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On commutativity of 3-prime near-rings with generalized (α, β)-derivations

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Abstract. Let $\mathcal{N}$ be a 3-prime near ring and $\alpha, \beta : \mathcal{N} \to \mathcal{N}$ be endomorphisms. In the present paper we amplify a few outcomes concerning generalized derivations and two-sided α-generalized derivations of 3-prime near rings to generalized (α, β)-derivations. Cases demonstrating the need of the 3-primeness speculation are given. When $\beta = id_{\mathcal{N}}$ (resp. $\alpha = \beta = id_{\mathcal{N}}$), one can easily obtain the main results of [1] (resp.[7]).

1 Introduction

In the present paper, $\mathcal{N}$ is a zero symmetric right near-ring i.e. non empty set together with two binary operations "$+$" and "." that satisfies $(\mathcal{N}, +, 0)$ is a group (not necessarily abelian), $(\mathcal{N}, .)$ is a semigroup, for all $x, y, z \in \mathcal{N}$: $(x+y)z = xz+yz$ ("right distributive law") and $n0 = 0$ for all $n \in \mathcal{N}$. $Z(\mathcal{N})$ is the multiplication center of $\mathcal{N}$, that is, $Z(\mathcal{N}) = \{x \in N \mid xy = yx \text{ for all } y \in \mathcal{N}\}$. Note that $0 \in Z(\mathcal{N})$, so $Z(\mathcal{N}) \neq \emptyset$. Usually $\mathcal{N}$ will be 3-prime near ring, that is, will have the property that $x\mathcal{N}y = \{0\}$ for $x, y \in \mathcal{N}$ implies $x = 0$ or $y = 0$. Nonempty subset I of $\mathcal{N}$ is called a semigroup right ideal or a semigroup left ideal if $I\mathcal{N} \subseteq I$ or $\mathcal{N}I \subseteq I$ respectively; and I is said to be a semigroup ideal if its both a semigroup right ideal and a semigroup left ideal. Recalling that $\mathcal{N}$ is 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. As in [8], an additive mapping $F : \mathcal{N} \to \mathcal{N}$ is a right or left generalized derivation with associated derivation d if $F(xy) = F(x)y + xd(y)$ or $F(xy) = d(x)y + xF(y)$ holds for all $x, y \in \mathcal{N}$ respectively.

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Let $\alpha, \beta : \mathcal{N} \to \mathcal{N}$ be endomorphisms, an additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called (α, β)-derivation, if $d(xy) = \alpha(x)d(y) + d(x)\beta(y)$ for all $x, y \in \mathcal{N}$, and or equivalently from [3] that $d(xy) = d(x)\beta(y) + \alpha(x)d(y)$, for all $x, y \in \mathcal{N}$.

Now we give an example of a (α, β)-derivation on a near-ring $\mathcal{N}$ which is not a derivation.

Example 1. Let S be a zero-symmetric near-ring. Define $\mathcal{N}$ and $d, \alpha, \beta : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in S \right\},\ d \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

$$\alpha \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \beta \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}.$$

Clearly $\mathcal{N}$ is a zero symmetric near-ring, d is a (α, β)-derivation on $\mathcal{N}$ but not a derivation.

Let $\alpha, \beta : \mathcal{N} \to \mathcal{N}$ be endomorphisms. An additive mapping $F : \mathcal{N} \to \mathcal{N}$ is called a right generalized (α, β)-derivation (resp. left generalized (α, β)-derivation) if there exists a (α, β)-derivation d such that $F(xy) = F(x)\beta(y) + \alpha(x)d(y)$ (resp. $F(xy) = d(x)\beta(y) + \alpha(x)F(y)$) for all $x, y \in \mathcal{N}$. Moreover, F is called a generalized (α, β)-derivation if F is both right generalized (α, β)-derivation and left generalized (α, β)-derivation. Clearly the notion of generalized (α, β)-derivations includes those of (α, β)-derivations (when $F = d$) of derivations (when $F = d$ and $\alpha = \beta = id_{\mathcal{N}}$, where $id_{\mathcal{N}}$ is the identity map on $\mathcal{N}$) and of generalized derivations (which is the case when $\alpha = \beta = id_{\mathcal{N}}$). Hence the concept of generalized (α, β)-derivations includes those of derivations, generalized derivations and (α, β)-derivations.

Now we give an example of a generalized (α, β)-derivation F associated with (α, β)-derivation d on a near-ring such that F is not a (α, β)-derivation of $\mathcal{N}$.

