpha + 2\beta)(-c_5x_{\alpha+\beta} - \frac{1}{2}c_3c_4x_\alpha x_\beta),\\
&\lambda_{\alpha+\beta} = \xi(3\alpha + 2\beta)(c_5x_{2\alpha+\beta} + \frac{1}{2}c_2c_5x_\alpha x_{\alpha+\beta} + \frac{1}{6}c_2c_3c_4x_\alpha^2x_\beta),\\
&\lambda_\beta = \xi(3\alpha + 2\beta)(-c_4x_{3\alpha+\beta} - \frac{1}{2}c_1c_5x_\alpha x_{2\alpha+\beta} - \frac{1}{6}c_1c_2c_5x_\alpha^2x_{\alpha+\beta} - \frac{1}{24}c_1c_2c_3c_4x_\alpha^3x_\beta),\\
&\lambda_\alpha = \xi(3\alpha + 2\beta)(c_1c_5x_\beta x_{2\alpha+\beta} - \frac{1}{2}c_2c_5x_{\alpha+\beta}^2 + \frac{1}{24}c_1c_2c_3c_4x_\alpha^2x_\beta^2).
\end{aligned}$$

One can immediately check that λ satisfies the required system of equations. On the other hand, given arbitrary $\lambda_{2\alpha+\beta}$, $\lambda_{\alpha+\beta}$, λ_β, one can put $x_\beta = \dfrac{\lambda_{3\alpha+\beta}}{c_4\xi(3\alpha+2\beta)}$,

$$\begin{aligned}
x_{\alpha+\beta} &= -\frac{\lambda_{2\alpha+\beta} + \frac{1}{2}c_3c_4x_\alpha x_\beta\xi(3\alpha + 2\beta)}{c_5\xi(3\alpha + 2\beta)} = -\frac{\lambda_{2\alpha+\beta} + \frac{1}{2}c_3\lambda_{3\alpha+\beta}x_\alpha}{c_5\xi(3\alpha + 2\beta)},\\
x_{2\alpha+\beta} &= \frac{\lambda_{\alpha+\beta} - \frac{1}{2}c_2c_5x_\alpha x_{\alpha+\beta}\xi(3\alpha + 2\beta) - \frac{1}{6}c_2c_3c_4x_\alpha^2x_\beta\xi(3\alpha + 2\beta)}{c_5\xi(3\alpha + 2\beta)}\\
&= \frac{\lambda_{\alpha+\beta} + \frac{1}{2}c_2x_\alpha(\lambda_{2\alpha+\beta} + \frac{1}{2}c_3\lambda_{3\alpha+\beta}x_\alpha) - \frac{1}{6}c_2c_3\lambda_{3\alpha+\beta}x_\alpha^2}{c_5\xi(3\alpha + 2\beta)}\\
&= \frac{\lambda_{\alpha+\beta} + \frac{1}{2}c_2x_\alpha\lambda_{2\alpha+\beta} + \frac{1}{12}c_2c_3\lambda_{3\alpha+\beta}x_\alpha^2}{c_5\xi(3\alpha + 2\beta)}.
\end{aligned}$$

It is straightforward to check that, for these values of x_β, $x_{\alpha+\beta}$ and $x_{2\alpha+\beta}$, one has

$$\lambda_\alpha = \frac{c_3\lambda_{\alpha+\beta}\lambda_{3\alpha+\beta}}{c_5\lambda_{3\alpha+2\beta}} - \frac{c_2\lambda_{2\alpha+\beta}^2}{2c_5\lambda_{3\alpha+2\beta}}.$$

Thus, $\mathcal{O}_{D,\xi} = \Omega_{D,\xi}$ is exactly the set of solutions of the required system of equations.

Cases 8–12 can be considered uniformly (in all these cases $|D| = 2$). In all cases, except case 11, D contains a basis root γ ($\gamma = \alpha$ or $\gamma = \beta$). Since the coadjoint orbit of $\xi(\gamma)e_\gamma^*$ is $\{\xi(\gamma)e_\gamma^*\}$, everything is evident. Case 11 is an easy exercise. □

We are now ready to prove our first main result, Theorem 1.4, which claims that, for $\Phi = G_2$, $\mathfrak{n}^* = \bigsqcup_{D,\xi}\mathcal{O}_{D,\xi}$, where the union is taken over all non-singular rook placements D and all maps $\xi\colon D \to \mathbb{C}^\times$.

Proof of Theorem 1.4. Using Proposition 3.1, one can check that each $\lambda \in \mathfrak{n}^*$ belongs to exactly one orbit $\mathcal{O}_{D,\xi}$. Namely, pick a linear form $\lambda \in \mathfrak{n}^*$. Then exactly one of the following cases can occur.

- $\lambda_\alpha = \lambda_\beta = \lambda_{\alpha+\beta} = \lambda_{2\alpha+\beta} = \lambda_{3\alpha+\beta} = \lambda_{3\alpha+2\beta} = 0$. Then $\lambda \in \mathcal{O}_{\varnothing,\xi}$.

