On birational automorphisms of Severi–Brauer surfaces

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Abstract. The generators of the group of birational automorphisms of any Severi-Brauer surface non-isomorphic over an algebraically non-closed field to the projective plane are explicitly described.

The classical theorem by M. Noether states that the group of birational automorphisms of the projective plane $\mathbf{P}^2_{\overline{k}}$ over an algebraically closed field $\overline{k}$ is generated by projective automorphisms and standard quadratic Cremona transformations (see [1, Ch. V, §§ 5,6]). In a generalization of this theorem to non-closed fields it is natural to consider, together with projective plane, the *Severi–Brauer surfaces*, i.e., surfaces defined over non-closed fields which become isomorphic to the projective plane if their field of definition is lifted up to its algebraic closure.

The arising situation has a sapid cohomologic interpretation. Let $\overline{G} = \mathrm{Gal}(\overline{k}/k)$ be the Galois group of the algebraic closure. Classes of Severi–Brauer surfaces over k can be identified with elements of the space $H^1(\overline{G}; PGL_3(\overline{k}))$, up to a k-isomorphism. It is known (see [14]) that the exact sequence of groups

$$1 \longrightarrow \overline{k}^\times \longrightarrow GL_3(\overline{k}) \longrightarrow PGL_3(\overline{k}) \longrightarrow 1$$

induces an embedding $H^1(\overline{G}; PGL_3(\overline{k})) \xrightarrow{\delta} \mathrm{Br}(k)$, where $\mathrm{Br}(k) = H^2(\overline{G}, \overline{k}^\times)$ is the Brauer group of classes (up to an equivalence) of central simple k-algebras. The image of the embedding δ consists exactly of the elements $\gamma \in \mathrm{Br}(k)$ with Schur index (see [12]) which divides 3, i.e., is equal to either 1 (the projective plane) or 3 (a Severi-Brauer surface).

Let $S_2(\gamma)$ be the set of Severi–Brauer surfaces corresponding to γ, and let $V \in S_2(\gamma)$. The main result of this paper is a description of generators of the group of birational automorphisms of V if $\gamma \neq 1$, i.e., of Severi–Brauer surface non-isomorphic over k to the projective plane. It is interesting that this description requires a reference to $V' \in S_2(\gamma^{-1})$. The group of birational automorphisms of V contains a group of biregular automorphisms of V (described by Theorem 3 of [14]) as a subgroup.

This work had been carried out in 1970 as my MS Diploma thesis at the Department of Mechanics and Mathematics, Moscow State University. D. Leites translated it and preprinted in proceedings of his "Seminar on Supersymmetries" (Reports of the Department of Mathematics, Stockholm University, 33/1989-2).

Recently the above-mentioned preprint of this text was cited in an interesting paper by C. Shramov [17]. Since the result of this old work of mine is still useful, I decided to update my preprint, make it available by putting it in arXiv, and add to it comments I got meanwhile. First of all, from Torsten Ekedahl.

1   Birational maps of Severi–Brauer surfaces corresponding to points of degree 3.

Fix $V \in S_2(\gamma)$ and $V' \in S_2(\gamma^{-1})$. Theorem 2 of [14] and the theory of central simple algebras easily imply that there always exist points of degree 3 on V and V', there are no points of lesser degree, and degrees of all closed points are multiples of 3.

The aim of this section is to associate with every point $x \in V$ of degree 3 a birational map $\varphi_3(x) : V \longrightarrow V'$. In what follows I always assume that the characteristics of the field k over which V is defined is not 2 or 3. Thus, let x be a point of degree 3 belonging to V. Let us perform a monoidal transformation $\mathrm{dil}_x : V_1 \longrightarrow V$ with the center at x.

An exceptional curve of the first kind L which is different from $\mathrm{dil}_x^{-1}(x)$ belongs to V_1. Contracting this curve leads again to a Severi-Brauer surface, and we thus obtain a birational map $\beta_\alpha(x) : V \longrightarrow V'_\alpha$. By Theorem 5 of [14] either $V'_\alpha \in S_2(\gamma)$ or $V'_\alpha \in S_2(\gamma^{-1})$.

Lemma 1. $V'_\alpha \in S_2(\gamma^{-1})$.