Example 2. Let S be a zero-symmetric near-ring. Let us define $\mathcal{N}$, d, F and $\alpha, \beta : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in S \right\},$$

$$d \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix},\quad F \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

$$\alpha \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \text{ and } \beta \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly $\mathcal{N}$ is a zero symmetric near ring, d is a (α, β)-derivation of $\mathcal{N}$, and F is a generalized (α, β)-derivation associated with d, but F is not a (α, β)-derivation of $\mathcal{N}$.

We will write, for all $x, y \in \mathcal{N}$,

$$[x, y] = xy - yx \quad \text{and} \quad x \circ y = xy + yx$$

for the Lie and Jordan products, respectively. Usually, we denote

$$[x,y]_{\alpha,\beta} := \alpha(x)y - y\beta(x) \quad \text{and} \quad (x\circ y)_{\alpha,\beta} := \alpha(x)y + y\beta(x),$$

for all $x,y\in\mathcal{N}$. In particular $[x,y]_{id_{\mathcal{N}},id_{\mathcal{N}}} = [x,y]$ and $(x\circ y)_{id_{\mathcal{N}},id_{\mathcal{N}}} = x\circ y$, for all $x,y\in\mathcal{N}$.

In the present paper, we generalize Theorems 3.1 and 3.5 of [1], Theorems 2.9, 2.10, 3.1, 3.2, 3.3 and 3.5 of [7].

2 Preliminaries

We begin with the following lemmas which are essential in the following two sections.

Lemma 1. *[6, Lemmas 1.2 (i), 1.2 (iii) & 1.3 (iii)]. Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) If $z\in Z(\mathcal{N})\setminus\{0\}$, then z is not a zero divisor.

(ii) If $z\in Z(\mathcal{N})\setminus\{0\}$ and $xz\in Z(\mathcal{N})$, then $x\in Z(\mathcal{N})$.

(iii) If z centralizes a non zero semigroup left ideal, then $z\in Z(\mathcal{N})$.

Lemma 2. *[6, Lemma 1.3 (i)]. Let $\mathcal{N}$ be a 3-prime near-ring. If I is a nonzero semigroup left ideal (resp. semigroup right ideal) and x is an element of $\mathcal{N}$ such that $xI=\{0\}$, (or $Ix=\{0\}$,) then $x=0$.*

Lemma 3. *[6, Lemma 1.4 (i)]. Let $\mathcal{N}$ be a 3-prime near-ring and I is a nonzero semigroup ideal of $\mathcal{N}$. If $x,y\in\mathcal{N}$ and $xIy=\{0\}$, then $x=0$ or $y=0$.*

Lemma 4. *[6, Lemma 1.5]. Let $\mathcal{N}$ be a 3-prime near-ring. If $Z(\mathcal{N})$ contains a nonzero semigroup right ideal or a semigroup left ideal, then $\mathcal{N}$ is a commutative ring.*

Lemma 5. *[3, Lemma 2.2]. Let d be a (α,β)-derivation on a near-ring $\mathcal{N}$. Then $\mathcal{N}$ satisfies the following partial distributive laws:*

(i) $z(\alpha(x)d(y)+d(x)\beta(y)) = z\alpha(x)d(y)+zd(x)\beta(y)$ for all $x,y,z\in\mathcal{N}$.

(ii) $z(d(x)\beta(y)+\alpha(x)d(y)) = z(d(x)\beta(y)+z\alpha(x)d(y)$ for all $x,y,z\in\mathcal{N}$.

Lemma 6. *[9, Lemma 4]. Let $\mathcal{N}$ be a 3-prime near ring and $d:\mathcal{N}\to\mathcal{N}$ be a nonzero (α,β)-derivation. If I is a nonzero semigroup left ideal or a semigroup right ideal, then $d(I)\neq\{0\}$.*

Lemma 7. *[9, Theorem 2]. Let $\mathcal{N}$ be a 3-prime near ring and I is a nonzero semigroup left ideal of $\mathcal{N}$. If $\mathcal{N}$ admitting a non-trivial (α,β)-derivation d such that $d(I)\subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 8. *Let $\mathcal{N}$ be a 3-prime near-ring and α, β maps of $\mathcal{N}$ such as α is additive. If $\mathcal{N}$ admits an additive mapping F, then the following assertions are equivalent:*

(i) $F(xy) = F(x)\beta(y)+\alpha(x)d(y)$ for all $x,y\in\mathcal{N}$,

(ii) $F(xy) = \alpha(x)d(y) + F(x)\beta(y)$ *for all* $x, y \in \mathcal{N}$.