- $\lambda_\beta = \lambda_{\alpha+\beta} = \lambda_{2\alpha+\beta} = \lambda_{3\alpha+\beta} = \lambda_{3\alpha+2\beta} = 0, \lambda_\alpha \neq 0$. Then $\lambda \in \mathcal{O}_{\{\alpha\},\xi}$.
- $\lambda_{\alpha+\beta} = \lambda_{2\alpha+\beta} = \lambda_{3\alpha+\beta} = \lambda_{3\alpha+2\beta} = 0, \lambda_\beta \neq 0$. If $\lambda_\alpha = 0$ then $\lambda \in \mathcal{O}_{\{\beta\},\xi}$, if $\lambda_\alpha \neq 0$ then $\lambda \in \mathcal{O}_{\{\alpha,\beta\},\xi}$.
- $\lambda_{2\alpha+\beta} = \lambda_{3\alpha+\beta} = \lambda_{3\alpha+2\beta} = 0, \lambda_{\alpha+\beta} \neq 0$. Then $\lambda \in \mathcal{O}_{\{\alpha+\beta\},\xi}$.
- $\lambda_{3\alpha+\beta} = \lambda_{3\alpha+2\beta} = 0, \lambda_{2\alpha+\beta} \neq 0$. If $\lambda_\beta = \dfrac{c_1\lambda_{\alpha+\beta}^2}{2c_2\lambda_{2\alpha+\beta}}$ then $\lambda \in \mathcal{O}_{\{2\alpha+\beta\},\xi}$, otherwise $\lambda \in \mathcal{O}_{\{\beta,2\alpha+\beta\},\xi}$.
- $\lambda_{3\alpha+2\beta} = 0, \lambda_{3\alpha+\beta} \neq 0$. If $\lambda_\beta = \dfrac{c_1c_2\lambda_{2\alpha+\beta}^2}{6c_3^2\lambda_{3\alpha+\beta}^2}$ and $\lambda_{\alpha+\beta} = \dfrac{c_2\lambda_{2\alpha+\beta}^2}{2c_3\lambda_{3\alpha+\beta}}$ then $\lambda \in \mathcal{O}_{\{3\alpha+\beta\},\xi}$. If $\lambda_\beta \neq \dfrac{c_1c_2\lambda_{2\alpha+\beta}^2}{6c_3^2\lambda_{3\alpha+\beta}^2}$ and $\lambda_{\alpha+\beta} = \dfrac{c_2\lambda_{2\alpha+\beta}^2}{2c_3\lambda_{3\alpha+\beta}}$ then $\lambda \in \mathcal{O}_{\{\beta,3\alpha+\beta\},\xi}$. If $\lambda_{\alpha+\beta} \neq \dfrac{c_2\lambda_{2\alpha+\beta}^2}{2c_3\lambda_{3\alpha+\beta}}$ then $\lambda \in \mathcal{O}_{\{\alpha+\beta,3\alpha+\beta\},\xi}$.
- $\lambda_{3\alpha+2\beta} \neq 0$. If $\lambda_\alpha = \dfrac{c_3\lambda_{\alpha+\beta}\lambda_{3\alpha+\beta}}{c_5\lambda_{3\alpha+2\beta}} - \dfrac{c_2\lambda_{2\alpha+\beta}^2}{2c_5\lambda_{3\alpha+2\beta}}$ then $\lambda \in \mathcal{O}_{\{3\alpha+2\beta\},\xi}$, otherwise $\lambda \in \mathcal{O}_{\{\alpha,3\alpha+2\beta\},\xi}$.

The proof is complete. □

Remark 3.2. i) In fact, $\mathcal{O}_{D,\xi} = \Omega_{D,\xi}$ for all D (and ξ), except case 11.

ii) There exists exactly one singular rook placement in G_2^+, namely, $D = \{\alpha, \alpha+\beta\}$. We do not consider this rook placement because

$$\Omega_{D,\xi} = \mathcal{O}_{D,\xi} = \Omega_{\{\alpha+\beta\},\xi_{\alpha+\beta}} = \mathcal{O}_{\{\alpha+\beta\},\xi_{\alpha+\beta}},$$

where $\xi_{\alpha+\beta} = \xi|_{\{\alpha+\beta\}}$.

iii) It follows from Proposition 3.1 that $\dim \mathcal{O}_{D,\xi} = |S(D)|$ does not depend on ξ, as for A_{n-1}^+.

4 Case $\Phi = F_4$

In this section we prove our second main result, Theorem 1.6. To do this, we firstly prove the following simple lemma. Let D be a non-singular orthogonal rook placement in Φ^+, where $\Phi = F_4$, and $\xi_1, \xi_2 \colon D \to \mathbb{C}^\times$ be a map. Assume that there is the unique maximal root β_0 in D (with respect to the natural order on Φ^+).

Lemma 4.1. *Let $\beta \in D \setminus \{\beta_0\}$ be such that $\gamma \not> \beta$ for all $\gamma \in D \setminus \{\beta_0\}$. Assume that $\beta_0 - \beta = \gamma_1 + \ldots + \gamma_k$ can be uniquely expressed as a sum of positive roots γ_j, $1 \leq j \leq k$. Further, assume that*

$$\beta + \sum_{j \in J} \gamma_j \in \Phi^+$$

for each subset $J \subset \{1, \ldots, k\}$. If $\xi_1(\beta) \neq \xi_2(\beta)$, then Ω_{D,ξ_1} and Ω_{D,ξ_2} do not coincide.

Proof. Suppose that $\Omega_{D,\xi_1} = \Omega_{D,\xi_2}$. Then $\xi_1(\beta_0) = \xi_2(\beta_0)$. Let $\xi\colon D \to \mathbb{C}^\times$ be a map. Pick an element $x \in \mathfrak{n}$ and denote $\mu = \exp x.f_{D,\xi}$. One has

$$\begin{aligned}
\mu(e_{\beta_0 - \gamma_j}) &= (\exp x.f_{D,\xi})(e_{\beta_0 - \gamma_j}) \\
&= f_{D,\xi}(e_{\beta_0 - \gamma_j} - [x, e_{\beta_0 - \gamma_j}] + \ldots) \\
&= a - f_{D,\xi}([x_{\gamma_j} e_{\gamma_j}, e_{\beta_0 - \gamma_j}]) \\
&= a - x_{\gamma_j} f_{D,\xi}([e_{\gamma_j}, e_{\beta_0 - \gamma_j}]) \\
&= a - x_{\gamma_j} \cdot c_j \cdot \xi(\beta_0),
\end{aligned}$$

where c_j is the nonzero scalar such that $[e_{\gamma_j}, e_{\beta_0 - \gamma_j}] = c_j e_{\beta_0}$, while

$$a = \begin{cases} \xi(\beta), & \text{if } \beta_0 - \gamma_j = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, all x_{γ_j} are uniquely defined by μ. Now, let S_k be the symmetric group on k letters. We obtain

$$\begin{aligned}
(\exp x.f_{D,\xi})(e_\beta) &= f_{D,\xi}\Big(e_\beta + (-1)^k \cdot \frac{1}{k!} \underbrace{[x, [x, \ldots [x, e_\beta] \ldots]]}_{k \text{ commutators}}\Big) \\
&= \xi(\beta) + f_{D,\xi}\Big((-1)^k \cdot \frac{1}{k!}\Big[\sum_{j=1}^k x_{\gamma_j} e_{\gamma_j}, \Big[\sum_{j=1}^k x_{\gamma_j} e_{\gamma_j}, \ldots \Big[\sum_{j=1}^k x_{\gamma_j} e_{\gamma_j}, e_\beta\Big] \ldots\Big]\Big]\Big) \\
&= \xi(\beta) + (-1)^k \cdot \frac{1}{k!} \prod_{j=1}^k x_{\gamma_j} \cdot f_{D,\xi}\Big(\sum_{\delta \in S_k} [e_{\gamma_{\delta(1)}}, [e_{\gamma_{\delta(2)}}, \ldots [e_{\gamma_{\delta(k)}}, e_\beta] \ldots]]\Big).
\end{aligned}$$