Proof. Let $k(x)$ be the field of quotients of the local ring at point x and K the minimal normal extension of k containing $k(x)$. Let $G = \mathrm{Gal}(K/k)$ and $G_1 = \mathrm{Gal}(K/k(x))$. The natural homomorphism $H^2(G; K^\times) \longrightarrow \mathrm{Br}(k)$ is an embedding whose image contains γ. Since it will not cause a misunderstanding, the preimage of γ will be also denoted by γ. Let $\gamma \in H^2(G; K^\times)$ be represented by a system of factors $C_{\sigma,\tau} \in K^\times$, where $\sigma, \tau \in G$. Let (K, G, γ) denote the k-algebra given by the system of relations

$$\begin{cases} u_\sigma u_\tau = u_{\sigma\tau} \cdot C_{\sigma,\tau} \\ cu_\sigma = u_\sigma \cdot c^\sigma, \qquad \text{where } c \in K^\times, \end{cases}$$

i.e., the twisted product of K by G relative the system $C_{\sigma,\tau}$.

Let M be an irreducible right (K, G, γ)-module. By Lemma 2 in §5 of [15] there exists $T \in M$ such that $Tu_{\sigma_1} = T \cdot c_{\sigma_1}$ for all $\sigma_1 \in G_1$, where $c_{\sigma_1} \in K^\times$. Set $T^{u_\rho} = T \cdot u_\rho$; we see that $M = \oplus_{\rho \in G \mod G_1} T^{u_\rho} \cdot K$.

Since char $k \neq 3$, the field $k(x)$ is separable over k, and hence the full system of representatives of classes $G \mod G_1$ contains 3 elements: ρ_0, ρ_1 and ρ_2. By setting $T_i := T^{u_{\rho_i}}$ we have $M = T_0 k \oplus T_1 k \oplus T_2 k$.

Let $\mathbf{P}_K^2 = \mathrm{Proj}\, K[T_0, T_1, T_2]$, let $\mathcal{O}(2)$ be the canonical invertible sheaf over $\mathbf{P}_K^2$ and let $\Gamma(\mathbf{P}_K^2, \mathcal{O}(2))$ be the K-vector space of its global sections. The subspace $M' \subset \Gamma(\mathbf{P}_K^2, \mathcal{O}(2))$, where

$$M' = T_0 T_1 K \oplus T_1 T_2 K \oplus T_0 T_2 K,$$

defines a linear system of conics on the plane and the choice of the basis in it given by $\{T_0 T_1, T_1 T_2, T_0 T_2\}$ defines a birational isomorphism $\overline{\beta} : \mathbf{P}_K^2 \longrightarrow \mathbf{P}_K^2$ which is the standard quadratic Cremona transformation with the center in the triple

$$x_1 = (0 : 0 : 1), \ x_2 = (0 : 1 : 0), \ x_3 = (1 : 0 : 0).$$

Let L_i be the line in the plane $\mathbf{P}_K^2$ containing 2 of these points different form x_i for $i = 1, 2, 3$. Evidently, the cycles $C_x = x_1 + x_2 + x_3$ and $C_L = L_1 + L_2 + L_3$ on the plane $\mathbf{P}_K^2$ are simple rational over k cycles with respect to the Galois group of the covering $\mathbf{P}_K^2 \cong V \otimes_k K \longrightarrow V$. Hence, the cycle C_x defines a closed point on V which can be considered coinciding with x. Results of [15, § 4] imply that the birational map $\overline{\beta}$ can be descended to a birational map $\beta : V \longrightarrow V''_\alpha$, where $V''_\alpha \in S_2(\gamma^{-1})$.

Due to the regularity of C_L it is clear that $\varphi_3(x)$ coincides, up to a biregular isomorphism $V'_\alpha \longrightarrow V''_\alpha$, with the map $\beta_\alpha(x) : V \longrightarrow V'_\alpha$ constructed earlier.   □

In March, 1985, A. Skorobogatov informed me of the following short proof of Lemma 1; in particular, this enables one to get rid of necessity to refer to [15].

By a theorem of Manin (see [9, Ch. IV]), $U = V \setminus C_L$ is a principal homogeneous space over 2-dimensional torus $T(K)$. Consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K^\times & \longrightarrow & GL_3(K) & \longrightarrow & PGL_3(K) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & K^\times & \longrightarrow & R_{K/k(G_m(K))} & \longrightarrow & T(K) & \longrightarrow & 1
\end{array}
$$

where $R_{K/k}$ is the Weil functor, $G_m(K)$ is the group of K-points of the multiplicative group, see [16]. Passing to the Galois cohomology we get, thanks to Hilbert's Satz 90, the following diagram

$$
\begin{array}{ccc}
1 \longrightarrow & H^1(G; PGL_3(K)) & \\
& \nearrow \qquad \searrow & \\
& & H^2(G; K^\times) \subset \mathrm{Br}(k) \\
& \searrow \qquad \nearrow & \\
1 \longrightarrow & H^1(G; T(K)) &
\end{array}
$$

The considered (Cremona) transformation acts on U and being lifted to $T(K)$ is the inversion $x \mapsto x^{-1}$. Hence, in the group of principal homogeneous spaces, $H^1(G; T(K))$, this transformation induces an inversion, and therefore it does the same in the Brauer group. Lemma 1 is proved once more.