Proof. $(i) \Rightarrow (ii)$ Assume that $F(xy) = F(x)\beta(y) + \alpha(x)d(y)$, for all $x, y \in \mathcal{N}$, so

$$\begin{aligned} F((x+x)y) &= F(x+x)\beta(y) + \alpha(x+x)d(y) \\ &= F(x)\beta(y) + F(x)\beta(y) + \alpha(x)d(y) + \alpha(x)d(y) \text{ for all } x, y \in \mathcal{N}, \end{aligned}$$

and

$$\begin{aligned} F((x+x)y) &= F(xy) + F(xy) \\ &= F(x)\beta(y) + \alpha(x)d(y) + F(x)\beta(y) + \alpha(x)d(y) \text{ for all } x, y \in \mathcal{N}. \end{aligned}$$

Comparing the two equations, then we get

$$F(x)\beta(y) + \alpha(x)d(y) = \alpha(x)d(y) + F(x)\beta(y) \text{ for all } x, y \in \mathcal{N}.$$

Similarly, we can prove the other implication. □

Lemma 9. *[10, Lemma 2.2]. Let* F *be a generalized* (α, β)*-derivation of near ring* $\mathcal{N}$ *associated with* d*. Then*

$$z(F(x)\beta(y) + \alpha(x)d(y)) = zF(x)\beta(y) + z\alpha(x)d(y) \text{ for all } x, y, z \in \mathcal{N}.$$

We need the following lemma in the next sections

Lemma 10. *Let* $\mathcal{N}$ *be a 2-torsion-free 3-prime near-ring and* I *is a nonzero semigroup ideal of* $\mathcal{N}$*. If* α *and* β *are automorphisms on* $\mathcal{N}$*, then there exists* $x, y \in I$ *such that* $(x \circ y)_{\alpha,\beta} \neq 0$.

Proof. We demonstrate by disagreement, we isolate the confirmation of this lemma into two sections, in the initial segment we demonstrate that $\mathcal{N}$ is a commutative ring, situated in this property in the second part we get the disagreement.

Assume on the contrary that $(x \circ y)_{\alpha,\beta} = 0$ for all $x, y \in I$, then $\alpha(x)y = -y\beta(x)$ for all $x, y \in I$. Replacing y by yz in the last equation and using it, we obtain

$$\begin{aligned} \alpha(x)yz &= -yz\beta(x) \\ &= (-y)(z\beta(x)) \\ &= (-y)(-\alpha(x)z) \\ &= (-y)(\alpha(-x)z) \text{ for all } x, y, z \in I \end{aligned}$$

which implies that

$$(\alpha(x)y + y\alpha(-x))I = \{0\} \text{ for all } x, y \in I.$$

Using Lemma 2, we get $\alpha(-x)y = y\alpha(-x)$ for all $x, y \in I$. Taking ny in place of y, where $n \in \mathcal{N}$, we obtain

$$\begin{aligned} \alpha(-x)ny &= ny\alpha(-x) \\ &= n\alpha(-x)y \text{ for all } x, y \in I, n \in \mathcal{N} \end{aligned}$$

which reduces to $[\alpha(-x), n]I = \{0\}$ for all $x \in I$, $n \in \mathcal{N}$. Using again Lemma 2, we get $\alpha(-x) \in Z(\mathcal{N})$, for all $x \in I$, i.e. $\alpha(-I) \subseteq Z(\mathcal{N})$. Since α is an automorphism of $\mathcal{N}$, then $-I \subseteq Z(\mathcal{N})$ and using the fact that $-I$ is a nonzero semigroup right ideal. Thus $\mathcal{N}$ is a commutative ring by Lemma 4. In this case, our hypothesis implies that

$$\begin{aligned} 0 &= \alpha(x)y + y\beta(x) \\ &= \alpha(x)y + \beta(x)y \\ &= (\alpha(x) + \beta(x))y \text{ for all } x, y \in I. \end{aligned}$$

It follows by Lemma 2 $\alpha(x) + \beta(x) = 0$ for all $x \in I$. i.e. $\beta(x) = -\alpha(x)$ for all $x \in I$. So for every $n \in \mathcal{N}$ and $x \in I$, we get

$$\begin{aligned} -\alpha(n)\alpha(x) &= -\alpha(nx) \\ &= \beta(nx) \\ &= \beta(n)\beta(x) \\ &= \beta(n)(-\alpha(x)) \\ &= -\beta(n)\alpha(x) \text{ for all } x \in I, n \in \mathcal{N}. \end{aligned}$$