Denote the second summand by F. Then F is uniquely defined by μ, because x_{γ_j} and $\xi(\beta_0)$ are uniquely defined by μ. If $\Omega_{D,\xi_1} = \Omega_{D,\xi_2}$ then there exist x_1, x_2, for which $\exp x_1.f_{D,\xi_1} = \exp x_2.f_{D,\xi_2}$. So, $(\exp x_1.f_{D,\xi_1})(e_\beta) = (\exp x_2.f_{D,\xi_2})(e_\beta)$, or equivalently, $\xi_1(\beta) + F = \xi_2(\beta) + F$, hence $\xi_1(\beta) = \xi_2(\beta)$, a contradiction. □

Now, we need a general construction, which can be applied to an arbitrary root system. Namely, let $\mathfrak{g}$, $\mathfrak{b}$ and $N = \exp(\mathfrak{n})$ be as in the introduction. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n}^-$ is the nilradical of the Borel subalgebra opposite to $\mathfrak{b}$, and let Φ^- be the set of negative roots. Then the root vectors e_α, $\alpha \in \Phi^-$, form a basis of $\mathfrak{n}^-$. Further, let $\alpha_1, \ldots, \alpha_n$ be the simple roots from Φ, and h_{α_i}, $1 \leq i \leq n$, be a basis of $\mathfrak{h}$ such that $\{e_\alpha, \alpha \in \Phi\} \cup \{h_{\alpha_i}, 1 \leq i \leq n\}$ is a Chevalley basis of $\mathfrak{g}$.

We fix a total order $\leq_t$ on this basis such that $e_\alpha <_t h_{\alpha_i} <_t e_{-\beta}$ for all $\alpha, \beta \in \Phi^+$, $1 \leq i \leq n$, and $e_\alpha <_t e_\beta$ if $\alpha, \beta \in \Phi$ and $\alpha > \beta$. This identifies $\mathfrak{gl}(\mathfrak{g})$ with the Lie algebra $\mathfrak{gl}_{\dim \mathfrak{g}}(\mathbb{C})$, and $\mathrm{ad}(\mathfrak{n})$ with a subalgebra of the Lie algebra $\mathfrak{u}$ of all the strictly upper-triangular matrices of $\mathfrak{gl}_{\dim \mathfrak{g}}(\mathbb{C})$.

Let $\mathrm{GL}(V)$ be the group of all invertible linear operators on a vector space V. Since we have fixed a basis for $\mathfrak{g}$, the group $\mathrm{GL}(\mathfrak{g})$ can be identified with the group $\mathrm{GL}_{\dim \mathfrak{g}}(\mathbb{C})$, and $\exp \mathrm{ad}(\mathfrak{n}) \cong N$ is identified with a subgroup of the group U of all upper-triangular matrices from $\mathrm{GL}_{\dim \mathfrak{g}}(\mathbb{C})$ with 1's on the diagonal. Furthermore, using the Killing form on $\mathfrak{g}$ and the trace form on $\mathfrak{gl}(\mathfrak{g})$, one can identify $\mathfrak{n}^*$ with the space $\mathfrak{n}_- = \langle e_{-\alpha}, \alpha \in \Phi^+ \rangle_{\mathbb{C}}$ and $\mathfrak{u}^*$ with the space $\mathfrak{u}_- = \mathfrak{u}^T$, where the superscript T denotes the transposed matrix. Under all these identifications, it is enough to check that the coadjoint U-orbits of the linear forms $\widetilde{f}_{D,\xi_1}$ and $\widetilde{f}_{D,\xi_2}$ are distinct. Here, given a map $\xi\colon D \to \mathbb{C}^\times$, we denote by $\widetilde{f}_{D,\xi}$ the matrix

$$\widetilde{f}_{D,\xi} = \left(\mathrm{ad}\left(\sum\nolimits_{\beta \in D} \xi(\beta) e_\beta\right)\right)^T \in \mathfrak{u}_- \cong \mathfrak{u}^*.$$

We will now study the matrix $f = \widetilde{f}_{D,\xi}$ in more detail. The rows and the columns of matrices from $\mathfrak{gl}(\mathfrak{g})$ are now indexed by the elements of the Chevalley basis fixed above. Given a matrix x from $\mathfrak{gl}(\mathfrak{g})$ and two basis elements a, b, we will denote by $x_{a,b}$ the entry of x lying in the ath row and the bth column. The following proposition was proved in [9]. For the reader's convenience, we reproduce the proof here, because our main technical tool used in the proof of Theorem 1.6 is based on similar ideas.

Proposition 4.2 ([9, Proposition 4.2]). *Let Φ be an irreducible root system, and D be a non-singular rook placement in Φ^+. Let β_0 be a root in D, ξ_1 and ξ_2 be maps from D to $\mathbb{C}^\times$ for which $\xi_1(\beta_0) \neq \xi_2(\beta_0)$. Assume that there exists a simple root $\alpha_0 \in \Delta$ satisfying $(\alpha_0, \beta_0) \neq 0$ and $(\alpha_0, \beta) = 0$ for all $\beta \in D$ such that $\beta \not< \beta_0$. Then $\Omega_{D,\xi_1} \neq \Omega_{D,\xi_2}$.*

Proof. Since

$$\mathrm{ad}_{e_{\beta_0}}(h_{\alpha_0}) = [e_{\beta_0}, h_{\alpha_0}] = -\frac{2(\alpha_0, \beta_0)}{(\alpha_0, \alpha_0)} e_{\beta_0},$$

we obtain $f_{h_{\alpha_0}, e_{\beta_0}} = -\xi(\beta_0)\dfrac{2(\alpha_0, \beta_0)}{(\alpha_0, \alpha_0)} \neq 0$. One may assume without loss of generality that $h_{\alpha_0} >_t h_{\alpha_i}$ for all $\alpha_i \neq \alpha_0$. We claim that

$$f_{h_{\alpha_0}, e_\alpha} = f_{e_{-\gamma}, e_{\beta_0}} = 0 \text{ for all } e_\alpha <_t e_{\beta_0} \text{ and all } e_{-\gamma}, \alpha, \gamma \in \Phi^+. \tag{1}$$

Indeed, if $\alpha \notin D$ then, evidently, $f_{h_{\alpha_0}, e_\alpha} = 0$. If $\alpha = \beta \in D$ and $e_\beta <_t e_{\beta_0}$ then $\beta \not< \beta_0$, hence

$$f_{h_{\alpha_0}, e_\beta} = -\xi(\beta)\frac{2(\alpha_0, \beta)}{(\alpha_0, \alpha_0)} = 0,$$

because $(\alpha_0, \beta) = 0$. On the other hand, if $f_{e_{-\gamma}, e_{\beta_0}} \neq 0$ for some $\gamma \in \Phi^+$ then $\beta_0 = \beta - \gamma$. This contradicts the condition $\beta_0 \notin S(\beta)$.