Now, take a biregular isomorphism $\psi_\alpha : V'_\alpha \longrightarrow V'$ and define a birational map

$$\varphi_3^\alpha(x) = \psi_\alpha \circ \beta_\alpha(x) : V \longrightarrow V'.$$

The element $\varphi_3^\alpha(x)$ of the set $\mathrm{Bir}(V, V')$ of birational maps is defined uniquely up to an action of the group $\mathrm{Aut}(V')$ of biregular automorphisms of V' on the set $\mathrm{Bir}(V, V')$.

Now, let $Z(V) = Z(\gamma)$ be the group of cycles on V (see [8]). It can be represented as a direct sum $Z(\gamma) = \mathrm{Pic}(V) \oplus Z^0(\gamma)$, where $Z_0(\gamma)$ is the group of 0-dimensional cycles on V.

It will be convenient for us to describe every 0-dimensional cycle on V as a simple rational cycle on $V \otimes_k \overline{k} \cong \mathbf{P}_{\overline{k}}^2$. The Picard group of V is isomorphic to a free cyclic group with the anticanonical class as a generator, i.e., $\mathrm{Pic}(V) \cong \mathbf{Z}(-\omega_\gamma)$.

Lemma 2. *Let $\varphi_3(x)_* : Z(\gamma) \longrightarrow Z(\gamma^{-1})$ be the homomorphism induced by $\varphi_3(x)$, let*

$$\alpha = -d\omega_\gamma - b(x_1 + x_2 + x_3) - \sum_{i \geqslant 4} b_i x_i \in Z(V).$$

Then,

$$\varphi_3(x)_*(\alpha) = -(2d - b)\omega_{\gamma^{-1}} - (3d - 2b)(x'_1 + x'_2 + x'_3) - \sum_{i \geqslant 4} b_i x'_i,$$

where $x'_1 + x'_2 + x'_3$ is a simple rational cycle on $V' \otimes_k \overline{k}$ and $V' \in S_2(\gamma^{-1})$; this curve is the image of the exceptional curve of the first kind under cont_L.

Proof. It is subject to a simple calculation, see [1]. $\qquad\square$

2   Birational maps of Severi–Brauer surfaces corresponding to points of degree 6

To every point $x \in V$ of degree 6 a birational map $\varphi_6(x) : V \longrightarrow V'$ can also be assigned, where V' is a fixed element of $S_2(\gamma^{-1})$. We will need the following statement.

Lemma 3. *Consider the fiber product $V \otimes_k \overline{k} \cong \mathbf{P}_{\overline{k}}^2$ and its projection onto the first factor $p : \mathbf{P}_{\overline{k}}^2 \to V$. Let $x \in V$ be a closed point of degree 6 and define $p^{-1}(x) := (x_0, \dots, x_5)$, where the x_i are closed points in $\mathbf{P}_{\overline{k}}^2$. Then, no 3 points x_i belong to one line and all 6 points do not belong to a conic.*

Proof. If no 3 points belong to one line and all 6 belong to a conic, then these points uniquely define this conic. This conic defines, on V, a simple rational cycle over k. The divisor corresponding to this cycle is of degree 2 contradicting Proposition 13 of [14].

Now suppose that there is a line in $\mathbf{P}_{\overline{k}}^2$ containing 3 points x_i. The totality of all lines with pair-wise distinct points x_i forms a cycle rational over k. The degree of the divisor corresponding to this cycle is equal to the number of these lines and due to [14, Proposition 13] should be a multiple of 3. Since the number of lines which connect pair-wise distinct points x_i does not depend on the order of these points on lines, it follows that from the combinatorial point of view only the following 9 cases are possible:

1) All 6 points belong to one line.

2) There is a line containing exactly 5 points.

3) There is a line containing exactly 4 points, and there is no line containing 3 points.

4) There exists a line containing exactly 4 points, and a line containing exactly 3 points.

5) There exist exactly two lines each of them containing 3 points, together they contain 5 points, and there are no lines containing more than 3 points different from these two lines.