Which implies that $\alpha(n)\alpha(x) = \beta(n)\alpha(x)$ for all $x \in I$, $n \in N$. So

$$\begin{aligned} (\alpha(n)\alpha(x) - \beta(n)\alpha(x)) &= 0 \\ &= (\alpha(n) - \beta(n))\alpha(x) \text{ for all } x \in I, n \in \mathcal{N}. \end{aligned}$$

Thus by Lemma (2), we get $\alpha(n) = \beta(n)$ for all $n \in \mathcal{N}$. But $\alpha(x) = -\beta(x)$ for all $x \in I$. So $\beta(x) = -\beta(x)$ for all $x \in I$, and using 2-torsion freeness of $\mathcal{N}$, we get $2\beta(x) = 0 = \beta(x)$ for all $x \in I$. Hence $\beta(I) = \{0\}$, but β is an automorphisms, which implies $I = \{0\}$; a contradiction. □

Lemma 11. *Let $\mathcal{N}$ be a 3-prime near ring, I is a nonzero semigroup left ideal and α, β be automorphisms on $\mathcal{N}$. If $x \in \mathcal{N}$ and $[x, y]_{\alpha,\beta} = 0$ for all $y \in I$, then $x \in Z(\mathcal{N})$.*

Proof. Let $x \in \mathcal{N}$ such that $[x, y]_{\alpha,\beta} = 0$ for all $y \in I$, then $\alpha(x)y = y\beta(x)$ for all $y \in I$. Replace y by ty, where $t \in \mathcal{N}$, we get

$$\begin{aligned} \alpha(x)ty &= ty\beta(x) \\ &= t\alpha(x)y \text{ for all } y \in I, t \in \mathcal{N}. \end{aligned}$$

Then $[\alpha(x), t]y = 0$ for all $y \in I$, $t \in N$. By Lemma 2, we obtain $\alpha(x) \in Z(\mathcal{N})$, but α is an automorphism, so $x \in Z(\mathcal{N})$. □

3 Commutativity conditions and (α, β)-derivations

In this section, $\mathcal{N}$ is assumed to be a zero symmetric near-ring and $\alpha, \beta : \mathcal{N} \to \mathcal{N}$ are automorphisms.

Our next theorem is a generalization of [1, Theorem 3.1] and [7, Theorem 2.9].

Theorem 1. *Let $\mathcal{N}$ be a 3-prime near-ring. If I is a nonzero semigroup ideal and d is a nonzero (α,β)-derivation on $\mathcal{N}$, then the following assertions are equivalent:*

(i) $[x,y]_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x,y \in I$*;*

(ii) $[d(x),y]_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x,y \in I$*;*

(iii) $\mathcal{N}$ *is a commutative ring.*

Proof. $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are obvious.

$(i) \Rightarrow (iii)$ Assume that

$$[x,y]_{\alpha,\beta} \in Z(\mathcal{N}) \quad \text{for all } \ x,y \in I. \tag{1}$$

Replacing y by $y\beta(x)$ in (1) and noting that $[x,y\beta(x)]_{\alpha,\beta} = [x,y]_{\alpha,\beta}\beta(x)$, we get

$$[x,y]_{\alpha,\beta}\beta(x) \in Z(\mathcal{N}) \quad \text{for all } \ x,y \in I. \tag{2}$$

By Lemma 1 (ii), we conclude that for each $x \in I$, we have

$$[x,y]_{\alpha,\beta} = 0 \ \text{ or } \ \beta(x) \in Z(\mathcal{N}) \quad \text{for all } \ x,y \in I. \tag{3}$$

But β is an automorphism, so (3 implies that

$$[x,y]_{\alpha,\beta} = 0 \ \text{ or } \ x \in Z(\mathcal{N}) \quad \text{for all } \ x,y \in I. \tag{4}$$

By Lemma 11, we get $x \in Z(\mathcal{N})$ for all $x \in I$, i.e $I \subseteq Z(\mathcal{N})$. Hence $\mathcal{N}$ is a commutative ring by Lemma 4.

The proof of $(ii) \Rightarrow (iii)$ is by the same way of the proof of $(i) \Rightarrow (iii)$, and use Lemma 7 instead of Lemma 4. □

It is worthy noticing that the results of Theorem 1 generalizes [1, Theorem 3.1], if we put $\beta = id_{\mathcal{N}}$, and [7, Theorem 2.9], if we put $\alpha = \beta = id_{\mathcal{N}}$.