Thus, $(\widetilde{f}_{D,\xi_1})_{h_{\alpha_0},e_\alpha}$ and $(\widetilde{f}_{D,\xi_2})_{h_{\alpha_0},e_\alpha}$ are different nonzero scalars, and (1) is satisfied both for $f = \widetilde{f}_{D,\xi_1}$ and for $f = \widetilde{f}_{D,\xi_2}$. Now it follows immediately from the proof of [1, Proposition 3] (or from Remark 2.7) that the coadjoint U-orbits of these matrices are distinct, and, consequently, $\Omega_{D,\xi_1} \neq \Omega_{D,\xi_2}$, as required. □

Our main technical tool generalizes the proposition above in the following way. Fix an order $\{\beta_1, \ldots, \beta_m\}$ on D and an order $\{\alpha_1, \ldots, \alpha_n\}$ on the simple roots in Φ^+ such that $h_{\alpha_i} <_t h_{\alpha_j}$ and $e_{\beta_i} <_t e_{\beta_j}$ for $i < j$. Note that

$$f_{h_{\alpha_i},e_{\beta_j}} = -\xi(\beta_j)\frac{2(\alpha_i, \beta_j)}{(\alpha_i, \alpha_i)}.$$

Given $J \in \{1, \ldots, m\}$ and $I \subset \{1, \ldots, n\}$ with $|I| = |J|$, denote by $\Delta_I^J(\xi)$ the minor of the matrix f with the set of rows $\{h_{\alpha_i}, i \in I\}$ and the set of columns $\{e_{\beta_j}, j \in J\}$. Furthermore, let $\widetilde{\Delta}_I^J$ be the determinant of the matrix, which (i, j)-th element equals $p_{i,j} = \dfrac{2(\alpha_i, \beta_j)}{(\alpha_i, \alpha_i)}$, so that $\Delta_I^J(\xi) = \pm \prod_{j \in J} \xi(\beta_j)\widetilde{\Delta}_I^J$.

Proposition 4.3. *Assume that there exist an m-tuple $I = (i_1, \ldots, i_m)$ such that, for all $1 \leq k \leq m$, $\widetilde{\Delta}_{I_k}^{J_k} \neq 0$, where $I_k = \{i_l \mid l \leq k, i_l \geq i_k\}$ and $J_k = \{j \mid i_j \in I_k\}$. Assume also that, for all $1 \leq k \leq m$, $\widetilde{\Delta}_{I'_l}^{J'_l} = 0$ for $l \notin \{i_1, \ldots, i_{k-1}\}$, $l > i_k$, where*

$$I'_l = \{l\} \cup \{i_s \mid s < k, i_s > l\} \quad \text{and} \quad J'_l = \{k\} \cup \{j \mid i_j \in I'_l \setminus \{l\}\}.$$

Let ξ_1 and ξ_2 be maps from D to $\mathbb{C}^\times$. If $\xi_1 \neq \xi_2$ then $\Omega_{D,\xi_1} \neq \Omega_{D,\xi_2}$.

Proof. For simplicity, we denote $\Phi_1^+ = \{\delta \mid e_\delta <_t e_\beta \text{ for all } \beta \in D\}$ and $\Phi_2^+ = \Phi^+ \setminus (D \cup \Phi_1^+)$.

First, note that $f_{e_{-\gamma},e_{\beta_j}} = 0$ and $f_{h_{\alpha_i},e_\delta} = 0$ for all $\gamma, \delta \in \Phi_1^+$, $\alpha_i \in \Delta$, $\beta_j \in D$. Indeed, $f_{e_{-\gamma},e_{\beta_j}}$ equals the coefficient of e_{β_j} in the expression $\sum_{\beta \in D} \xi(\beta)[e_\beta, e_{-\gamma}]$. But if this coefficient is nonzero then $\beta - \gamma = \beta_j$ for some $\beta \in D$, which contradicts the non-singularity of D. On the other hand, $f_{h_{\alpha_i},e_\delta}$ equals the coefficient of e_δ in the expression $\sum_{\beta \in D} \xi(\beta)[e_\beta, h_{\alpha_i}]$ which is clearly zero, because $[e_\beta, h_{\alpha_i}]$ is parallel to e_β for each $\beta \in D$, while $\delta \notin D$. On the picture below we draw schematically the matrix f. Marks Φ_1^+, D, Φ_2^+, Δ, Φ^- mean that the corresponding rows and columns of the matrix f are indexed by e_δ for $\delta \in \Phi_1^+$, e_{β_j} for $\beta_j \in D$, e_γ for $\gamma \in \Phi_2^+$, h_{α_i} for $\alpha_i \in \Delta$, e_α for $\alpha \in \Phi^-$ respectively. We replaced by big zeroes the blocks $\Delta \times \Phi_1^+$ and $\Phi^- \times D$ filled in zero entries. The minors $\Delta_{I_l}^{J_l}$ are the determinants of submatrices of the grey block $\Delta \times D$.

	Φ_1^+	D	Φ_2^+	Δ	Φ^-
Φ_1^+					
D					
Φ_2^+					
Δ	0				
Φ^-		0			

The Lie algebra $\mathfrak{u}$ corresponds to the root system A_{N-1}, where $N = |\Phi| + \operatorname{rk}\Phi$. Let $\widetilde{D}_i$ be the subset of A_{N-1}^+ and $\widetilde{\xi}_i\colon \widetilde{D}_i \to \mathbb{C}^\times$ be the map such that $f_i = \widetilde{f}_{D,\xi_i}$ (as an element of $\mathfrak{u}^*$) belongs to the basic subvariety $\mathcal{O}_{\widetilde{D}_i,\widetilde{\xi}_i}$ of $\mathfrak{u}^*$ defined in Section 2, $i = 1, 2$. Put $\mathcal{J} = \{e_\alpha, \alpha \in \Phi\} \cup \{h_{\alpha_i}, \alpha_i \in \Delta\}$. Each pair $(x, y) \in \mathcal{J} \times \mathcal{J}$ such that the (x, y)-th entry of f_i lies under the diagonal corresponds to the unique root $\varepsilon_y - \varepsilon_x \in A_{N-1}^+$. We denote the inverse map from A_{N-1}^+ to $\mathcal{J} \times \mathcal{J}$ by τ.