5+i) There exist exactly i lines, where $i = 1, 2, 3$ or 4, each of them containing exactly 3 points, there are no lines containing more points, and the case 5) fails.

Let n_j be the number of lines connecting pairs of points of the set in case j). Then, it is easy to see that the following relations hold:

j	1	2	3	4	5	6	7	8	9
n_j	1	6	10	8	11	13	11	9	7

Due to the above, it follows that only cases 2) and 8) can hold.

Case 2). Let $x_1, \ldots, x_5$ be points belonging to the line L. By transitivity of the $\mathrm{Gal}(\overline{k}/k)$-action on $(x_1, \ldots, x_6)$ we can find $g \in \mathrm{Gal}(\overline{k}/k)$ such that $g(x_1) = x_6$. Then, g transforms L into a line passing through x_6. But L contains 5 points and not all 6 points belong to one line, hence a contradiction.

Case 8). Let L_1, L_2, and L_3 be lines passing through x_1, x_2, x_3, through x_3, x_4, x_5, and through x_5, x_6, x_1, respectively. We have $g(L_i) = L_j$, where $i, j = 1, 2, 3$, for any $g \in \mathrm{Gal}(\overline{k}/k)$. Moreover, $g(L_i \cap L_j) = L_{i'} \cap L_{j'}$ since if it is not so, then through the point not representable in the form $L_i \cap L_j$ two lines from the set $\{L_1, L_2, L_3\}$ pass, but then $\mathrm{Gal}(\overline{k}/k)$ does not act transitively on $\{x_1, \ldots, x_6\}$ contradicting the simplicity of the cycle $x_1 + \ldots + x_6$. □

By Lemma 3 to any point $x \in V$ such that $\deg_k(x) = 6$ we can assign a birational map $\varphi_6(x) : V \longrightarrow V'$.

Let $\mathrm{dil}_x : \overline{V} \longrightarrow \mathbf{P}^2_{\overline{k}}$ be a monoidal transformation with the center at the points $x_1, x_2, \ldots, x_6$, and let Q_i be a conic in $\mathbf{P}^2_{\overline{k}}$ containing $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_6\}$. Let S_i be the proper preimage of Q_i with respect to dil_x. The curve $S = \underset{0 \leqslant i \leqslant 6}{\cup} S_i$ is contractible since it is an exceptional curve of the first kind. Let $\mathrm{cont}_S : V \longrightarrow \mathbf{P}^2_{\overline{k}}$ be a contraction morphism of S and $\overline{x}_i = \mathrm{cont}_S(S_i)$. Thus, we have a birational isomorphism

$$\overline{\beta} = \mathrm{cont}_S \cdot \mathrm{dil}_x^{-1} : \mathbf{P}^2_{\overline{k}} \longrightarrow \mathbf{P}^2_{\overline{k}}.$$

It is easy to compute the value of the homomorphism $\overline{\varphi}_6(x)_* : Z(\mathbf{P}^2_{\overline{k}}) \longrightarrow Z(\mathbf{P}^2_{\overline{k}})$ at the anticanonical class $-\omega_{\mathbf{P}^2_{\overline{k}}} = -\omega$:

$$\overline{\beta}_*(-\omega) = -5\omega - 6(\overline{x}_1 + \ldots + \overline{x}_6). \tag{1}$$

Since $Q_1 + \ldots + Q_6$ is a simple and rational cycle over k, then so is $S_1 + \cdots + S_6$. Hence, $\overline{\varphi}_6(x)$ can descend to a birational isomorphism $\beta_\alpha(x) : V \longrightarrow V'_\alpha$ which is a composition of the blowing up of the point $x \in V$ such that $\deg_k(x) = 6$ and a contraction of the exceptional curve S of the first kind. By Theorem 5 of [14] either $V'_\alpha \in S_2(\gamma)$ or $V'_\alpha \in S_2(\gamma^{-1})$.

Lemma 4. $V'_\alpha \in S_2(\gamma^{-1})$.

Proof. Let $k(x)$ be the field of quotients of the local ring at point x. Consider the minimal normal extension K of the field k such that $k \subset k(x) \subset K$. Since $[k(x) : k] = 6$, it follows that $\mathrm{Gal}(K/k)$ contains at least one Sylow 3-subgroup. Let $G(3) \subset \mathrm{Gal}(\overline{k}/k)$ be a Sylow 3-subgroup, and $K_3 \subset K$ the subfield of elements fixed under $G(3)$.