If $\mathcal{N}$ is 2-torsion free, Theorem 1 stays legitimate if we replace $[x,y]_{\alpha,\beta}$ by $(x \circ y)_{\alpha,\beta}$. In fact, we obtain the following result:

The next theorem is a generalization of [1, Theorem 3.5] and [7, Theorem 2.10].

Theorem 2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If I is a nonzero semigroup ideal and d is a nonzero (α,α)-derivation on $\mathcal{N}$, then the following assertions are equivalent:*

(i) $(x \circ y)_{\alpha,\alpha} \in Z(\mathcal{N})$ *for all* $x,y \in I$*;*

(ii) $(d(x) \circ y)_{\alpha,\alpha} \in Z(\mathcal{N})$ *for all* $x,y \in I$*;*

(iii) $\mathcal{N}$ *is a commutative ring.*

Proof. $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are obvious.

The proof of part $(i) \Rightarrow (ii)$ of Theorem 2 is the same as the proof of $(i) \Rightarrow (iii)$ of Theorem 1 with the same steps.

$(ii) \Rightarrow (iii)$ Assume that

$$(d(x) \circ y)_{\alpha,\alpha} \in Z(\mathcal{N}) \quad \text{for all} \ \ x, y \in I. \tag{5}$$

As above replacing y by $y\alpha(d(x))$ in (5), we get

$$(d(x) \circ y)_{\alpha,\alpha}\alpha(d(x)) \in Z(\mathcal{N}) \quad \text{for all} \ \ x, y \in I. \tag{6}$$

By Lemma (1) (ii), we conclude that

$$(d(x) \circ y)_{\alpha,\alpha} = 0 \text{ or } \alpha(d(x)) \in Z(\mathcal{N}) \quad \text{for all} \ \ x, y \in I. \tag{7}$$

Again (7) implies that

$$(d(x) \circ y)_{\alpha,\alpha} = 0 \text{ or } \ d(x) \in Z(\mathcal{N}) \quad \text{for all} \ \ x, y \in I. \tag{8}$$

Assume there exists $x_0 \in I$ such that $d(x_0) \in Z(\mathcal{N})$. Since α is an automorphism of $\mathcal{N}$, $\alpha(d(x_0)) \in Z(\mathcal{N})$. Then (5) implies $(y+y)\alpha(d(x_0)) \in Z(\mathcal{N})$ for all $y \in I$. By Lemma 1 (ii), we obtain $\alpha(d(x_0)) = 0$ or $y + y \in Z(\mathcal{N})$ for all $y \in I$ which implies that $d(x_0) = 0$ or $(y + y)y = y^2 + y^2 \in Z(\mathcal{N})$ for all $y \in I$. Using again Lemma 1 (ii) with 2-torsion freeness of $\mathcal{N}$, we get $d(x_0) = 0$ or $y \in Z(\mathcal{N})$ for all $y \in I$ which means that $d(x_0) = 0$ or $I \subseteq Z(\mathcal{N})$. By Lemma 4, we conclude that $d(x_0) = 0$ or $\mathcal{N}$ is a commutative ring. In this case (8) becomes

$$(d(x) \circ y)_{\alpha,\alpha} = 0 \quad \text{for all} \ \ x, y \in I \ \ \text{or} \ \mathcal{N} \ \ \text{is a commutative ring.}$$

If $(d(x) \circ y)_{\alpha,\alpha} = 0$ for all $x, y \in I$. We get $\alpha(d(x))y = -y\alpha(d(x))$ for all $x, y \in I$. Putting yt in place of y, we obtain

$$\begin{aligned} \alpha(d(-x))yt &= yt\alpha(d(x)) \\ &= y(t\alpha(d(x)) \\ &= y\alpha(d(-x))t \quad \text{for all} \ \ x, y, t \in I, \end{aligned}$$

which implies that $\alpha(d(-x))y - y\alpha(d(-x))I = \{0\}$ for all $x, y \in I$. As a consequence, $\alpha(d(-x))y = y\alpha(d(-x))$ for all $x, y \in I$. Replacing y by ny, where $n \in \mathcal{N}$ in the last expression and using it again, we arrive at $\alpha(d(-x)) \in Z(\mathcal{N})$ for all $x \in I$. Since α is an automorphism of $\mathcal{N}$, we obtain $d(-x) \in Z(\mathcal{N})$ for all $x \in I$. i.e. $d(-I) \subseteq Z(\mathcal{N})$ and $\mathcal{N}$ is a commutative ring by Lemma 7. □

Note that we can be obtain [1, Theorem 3.5] and [7, Theorem 2.10] from Theorem 2 by choosing $\alpha = id_{\mathcal{N}}$.