Put $L = \{e_\alpha, \alpha \in \Phi^-\} \times \{e_{\delta,\delta\in\Phi_1^+}\}$ and $\widetilde{D}_i^L = \tau(\widetilde{D}_i) \cap L$. According to André's theory, we may assume without loss of generality that $\widetilde{D}_1^L = \widetilde{D}_2^L$ and $\widetilde{\xi}_1(\varepsilon_y - \varepsilon_x) = \widetilde{\xi}_2(\varepsilon_y - \varepsilon_x)$ for each $(x, y) \in \widetilde{D}_i^L$ (if not, then $\mathcal{O}_{\widetilde{D}_1,\widetilde{\xi}_1} \neq \mathcal{O}_{\widetilde{D}_2,\widetilde{\xi}_2}$ and, consequently, $\Omega_{D,\xi_1} \neq \Omega_{D,\xi_2}$.) We will prove that $\xi_1(\beta_j) = \xi_2(\beta_j)$ for all $1 \le j \le m$ by induction on j. The case $j = 0$ (with $I_0 = \varnothing$) can be considered as an evident inductive base case.

Let $j \geq 1$. Note that each β_l, $1 \le l \le m$, belongs to $\widetilde{D}_i$, and the intersection of $\tau(\widetilde{D}_i)$ with $\{h_{\alpha_i}, i \in \Delta\} \times \{e_\beta, \beta \in D\}$ (i.e., with the "grey" area) coincides with $\{(h_{\alpha_{i_l}}, \beta_l)\}_{l=1}^m$ (this follows immediately from Remark 2.7). Furthermore, recall the notion of $\Delta_{\widetilde{\alpha}}^{\widetilde{D}_i}(f_i)$ for $\widetilde{\alpha} \in A_{N-1}^+$ from Section 2, where $\widetilde{f}_i$ is considered as an element of $\mathfrak{u}^*$. It also follows from Remark 2.7 that, for each l from 1 to m and for $i = 1, 2$,

$$\Delta^{\widetilde{D}_i}_{\tau^{-1}(h_{\alpha_{i_l}}, e_{\beta_l})} = \pm \prod\nolimits_{\widetilde{\alpha}\in\widetilde{D}_i, \tau(\widetilde{\alpha})\in\widetilde{D}_i^L} \widetilde{\xi}_i(\widetilde{\alpha}) \Delta_{I_l}^{J_l} = \mathrm{const}_{i,l} \widetilde{\Delta}_{I_l}^{J_l},$$

where $\mathrm{const}_{i,l}$ is a scalar depending only on $D_{i,l} = \widetilde{D}_i^L \cup \{(h_{\alpha_{i_s}}, e_{\beta_s}), s < l, h_{\alpha_{i_s}} >_t h_{\alpha_{i_l}}\}$ and on $\widetilde{\xi}_i\Big|_{D_{i,l}}$. By the inductive assumption, $D_{1,j} = D_{2,j}$ and $\widetilde{\xi}_2\Big|_{D_{i,j}} = \widetilde{\xi}_2\Big|_{D_{2,j}}$. We conclude that $\widetilde{\xi}_1(\beta_j) = \widetilde{\xi}_2(\beta_j)$, as required and the proof is complete. □

Remark 4.4. It follows from the conditions of Proposition 4.3 that if such an m-tuple I exists then it is unique.

From now on, let $\Phi = F_4$. Recall that the set Δ of simple roots can be identified with the following subset of $\mathbb{R}^4$:

$$\Delta = \{\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}.$$

Here $\{\varepsilon_i\}_{i=1}^4$ is the standard basis of $\mathbb{R}^4$ (with the standard inner product). The set of positive roots is as follows:

$$\begin{aligned}\Phi^+ = \{&\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_2+2\alpha_3,\ \alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3,\\ &\alpha_1+\alpha_2+2\alpha_3,\ \alpha_1+2\alpha_2+2\alpha_3,\ \alpha_2+\alpha_3+\alpha_4,\ \alpha_2+2\alpha_3+\alpha_4,\ \alpha_2+2\alpha_3+2\alpha_4,\\ &\alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4,\\ &\alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\ \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4\alpha_1+2\alpha_2+3\alpha_3+\alpha_4,\\ &\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4,\ \alpha_1+2\alpha_2+4\alpha_3+2\alpha_4,\ \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\\ &2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4\}.\end{aligned}$$

We will apply Proposition 4.3 above to the following rook placements.

Proposition 4.5. *Let $\Phi = F_4$, and $D = \{\beta_1, \ldots, \beta_m\}$ be one of rook placements from the table below. Then the orders on Δ and D and the sets $I_j = \{i_1, \ldots, i_j\}$, $1 \le j \le m$, from the table below satisfy the conditions of Proposition* 4.3.