Set $G = \mathrm{Gal}(\overline{k}/k)$ and $G_1 = \mathrm{Gal}(\overline{k}/K_3)$. Consider the restriction homomorphism

$$\mathrm{res} : H^2(G; \overline{k}^\times) \longrightarrow H^2(G_1; \overline{k}^\times).$$

Let
$$\mathrm{res}(\gamma) = \gamma_1.$$
Since $[K_3 : k] \not\equiv 0 \mod 3$, then $\gamma_1 \neq 1$. Setting $V_1 = V \otimes_k K_3$ we see that $V_1 \in S_2(\gamma_1)$. The fiber over x of the projection $p_1 : V_1 \longrightarrow V$ is isomorphic to
$$\mathrm{Spec}(K_3) \underset{\mathrm{Spec}(k)}{\times} \mathrm{Spec}(k(x)) = \mathrm{Spec}(K_3 \otimes_k k(x)).$$
It follows from the construction that
$$\mathrm{Spec}(K_3 \otimes_k k(x)) = \mathrm{Spec}(K_1) \amalg \mathrm{Spec}(K_2),$$
where $[K_i : K_3] = 3$ for $i = 1, 2$. Thus, $p^{-1}(x_0) = \{y_1, y_2\}$ and $\deg_{K_3}(y_i) = 3$ for $i = 1, 2$. Since the diagram

$$
\begin{array}{ccc}
 & \mathbf{P}^2_{\overline{k}} & \\
 {\scriptstyle p}\swarrow & & \searrow{\scriptstyle p} \\
V_2 & \xrightarrow{\ \ p_1\ \ } & V_2
\end{array}
$$

is commutative, we can assume that
$$p^{-1}(y_1) = \{x_1, x_2, x_3\} \text{ and } p^{-1}(y_2) = \{x_4, x_5, x_6\}.$$

Now suppose Lemma 4 fails. Then, $V'_\alpha \in S_2(\gamma)$ and we have a birational isomorphism
$$\beta^1_\alpha = \beta_a(x) \otimes_k K_3 : V_1 \longrightarrow X_1,$$
where $X_1 \in S_2(\gamma_1)$. By eq.(1)
$$\beta^1_{\alpha*}(-\omega_{\gamma_1}) = -5\omega_{\gamma_1} - 6(\overline{x}_1 + \ldots + \overline{x}_6).$$
For $X'_1 \in S_2(\gamma_1^{-1})$ and $\overline{y}_0 = \{\overline{x}_1, \overline{x}_2, \overline{x}_3\} \in X_1$, if follows from Lemma 2 and the birational isomorphism $\psi_3(\overline{y}_0) : X_1 \to X'_1$ that
$$\varphi_3(\overline{y}_0)_* \circ \beta'_{\alpha*}(-\omega_{\gamma_1}) = -4\omega_{\gamma_1^{-1}} - 3(\overline{x}'_1 + \overline{x}'_2 + \overline{x}'_3) - 6(\overline{x}'_4 + \overline{x}'_5 + \overline{x}'_6).$$
Then, for the birational isomorphism $\varphi_3(\overline{y}'_0) : X'_1 \to X_1$, where $\overline{y}'_0 = \{\overline{x}'_4, \overline{x}'_5, \overline{x}'_6\}$, we have
$$\varphi_3(\overline{y}'_0)_* \circ \varphi_3(\overline{y}_0)_* \circ \beta'_{\alpha*}(-\omega_{\gamma_1}) = -2\omega_{\gamma_1} - 3(\overline{x}''_1 + \overline{x}''_2 + \overline{x}''_3).$$
Finally, for the birational isomorphism $\varphi_3(\overline{y}''_0) : X_1 \to X'_1$, where $y''_0 = \{\overline{x}''_1, \overline{x}''_2, \overline{x}''_3\}$, we have
$$\varphi_3(\overline{y}''_0) \circ \varphi_3(\overline{y}'_0)_* \circ \varphi_3(\overline{y}_0)_* \circ \beta^1_{\alpha*}(-\omega_{\gamma_1}) = -\omega_{\gamma_1^{-1}}.$$
Thus, we got a biregular isomorphism $V_1 \longrightarrow X'_1$, where $X'_1 \in S_2(\gamma_1^{-1})$. This is a contradiction. $\qquad\square$

As it had been done with points of degree 3, we will associate with every $x \in V$ a birational map $\varphi_6(x) : V \longrightarrow V'$.