The following example shows that one cannot discard the 3-primeness hypothesis in Theorems 1 and 2.

Example 3. Let S be a 2-torsion free zero-symmetric near-ring which is not abelian.

Let us defined $\mathcal{N}, I$ and $d, \alpha, \beta : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \left(\begin{array}{ccc} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{array} \right) \middle| \, x, y \in S \right\}, \quad I = \left\{ \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right) \middle| \, y \in S \right\},$$

$$d \left(\begin{array}{ccc} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{array} \right) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right),$$

$$\alpha \left(\begin{array}{ccc} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{array} \right) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ y & 0 & x \\ 0 & 0 & 0 \end{array} \right) \text{ and } \beta = id_{\mathcal{N}}.$$

It is clear that $\mathcal{N}$ is a 2-torsion free non 3-prime near-ring and I is a nonzero semigroup ideal of $\mathcal{N}$. Moreover, d is a nonzero (α, β)-derivation of $\mathcal{N}$ satisfying the conditions:

$$[A, B]_{\alpha,\beta},\ [d(A), B]_{\alpha,\beta},\ (A \circ B)_{\alpha,\beta},\ (d(A) \circ B)_{\alpha,\beta} \in Z(\mathcal{N}) \quad \text{for all} A, B \in I,$$

but $\mathcal{N}$ is not a commutative ring.

4 Commutativity conditions and generalized (α, β)-derivations

In this section, $\mathcal{N}$ is assumed to be a zero symmetric near-ring and $\alpha, \beta : \mathcal{N} \to \mathcal{N}$ are automorphisms.

The next theorem is a generalization of [7, Theorem 3.1].

Theorem 3. *Let $\mathcal{N}$ be a 3-prime near-ring and I is a nonzero semigroup ideal. If $\mathcal{N}$ admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $F([x, y]) = [d(x), \beta(y)]$ for all $x, y \in I$, then $\mathcal{N}$ is a commutative ring.*

Proof. Assume that

$$F([x, y]) = [d(x), \beta(y)] \quad \text{for all } x, y \in I. \tag{9}$$

Replacing y by yx in (9), we get

$$[d(x), \beta(yx)] = F([x, yx]) = F([x, y]x) \quad \text{for all } x, y \in I. \tag{10}$$

Moreover, since $[d(x), \beta(x)] = 0$ for all $x \in I$. So

$$[d(x), \beta(yx)] = [d(x), \beta(y)]\beta(x) = F([x, y])\beta(x) \quad \text{for all } x, y \in I. \tag{11}$$

From (10) and (11), we get

$$F([x, y]x) = F([x, y])\beta(x) = F([x, y])\beta(x) + \alpha([x, y])d(x), \quad \text{for all } x, y \in I.$$

So $\alpha([x, y])d(x) = 0$ for all $x, y \in I$. But α is an automorphism, so

$$([x, y])\alpha^{-1}(d(x)) = 0 \quad \text{for all } x, y \in I. \tag{12}$$

Substituting zy for y in (12), where $z \in \mathcal{N}$, and use it to get

$$\begin{aligned}(xzy\alpha^{-1}(d(x)) &= zyx\alpha^{-1}(d(x)) \\ &= zxy\alpha^{-1}(d(x)) \quad \text{for all } \ x,y \in I, z \in \mathcal{N}.\end{aligned}$$

So $[x,z]I\alpha^{-1}(d(x)) = 0$ for all $x \in I, z \in \mathcal{N}$. It follows that

$$x \in Z(\mathcal{N}) \text{ or } \ d(x) = 0 \quad \text{for all } \ x \in I. \tag{13}$$

Suppose there is $x_0 \in I$ such that $x_0 \in I \cap Z(\mathcal{N})$, then from (9), it is clear that $0 = F([x_0, y]) = [d(x_0), \beta(y)]$ for all $y \in I$. So $d(x_0)\beta(y) = \beta(y)d(x_0)$ for all $y \in I$. Since β is an automorphism, then $d(x_0)y = yd(x_0)$ for all $y \in I$ which implies that $d(x_0)$ centralizes I and $d(x_0) \in Z(\mathcal{N})$ by Lemma 1(iii). According to (13), we conclude that $d(I) \subseteq Z(\mathcal{N})$, and hence $\mathcal{N}$ is a commutative ring by application of Lemma 7. □

Take $F = d$ in Theorem 3, we obtain the following corollary:

Corollary 1. *Let $\mathcal{N}$ be a 3-prime near-ring and I is a nonzero semigroup ideal. If $\mathcal{N}$ admits a nonzero (α,β)-derivation d such that $d([x,y]) = [d(x),\beta(y)]$ for all $x,y \in I$, then $\mathcal{N}$ is a commutative ring.*

If we put $\beta = id_{\mathcal{N}}, F = d$ in Theorem 3, we obtain the following result:

Corollary 2. *Let $\mathcal{N}$ be a 3-prime near-ring and I is a nonzero semigroup ideal. If $\mathcal{N}$ admits a nonzero $(\alpha,1)$-derivation d, such that $d([x,y]) = [d(x),y]$ for all $x,y \in I$, then $\mathcal{N}$ is a commutative ring.*

Note that if we take $\alpha = \beta = id_{\mathcal{N}}$ in Theorem 3, we get [7, Theorem 4.1]. The next theorem is a generalization of [7, Theorem 3.2].

Theorem 4. *Let $\mathcal{N}$ be a 3-prime near-ring and I is a nonzero semigroup ideal. If $\mathcal{N}$ admits a generalized (α,β)-derivation F associated with a nonzero (α,β)-derivation d such that $d([x,y]) = [F(x),\beta(y)]$ for all $x,y \in I$, then $\mathcal{N}$ is a commutative ring.*

Proof. As in the proof of Theorem 3, we get $[x,z]I\alpha^{-1}(d(x)) = 0$ for all $x \in I$ and $z \in \mathcal{N}$. Therefore

$$x \in Z(\mathcal{N}) \text{ or } \ d(x) = 0 \quad \text{for all } \ x \in I. \tag{14}$$

Suppose there exists $x_0 \in I \cap Z(\mathcal{N})$, then $F(x_0) \in Z(\mathcal{N})$ and $F(x_0^2) \in Z(\mathcal{N})$. So $F(x_0^2) = F(x_0)\beta(x_0) + \alpha(x_0)d(x_0) \in Z(\mathcal{N})$. But $\alpha(x_0), \beta(x_0)$ and $F(x_0)$ are in $Z(\mathcal{N})$ for all $x \in I \cap Z(\mathcal{N})$. Thus by lemmas 8 and 9, we get $\alpha(x_0)d(x_0) \in Z(\mathcal{N})$. By Lemma 1 (ii), we obtain either $\alpha(x_0) = 0$ or $d(x_0) \in Z(\mathcal{N})$. Since α is an automorphism, then (14) becomes $d(x) \in Z(\mathcal{N})$ for all $x \in I$. So $d(I) \subseteq Z(\mathcal{N})$ and $\mathcal{N}$ is a commutative ring by Lemma 7. □

Not that if we take $\alpha = \beta = id_{\mathcal{N}}$ in Theorem (4), we obtain [7, Theorem 4.2]. We now concentrate practically equivalent to conditions including anticommutators $x \circ y$. Our next theorem is a generalization of [7, Theorem 3.3].

Theorem 5. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and I a nonzero semigroup ideal. Then $\mathcal{N}$ admits no generalized (α,β)-derivation F with associated an (α,β)-derivation d such that $d(Z(\mathcal{N})) \neq \{0\}$ and $d(x \circ y) = F(x) \circ \beta(y)$ for all $x, y \in I$.*

Proof. Assume that

$$d(x \circ y) = F(x) \circ \beta(y) \quad \text{for all } \ x, y \in I. \tag{15}$$

Let $z \in Z(\mathcal{N})$ such that $d(z) \neq 0$. Replace y by zy in (15), so we obtain

$$(F(x) \circ \beta(y))\beta(z) = d((x \circ y)z) \quad \text{for all } \ x, y \in I. \tag{16}$$

So we get

$$\begin{aligned} d(x \circ y)\beta(z) &= d((x \circ y)z) \\ &= d(x \circ y)\beta(z) + \alpha(x \circ y)d(z) \quad \text{for all } \ x, y \in I \end{aligned}$$

So that $\alpha(x \circ y)d(z) = 0$ for all $x, y \in I$. But $d(z) \in Z(\mathcal{N}) - \{0\}$, then $\alpha(x \circ y) = 0$ for all $x, y \in I$ i.e $x \circ y = 0$ for all $x, y \in I$, so with tensionless this contradicts with [7, Lemma 2.8]. □

The following theorem is a generalization of [7, Theorem 3.5].