	Rook placement D	*Order on* Δ	$i_1, \ldots, i_m$
1	$\beta_1 = \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\ \beta_2 = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4,$ $\beta_3 = \alpha_1+\alpha_2+2\alpha_3$	$\alpha_1, \alpha_4, \alpha_2, \alpha_3$	$3, 2, 1$
2	$\beta_1 = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4,\ \beta_2 = \alpha_1+\alpha_2+\alpha_3$	$\alpha_2, \alpha_1, \alpha_4, \alpha_3$	$3, 2$
3	$\beta_1 = \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\ \beta_2 = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4,$ $\beta_3 = \alpha_1+\alpha_2+2\alpha_3,\ \beta_4 = \alpha_1+\alpha_2$	$\alpha_1, \alpha_2, \alpha_4, \alpha_3$	$2, 3, 4, 1$
4	$\beta_1 = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4,\ \beta_2 = \alpha_1+\alpha_2+2\alpha_3,$ $\beta_3 = \alpha_1+\alpha_2$	$\alpha_1, \alpha_2, \alpha_4, \alpha_3$	$3, 4, 2$
5	$\beta_1 = \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\ \beta_2 = \alpha_1+\alpha_2+2\alpha_3,$ $\beta_3 = \alpha_1+\alpha_2$	$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	$2, 4, 3$
6	$\beta_1 = \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\ \beta_2 = \alpha_1+\alpha_2+2\alpha_3+\alpha_4,$ $\beta_3 = \alpha_1+\alpha_2$	$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	$2, 3, 1$
7	$\beta_1 = \alpha_1+\alpha_2+2\alpha_3+\alpha_4,\ \beta_2 = \alpha_1+\alpha_2$	$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	$3, 2$
8	$\beta_1 = \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\ \beta_2 = \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4,$ $\beta_3 = \alpha_1+2\alpha_2+2\alpha_3,\ \beta_4 = \alpha_1$	$\alpha_2, \alpha_3, \alpha_4, \alpha_1$	$2, 3, 1, 4$
9	$\beta_1 = \alpha_1+2\alpha_2+4\alpha_3+2\alpha_4,\ \beta_2 = \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,$ $\beta_3 = \alpha_1$	$\alpha_3, \alpha_2, \alpha_1, \alpha_4$	$2, 1, 3$
10	$\beta_1 = \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4,\ \beta_2 = \alpha_3+\alpha_4$	$\alpha_4, \alpha_3, \alpha_2, \alpha_1$	$3, 2$
11	$\beta_1 = 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\ \beta_2 = \alpha_2+\alpha_3+\alpha_4,$ $\beta_3 = \alpha_2+2\alpha_3$	$\alpha_1, \alpha_3, \alpha_4, \alpha_2$	$1, 4, 3$
12	$\beta_1 = \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\ \beta_2 = \alpha_2+2\alpha_3+2\alpha_4,$ $\beta_3 = \alpha_2+2\alpha_3$	$\alpha_1, \alpha_4, \alpha_3, \alpha_2$	$4, 2, 3$
13	$\beta_1 = \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\ \beta_2 = \alpha_1+\alpha_2+\alpha_3+\alpha_4,$ $\beta_3 = \alpha_1+\alpha_2+2\alpha_3$	$\alpha_3, \alpha_4, \alpha_1, \alpha_2$	$4, 3, 2$
14	$\beta_1 = \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\ \beta_2 = \alpha_1+\alpha_2+2\alpha_3,$ $\beta_3 = \alpha_3+\alpha_4$	$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	$2, 4, 3$
15	$\beta_1 = \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\ \beta_2 = \alpha_1+\alpha_2,\ \beta_3 = \alpha_4$	$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	$2, 3, 4$

16	$\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4,\ \beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4,$ $\beta_3 = \alpha_1 + \alpha_2$	$\alpha_2, \alpha_3, \alpha_1, \alpha_2$	$2, 4, 3$
17	$\beta_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4,\ \beta_2 = \alpha_2 + \alpha_3 + \alpha_4,$ $\beta_3 = \alpha_1 + \alpha_2$	$\alpha_4, \alpha_1, \alpha_2, \alpha_3$	$3, 4, 2$
18	$\beta_1 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4,\ \beta_2 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$ $\beta_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3$	$\alpha_3, \alpha_2, \alpha_4, \alpha_1$	$2, 4, 3$
19	$\beta_1 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4,\ \beta_2 = \alpha_2 + 2\alpha_3 + \alpha_4,$ $\beta_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3$	$\alpha_1, \alpha_2, \alpha_4, \alpha_3$	$2, 4, 3$
20	$\beta_1 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4,\ \beta_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$	$\alpha_2, \alpha_3, \alpha_1, \alpha_4$	$2, 1$
21	$\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4,\ \beta_2 = \alpha_3 + \alpha_4,\ \beta_3 = \alpha_1$	$\alpha_3, \alpha_4, \alpha_2, \alpha_1$	$3, 2, 4$
22	$\beta_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4,\ \beta_2 = \alpha_2 + 2\alpha_3 + 2\alpha_4,$ $\beta_3 = \alpha_2 + 2\alpha_3$	$\alpha_2, \alpha_1, \alpha_3, \alpha_4$	$3, 4, 2$
23	$\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4,\ \beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4,$ $\beta_3 = \alpha_1 + \alpha_2 + 2\alpha_3$	$\alpha_2, \alpha_1, \alpha_3, \alpha_4$	$3, 4, 2$
24	$\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4,\ \beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3,$ $\beta_3 = \alpha_1 + \alpha_2$	$\alpha_2, \alpha_1, \alpha_3, \alpha_4$	$4, 3, 2$

Proof. The proof is case-by-case and is completely straightforward. As an example, consider the 17th rook placement D.

Clearly, the root β_3 (respectively, β_1) is orthogonal to the unique simple root, namely, to α_3 (respectively, to α_4). There are no simple roots orthogonal to β_2. Write out the minor of the matrix f, which rows correspond to h_{α_i}, $\alpha_i \in \Delta$, and columns correspond to e_{β_j}, $\beta_j \in D$. Recall the notion of $p_{i,j}$ introduced before Proposition 4.3.

$$\begin{array}{cccc} & \beta_3 & \beta_2 & \beta_1 \\ \alpha_4 & p_{1,1} & p_{1,2} & 0 \\ \alpha_1 & p_{2,1} & p_{2,2} & p_{2,3} \\ \alpha_2 & p_{3,1} & p_{3,2} & p_{3,3} \\ \alpha_3 & 0 & p_{4,2} & p_{4,3} \end{array}$$

Obviously, $\widetilde{\Delta}_{\{4\}}^{\{1\}} = |p_{4,1}| = 0$, while, for $i_1 = 3$,

$$\widetilde{\Delta}_{I_1}^{J_1} = |p_{3,1}| = \frac{2(\alpha_2, \beta_3)}{(\alpha_2, \alpha_2)} = -1 \neq 0.$$

Hence, in fact we have the only possibility for i_1: $i_1 = 3$. Next, for $i_2 = 4$, one has

$$\widetilde{\Delta}_{I_2}^{J_2} = |p_{4,2}| = \frac{2(\alpha_3, \beta_2)}{(\alpha_3, \alpha_3)} = -1 \neq 0.$$

Therefore, we have to put $i_2 = 4$. Finally, for $i_3 = 2$, we obtain

$$\widetilde{\Delta}_{I_3}^{J_3} = \begin{vmatrix} p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \\ 0 & p_{4,2} & p_{4,3} \end{vmatrix} = \frac{8}{(\alpha_1,\alpha_1)(\alpha_2,\alpha_2)(\alpha_3,\alpha_3)} \begin{vmatrix} (\alpha_1,\beta_3) & (\alpha_1,\beta_2) & (\alpha_1,\beta_1) \\ (\alpha_2,\beta_3) & (\alpha_2,\beta_2) & (\alpha_2,\beta_1) \\ 0 & (\alpha_3,\beta_2) & (\alpha_3,\beta_1) \end{vmatrix}$$
$$= 2 \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & -\frac{1}{2} & -1 \end{vmatrix} = 4 \neq 0.$$

Thus, there is the only candidate for i_3: $i_3 = 2$. It is easy to check that the sequence $(3, 4, 2)$ satisfies the conditions of Proposition 4.3.