Lemma 5. *Let $\varphi_6(x)_* : Z(\gamma) \longrightarrow Z(\gamma^{-1})$ be the homomorphism induced by $\varphi_6(x)$. Then,*

$$\varphi_6(x)_*\Big(-d\omega_\gamma - b(x_1 + \ldots + x_6) - \sum_{i \geq 7} b_i x_i\Big)$$
$$= -(5d - 4b)\omega_{\gamma^{-1}} - (6d - 5b)(x'_1 + \ldots + x'_6) - \sum_{i \geq 7} b_i x'_i,$$

where $s = x'_1 + \ldots + x'_6$ is a simple rational over k cycle on $V' \otimes_k k'$ with $V' \in S_2(\gamma^{-1})$ which is the image of contraction of the exceptional curve of the first kind defined by s.

Proof. It suffices to represent the map $\overline{\varphi}_6(x) : \mathbf{P}^2_{\overline{k}} \longrightarrow \mathbf{P}^2_{\overline{k}}$ as a product of 3 quadratic Cremona transformations. $\qquad\square$

$\square$

3   Proof of the main theorem

Let $V \in S_2(\gamma)$ and $V' \in S_2(\gamma^{-1})$ be fixed surfaces. The said earlier can be summed up as follows: To every point $x \in V$ such that $\deg x = 3$ or 6 there uniquely corresponds an orbit of the left action of the group $\mathrm{Aut}(V')$ of biregular automorphisms on the set $\mathrm{Bir}(V, V')$ of birational maps $V \longrightarrow V'$. For every such point, we will choose, once and for all, an element of the corresponding orbit. It is a birational map $\varphi_i(x) : V \longrightarrow V'$, where $i = \deg_k x$. Similarly, for any $y \in V'$, we will construct a map $\varphi_j(y) : V' \longrightarrow V$.

Our goal is the proof of the following statement.

Theorem 1. *The group* $\mathrm{Bir}(V)$ *of birational automorphisms of* V *is generated by its subgroup of biregular automorphisms* $\mathrm{Aut}(V)$, *and automorphisms* $\varphi_i(x)$ *and* $\varphi_i(y)$ *for all* $x \in V$ *and* $y \in V'$ *described above.*

The proof of this theorem follows from a series of lemmas. Let $g \in \mathrm{Bir}(V)$ be an arbitrary birational automorphism and let the value of the homomorphism $f_* : Z(\gamma) \to Z(\gamma)$ at $-\omega_\gamma$ be equal to $-d\omega_\gamma - \sum b_i x_i$.

Lemma 6. *The following relations hold:*

$$9d^2 - \sum b_i^2 = 9; \tag{2}$$

$$9d - \sum b_i = 9. \tag{3}$$

Proof. Let us use the fact that f_* preserves the arithmetic genus and the index of intersection of cycles on V, see [8]. We have

$$\left(-d\omega_\gamma - \sum b_i x_i\right)^2 = 9d^2 - \sum b_i^2 = (-\omega_\gamma)^2 = 9$$

yielding relation (2). Further,

$$p_a\left(-d\omega_\gamma - \sum b_i x_i\right) = \frac{9}{2}(d^2 - d) + 1 - \frac{1}{2}\sum b_i(b_i - 1) = p_a(-\omega_\gamma) = 1$$

or $9d^2 - \sum b_i^2 = 9d - \sum b_i$. Taking relation (2) into account we get relation (3). □

Lemma 7. *Let* $b = \max b_i$. *Then,*

$$b \geqslant d + 1 \tag{4}$$

Proof. Indeed, relation (2) implies $9d^2 - b\sum b_i \leqslant 9$. Taking relation (3) into account we see that $9d^2 - 9 \leqslant b(9d - 9)$ yielding inequality (4). □

Lemma 8. *Let* x_{i_0} *be a point of the cycle* $f_*(-\omega_\gamma)$ *whose coefficient is* b, *i.e.,* x_{i_0} *is a point of maximal multiplicity. Then,* $\deg_k x_{i_0} < 9$.