Theorem 6. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and I a nonzero semigroup ideal. Then there exists no generalized (α,β)-derivation F with associated nonzero (α,β)-derivation d such that $[d(x), \beta(x)] = 0$ and $d(x) \circ \beta(y) = F(x \circ y)$ for all $x, y \in I$.*

Proof. Assume that

$$[d(x), \beta(x)] = 0 \text{ and } \ d(x) \circ \beta(y) = F(x \circ y) \quad \text{for all } \ x, y \in I. \tag{17}$$

Replacing y by yx in (17), we get

$$d(x) \circ \beta(yx) = F((x \circ y)x) \quad \text{for all } \ x, y \in I. \tag{18}$$

Since $F((x \circ y)x) = F((x \circ y))\beta(x) + \alpha((x \circ y))d(x)$ for all $x, y \in I$. So (17) and (18) yields

$$\begin{aligned} d(x) \circ \beta(yx) &= (d(x) \circ \beta(y))\beta(x) \\ &= (d(x) \circ \beta(y))\beta(x) + \alpha((x \circ y))d(x) \end{aligned}$$

Which reduces to

$$xy\alpha^{-1}(d(x)) = -yx\alpha^{-1}(d(x)) \quad \text{for all } \ x, y \in I. \tag{19}$$

Replacing y by zy in (19), where $z \in \mathcal{N}$, and use it to get

$$\begin{aligned} -xzy\alpha^{-1}(d(x)) &= zyx\alpha^{-1}(d(x)) \\ &= z(-xy\alpha^{-1}(d(x))) \\ &= z(-x)y\alpha^{-1}(d(x)) \quad \text{for all } \ x, y \in I, z \in \mathcal{N} \end{aligned}$$

which implies that

$$[-x, z]I\alpha^{-1}(d(x)) = \{0\} \quad \text{for all } \ x \in I, z \in \mathcal{N}.$$

It follows that

$$-x \in Z(\mathcal{N}) \text{ or } \ d(x) = 0 \quad \text{for all } \ x \in I. \tag{20}$$

Suppose there exists $x_0 \in I$ such that $-x_0 \in Z(\mathcal{N})$. Using our hypothesis, we obtain $d(x_0) \circ \beta(x_0^2) = F((x_0 \circ x_0)x_0)$ which implies that

$$(d(x_0) \circ \beta(x_0))\beta(x_0) = F(x_0 \circ x_0)\beta(x_0) + \alpha(x_0 \circ x_0)d(x_0).$$

Using (17) it is easy to get $\alpha(x_0 \circ x_0)d(x_0)$. By 2-torsion freeness together with the fact that α is an automorphism of $\mathcal{N}$, we can conclude that $x_0^2\alpha^{-1}(d(x_0)) = 0$. Since $-x_0 \in Z(\mathcal{N})$, it is clear that $(-x_0)^2 = x_0^2$ it follows that $(-x_0)^2\alpha^{-1}(d(x_0)) = 0$, so $(-x_0)\mathcal{N}(-x_0)\mathcal{N}\alpha^{-1}(d(x_0)) = \{0\}$. By 3-primeness of $\mathcal{N}$, it is obvious that $\alpha^{-1}(d(x_0)) = 0$ and therefore $d(x_0) = 0$. In all cases $d(x) = 0$ for all $x \in I$ which is a contradiction with our assumption. □

The following example shows that the 3-primeness hypothesis in Theorems 3–6 cannot be discarded.

Example 4. Let S be a 2-torsion free zero-symmetric near-ring which is not abelian. Let us defined $\mathcal{N}, I$ and $d, F, \alpha, \beta : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} \middle| \, x, y \in S \right\}, \quad I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} \middle| \, y \in S \right\},$$

$$F = d, \quad d \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix},$$

$$\alpha = id_{\mathcal{N}} \quad \text{and} \quad \beta \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} = \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & 0 \\ 0 & x & 0 \end{pmatrix}.$$

It is clear that $\mathcal{N}$ is a 2-torsion free non 3-prime near-ring, I a nonzero semigroup ideal of $\mathcal{N}$ and F is a generalized (α, β)-derivation associated with a nonzero (α, β)-derivation d such that:

$$F([A, B]) = [d(A), \beta(B)], \quad d([A, B]) = [F(A), \beta(B)], \quad [d(A), \beta(A)] = 0,$$
$$F(A \circ B) = d(A) \circ \beta(B), \quad d(A \circ B) = F(A) \circ \beta(B),$$

for all $A, B \in I$, but $\mathcal{N}$ is not a commutative ring.

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