All other rook placements from the table above can be considered similarly. □

We are now ready to prove our second main result, Theorem 1.6, which claims that, for a non-singular orthogonal rook placement $D \subset F_4^+$ and two distinct maps ξ_1, ξ_2 from D to $\mathbb{C}^\times$, the associated coadjoint orbits Ω_{D,ξ_1}, Ω_{D,ξ_2} do not coincide.

Proof of Theorem 1.6. This proof is based on a case-by-case analysis. Namely, we split the rook placements in F_4^+ into several "classes" and then apply Lemma 4.1 and Propositions 4.2, 4.3 to these classes. We start with the maximal (possibly, singular) rook placements. It is easy to check that there are 24 maximal rook placements in F_4^+:

$D_1 = \{\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \beta_2 = \alpha_1 + \alpha_2 + \alpha_3, \beta_3 = \alpha_2 + \alpha_3, \beta_4 = \alpha_3\}$,
$D_2 = \{\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \beta_2 = \alpha_1 + \alpha_2 + \alpha_3, \beta_3 = \alpha_2 + 2\alpha_3, \beta_4 = \alpha_2\}$,
$D_3 = \{\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \beta_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3, \beta_3 = \alpha_3, \beta_4 = \alpha_1\}$,
$D_4 = \{\beta_1 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \beta_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \beta_3 = \alpha_1 + \alpha_2 + \alpha_3,$
$\beta_4 = \alpha_2 + \alpha_3\}$,
$D_5 = \{\beta_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \beta_2 = \alpha_2 + 2\alpha_3 + 2\alpha_4, \beta_3 = \alpha_2 + \alpha_3, \beta_4 = \alpha_3\}$,
$D_6 = \{\beta_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \beta_2 = \alpha_2 + 2\alpha_3 + 2\alpha_4, \beta_3 = \alpha_2 + 2\alpha_3, \beta_4 = \alpha_2\}$,
$D_7 = \{\beta_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \beta_2 = \alpha_2 + 2\alpha_3 + \alpha_4, \beta_3 = \alpha_4, \beta_4 = \alpha_2\}$,
$D_8 = \{\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \beta_2 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \beta_3 = \alpha_2 + 2\alpha_3 + 2\alpha_4,$
$\beta_4 = \alpha_2\}$,
$D_9 = \{\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \beta_2 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \beta_3 = \alpha_2 + \alpha_3 + \alpha_4,$
$\beta_4 = \alpha_3 + \alpha_4\}$,
$D_{10} = \{\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \beta_3 = \alpha_2 + 2\alpha_3 + \alpha_4,$
$\beta_4 = \alpha_4\}$,
$D_{11} = \{\beta_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3, \beta_3 = \alpha_2 + \alpha_3, \beta_4 = \alpha_1 + \alpha_2\}$,
$D_{12} = \{\beta_1 = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \beta_3 = \alpha_1 + \alpha_2 + \alpha_3,$
$\beta_4 = \alpha_3\}$,
$D_{13} = \{\beta_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \beta_2 = \alpha_2 + \alpha_3 + \alpha_4, \beta_3 = \alpha_3 + \alpha_4, \beta_4 = \alpha_2 + 2\alpha_3\}$,
$D_{14} = \{\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \beta_3 = \alpha_2 + 2\alpha_3 + 2\alpha_4,$

$$\begin{aligned}
&\beta_4=\alpha_2+2\alpha_3\},\\
D_{15}&=\{\beta_1=\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\beta_2=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4,\beta_3=\alpha_1+\alpha_2+2\alpha_3,\\
&\beta_4=\alpha_1+\alpha_2\},\\
D_{16}&=\{\beta_1=\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\beta_2=\alpha_1+\alpha_2+\alpha_3+\alpha_4,\beta_3=\alpha_1+\alpha_2+2\alpha_3,\\
&\beta_4=\alpha_3+\alpha_4\},\\
D_{17}&=\{\beta_1=\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\beta_2=\alpha_1+\alpha_2+2\alpha_3+\alpha_4,\beta_3=\alpha_1+\alpha_2,\beta_4=\alpha_4\},\\
D_{18}&=\{\beta_1=\alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\beta_2=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4,\beta_3=\alpha_2+2\alpha_3+\alpha_4,\\
&\beta_4=\alpha_1+\alpha_2+2\alpha_3\},\\
D_{19}&=\{\beta_1=\alpha_1+2\alpha_2+3\alpha_3+\alpha_4,\beta_2=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4,\beta_3=\alpha_2+\alpha_3+\alpha_4,\\
&\beta_4=\alpha_1+\alpha_2\},\\
D_{20}&=\{\beta_1=\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4,\beta_2=\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4,\beta_3=\alpha_1+2\alpha_2+2\alpha_3,\\
&\beta_4=\alpha_1\},
\end{aligned}$$

$$\begin{aligned}
D_{21}&=\{\beta_1=\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4,\beta_2=\alpha_1+\alpha_2+\alpha_3+\alpha_4,\beta_3=\alpha_1+2\alpha_2+2\alpha_3,\\
&\beta_4=\alpha_2+\alpha_3+\alpha_4\},\\
D_{22}&=\{\beta_1=\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4,\beta_2=\alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\beta_3=\alpha_4,\beta_4=\alpha_1\},\\
D_{23}&=\{\beta_1=\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4,\beta_2=\alpha_1+\alpha_2+2\alpha_3+\alpha_4,\beta_3=\alpha_2+2\alpha_3+\alpha_4,\\
&\beta_4=\alpha_1+2\alpha_2+2\alpha_3\},\\
D_{24}&=\{\beta_1=\alpha_1+2\alpha_2+3\alpha_3+\alpha_4,\beta_2=\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4,\beta_3=\alpha_3+\alpha_4,\beta_4=\alpha_1\}.
\end{aligned}$$

The first root β_1 is maximal among all roots in each of these rook placements. In the rook placements

$$D_8, D_{14}, D_{18}, D_{19}, D_{24}$$

the second root β_2 is maximal, too. As we mentioned above, if D is a subset of D_i containing a maximal root β from D_i and $\Omega_{D,\xi_1}=\Omega_{D,\xi_2}$ then $\xi_1(\beta)=\xi_2(\beta)$ (here ξ_1, ξ_2 are maps from D to $\mathbb{C}^\times$).