Proof. Let $n = \deg_k x_{i_0}$. Then, relation (3) can be rewritten as $9d - nb - \sum_{i \neq i_0} b_i = 9$. Thanks to inequality (4) we get

$$9d - n(d+1) - \sum_{i \neq i_0} b_i \geq 9 \quad \text{or} \quad -\sum_{i \neq i_0} b_i \geqslant 9 - 9d - n(d+1).$$

Setting $n = 9 + 3l$ we rewrite the latter expression in the form

$$-\sum_{i \neq i_0} b_i \geqslant 18 + 3l(d+1)$$

which is false if $l \geqslant 0$ since $b_i \geqslant 0$, see [8, Corollary 1.18]. Hence, $l < 0$, and then $\deg_k x_{i_0} = n \leqslant 6$, as was required. □

Proof of Theorem 1. It is well known ([1]) that the point x of maximal multiplicity of the cycle $g_*(-\omega_\gamma)$ belongs to V. By Lemma 8 the degree of this point is equal to either 3 or 6. Therefore, applying the homomorphism $\varphi_i(x)$ to $g_*(-\omega_\gamma)$ we will diminish, thanks to Lemmas 2, 5, 7, the absolute value of the coefficient of $\omega_{\gamma^{-1}}$. Repeatedly applying this procedure we will diminish the degree d to 1. Then, $b_i = 0$ for all i, as follows from relation (2). Finally, we get either

$$\left\{ \prod f_{j_k, i_k}(y, x)_* \right\} \circ g_*(-\omega_\gamma) = -\omega_\gamma \tag{5}$$

or

$$\left\{ \varphi_i(x) \circ \prod f_{j_k, i_k}(y, x)_* \right\} \circ g_*(-\omega_\gamma) = -\omega_\gamma. \tag{6}$$

Formula (6) leads to a contradiction since V and $V' \in S_2(\gamma^{-1})$ are not biregularly isomorphic.

Formula (5) implies that $\{\prod f_{j_k, i_k}(y, x)\} \circ f \in \mathrm{Aut}(V)$. Applying transformations inverse to $f_{j_k, i_k}(y, x)$ and isomorphisms φ_i to the left-hand side, we get the required. $\qquad\square$

Remark 1. On automorphisms of similar (Del Pezzo) surfaces, see [7].

4   Appendix. T. Ekedahl's comments

Most part of your paper is devoted to proof of Lemmas 1 and 4. The proof is overcomplicated with a long and rather ugly division into cases in the proof of Lemma 3. It is possible to give a short, uniform and conceptual treatment of these lemmas.

The main idea is to exploit the simple fact the Picard group is of rank 2 for any k-surface X obtained by blowing up a closed point on a Severi–Brauer surface V. If we first look at Lemma 3, then X is obtained blowing up a closed point of order 6.

The statement of the lemma is equivalent to, and may be replaced by, any of the statements:

a) $-K_X$ is ample,

b) $-K_X$ is very ample,

c) X is isomorphic to a cubic surface (cf. the English version of Manin's book [9, Ch. IV, § 24]).

It is easy to verify the statement a) by using the ampleness criterion of Moishezon and Nakai (cf. proof of Statement 24.5.2 in op. cit. or p. 365 in Hartshorne's book [5]) and the fact that $\mathrm{rk}\,\mathrm{Pic}\,X = 2$.

It is well known (cf. the comment on "Schäfli's double-six" in Hartshorne's book [5]) that there is a natural set of six (conjugated) lines on X complementing the six exceptional $\bar{k}$-lines of $X \longrightarrow V$. In Lemma 4 you study the surface V' obtained by contracting these complementary lines. This is (cf. Manin's book [9]) a del Pezzo k-surface of degree 9, i.e., a Severi–Brauer k-surface. We have, therefore, two elements $\{V\}$ and $\{V'\}$ in $H^1_{\text{ét}}(k, PGL_3)$ corresponding to the k-isomorphism classes of V and V' (cf. Milne's book [11, p. 134]).

In Lemma 4 you consider the images $\langle V \rangle$ and $\langle V' \rangle$ of the elements $\{V\}$ and $\{V'\}$ in $H^2_{\text{ét}}(K, G_m)$ in the cohomology sequence corresponding to the exact sequence of étale sheaves (cf. [11, p. 142]):

$$1 \longrightarrow G_m \longrightarrow GL_3 \longrightarrow PGL_3 \longrightarrow 1.$$

You prove that these images are inverses of each other.

It is possible to give a much more natural approach and prove Lemmas 1 and 4 at the same time. To begin with, the definition of the Brauer–Severi schemes implies that $\langle V \rangle$ (resp. $\langle V' \rangle$) belongs to the kernel of the natural map from $H^2_{\text{ét}}(K, G_m)$ to $H^2_{\text{ét}}(V, G_m)$ (resp. to $H^2_{\text{ét}}(V', G_m)$): use the fact that the image of $\{V\}$ in $H^1_{\text{ét}}(V, PGL_3)$ comes from $H^1_{\text{ét}}(V, GL_3)$. But it is known ([11, p. 106]) that $H^2_{\text{ét}}(V, G_m)$ injects into $H^2_{\text{ét}}(K(V), G_m) = H^2_{\text{ét}}(\bar{k}(V'), G_m)$ and this implies that $\langle V' \rangle \in \mathrm{Ker}(H^2_{\text{ét}}(k, G_m) \longrightarrow H^2_{\text{ét}}(V, G_m))$.