Next, it is straightforward to check that the following maximal rook placements D_i (together with a simple root α_0 and a distinguished root $\beta_0\in D_i$) satisfy the conditions of Proposition 4.2, except the non-singularity of D_i:

$$\begin{array}{lll}
D_2,\beta_0=\beta_3,\alpha_0=\alpha_1; & D_2,\beta_0=\beta_4,\alpha_0=\alpha_2; & D_3,\beta_0=\beta_2,\alpha_0=\alpha_2;\\
D_3,\beta_0=\beta_4,\alpha_0=\alpha_1; & D_4,\beta_0=\beta_2,\alpha_0=\alpha_4; & D_4,\beta_0=\beta_3,\alpha_0=\alpha_1;\\
D_5,\beta_0=\beta_2,\alpha_0=\alpha_4; & D_5,\beta_0=\beta_3,\alpha_0=\alpha_2; & D_6,\beta_0=\beta_2,\alpha_0=\alpha_4;\\
D_6,\beta_0=\beta_3,\alpha_0=\alpha_3; & D_6,\beta_0=\beta_4,\alpha_0=\alpha_2; & D_7,\beta_0=\beta_2,\alpha_0=\alpha_3;\\
D_7,\beta_0=\beta_4,\alpha_0=\alpha_2; & D_8,\beta_0=\beta_4,\alpha_0=\alpha_2; & D_{11},\beta_0=\beta_2,\alpha_0=\alpha_1;\\
D_{12},\beta_0=\beta_2,\alpha_0=\alpha_2; & D_{13},\beta_0=\beta_2,\alpha_0=\alpha_2; & D_{15},\beta_0=\beta_2,\alpha_0=\alpha_4;\\
D_{15},\beta_0=\beta_3,\alpha_0=\alpha_3; & D_{17},\beta_0=\beta_2,\alpha_0=\alpha_3; & D_{20},\beta_0=\beta_2,\alpha_0=\alpha_4;\\
D_{20},\beta_0=\beta_4,\alpha_0=\alpha_1; & D_{21},\beta_0=\beta_2,\alpha_0=\alpha_1; & D_{22},\beta_0=\beta_4,\alpha_0=\alpha_1;\\
D_{23},\beta_0=\beta_2,\alpha_0=\alpha_1; & D_{24},\beta_0=\beta_4,\alpha_0=\alpha_1. &
\end{array}$$

This implies that if D is a non-singular rook placement contained in one of these maximal rook placements and containing the root β_0, then D, β_0, α_0 satisfy the conditions of Proposition 4.2. Hence, if $\Omega_{D,\xi_1} = \Omega_{D,\xi_2}$ then $\xi_1(\beta_0) = \xi_2(\beta_0)$.

Now, let D be a non-singular subset of one of the rook placements $D_1, \ldots, D_{10}$. Assume that $D \subset D_1$. Note that $\beta_i \in S(\beta_j)$ for all $2 \leq i \leq 4$ and $1 \leq j < i$. Hence, $|D| = 1$, and there is nothing to prove. If $D \subset D_2$ contains β_1 then $\beta_2 \notin D$, because $\beta_2 \in S(\beta_1)$. On the other hand, if $\beta_1 \notin D$ and $\beta_2 \in D$ then β_2 is maximal in D. For β_3, β_4 see the previous paragraph. Another example: assume that $D \subset D_3$. If $\beta_1 \in D$ then $\beta_3 \notin D$, because $\beta_3 \in S(\beta_1)$. If $\beta_1 \notin D$ and $\beta_3 \in D$ then D, $\beta_0 = \beta_3$, $\alpha_0 = \alpha_3$ satisfy the condition of Proposition 4.2. For the roots β_2, β_4, see the previous paragraph. All other rook placements $D_4, \ldots, D_{10}$ can be considered in a similar way.

Most of the remaining rook placements (i.e., non-singular subsets of $D_{11}, \ldots, D_{24}$) can be considered by completely similar arguments. The exceptions are the 24 rook placements from Proposition 4.5 and the 8 following rook placements:

$$D_{25} = \{\beta_1 = \alpha_1 + \alpha_2 + 2\alpha_3, \beta_2 = \alpha_2 + \alpha_3, \beta_3 = \alpha_1 + \alpha_2\};$$
$$D_{26} = \{\beta_1 = \alpha_1 + \alpha_2 + 2\alpha_3, \beta_2 = \alpha_2 + \alpha_3\};$$
$$D_{27} = \{\beta_1 = \alpha_1 + \alpha_2 + 2\alpha_3, \beta_2 = \alpha_1 + \alpha_2\};$$
$$D_{28} = \{\beta_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \beta_2 = \alpha_2 + 2\alpha_3 + \alpha_4, \beta_3 = \alpha_1 + \alpha_2 + 2\alpha_3\};$$
$$D_{29} = \{\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \beta_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3, \beta_3 = \alpha_1\};$$
$$D_{30} = \{\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \beta_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3\};$$
$$D_{31} = \{\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \beta_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3\};$$
$$D_{32} = \{\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \beta_2 = \alpha_2 + 2\alpha_3 + \alpha_4, \beta_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3\}.$$

Proposition 4.3 completes the proof for 24 rook placements from Proposition 4.5. For the rook placements $D_{25}, \ldots, D_{32}$, one can apply Lemma 4.1 with $\beta = \beta_2$ or β_3 for D_{25}, $\beta = \beta_2$ for D_{26}, $\beta = \beta_2$ for D_{27}, $\beta = \beta_2$ or β_3 for D_{28}, $\beta = \beta_2$ for D_{29}, $\beta = \beta_2$ for D_{30}, $\beta = \beta_3$ for D_{31}, $\beta = \beta_3$ for D_{32}. All other roots from these rook placements either are maximal or satisfy the conditions of Proposition 4.2 for an appropriate simple root α_0, as we mentioned above. This completes the proof. □

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