The Hochschild–Serre spectral sequence

$$H^p_{\text{ét}}(k, H^q_{\text{ét}}(\overline{V}, G_m)) \Rightarrow H^{p+q}_{\text{ét}}(V, G_m),$$

see [11, p. 105], yields that

$$\mathrm{Ker}(H^2_{\text{ét}}(k, G_m) \longrightarrow H^2_{\text{ét}}(V, G_m)) = \mathbb{Z}/3\mathbb{Z},$$

and since $H^1_{\text{ét}}(k, GL_3) = 1$ (cf. [11, p. 124]) and $V(k) = V'(k) = \emptyset$, it follows that $\langle V \rangle$ and $\langle V' \rangle$ are non-trivial in $H^2_{\text{ét}}(k, G_m)$.

It suffices to prove that $\langle V \rangle \neq \langle V' \rangle$ in $H^2_{\text{ét}}(k, G_m)$. To show this, use the following commutative diagram of étale sheaves over V, V' and X (cf. [11, p. 143]):

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mu_3 & \longrightarrow & SL_3 & \longrightarrow & PGL_3 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & G_m & \longrightarrow & GL_3 & \longrightarrow & PGL_3 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle \det} & & & & \\
 & & G_m & = & G_m & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \qquad (7)$$

We then consider the image of $\{V\} \in H^1_{\text{ét}}(k, PGL_3)$ under the composite map

$$H^1_{\text{ét}}(k, PGL_3) \longrightarrow H^1_{\text{ét}}(V, PGL_3) \longrightarrow H^2_{\text{ét}}(V, \mu_3).$$

This image lies in the kernel of the map $H^2_{\text{ét}}(V, \mu_3) \longrightarrow H^2_{\text{ét}}(V, G_m)$ since the pullback of $\{V\}$ in $H^1_{\text{ét}}(V, PGL_3)$ comes from an element in $H^1_{\text{ét}}(V, GL_3)$. We have, therefore, by the first column a well-defined element $[D_\gamma]$ in $\operatorname{Pic} V / 3 \operatorname{Pic} V$ corresponding to the image of $\{V\}$ in $\ker(H^2_{\text{ét}}(V, \mu_3) \longrightarrow H^2_{\text{ét}}(V, G_m))$ and we obtain in the same way an element $[D_{\gamma'}]$ in $\operatorname{Pic} V' / 3 \operatorname{Pic} V'$ from $\{V'\}$. If $\{V\} = \{V'\}$ in $H^1_{\text{ét}}(k, PGL_3)$, then the pullbacks of $[D_\gamma]$ and $[D_{\gamma'}]$ in $\operatorname{Pic} X / 3 \operatorname{Pic} X$ must coincide.

But it is easy to compute $[D_\gamma]$ and $[D_{\gamma'}]$. We already noted that the image of $\{V\}$ in $H^1_{\text{ét}}(V, PGL_3)$ comes from $H^1_{\text{ét}}(V, GL_3) = H^1_{\text{zar}}(V, GL_3)$ and it is known (cf., e.g., the end of Quillen's article [13]) that one may choose an element in $H^1_{\text{zar}}(V, GL_3)$ corresponding to a vector bundle J_γ coming from a natural extension

$$0 \longrightarrow \Omega_\gamma \longrightarrow J_\gamma \longrightarrow \mathcal{O}_\gamma \longrightarrow 0,$$

where Ω_γ is the cotangent bundle of V. This implies by the diagram (7) that $[D_\gamma]$ is equal to the class of the line bundle $\det(J_\gamma)$, i.e., to the image $[-K_\gamma] \in \operatorname{Pic} V / 3 \operatorname{Pic} V$ of the anti-canonical class $-K_\gamma$. But it is easy to see that the images of $[-K_\gamma]$ and $[-K_{\gamma'}]$ in $\operatorname{Pic} X / 3 \operatorname{Pic} X$ do not coincide (here it is essential to consider $\operatorname{Pic} X / 3 \operatorname{Pic} X$ and not $\operatorname{Pic} \overline{X} / 3 \operatorname{Pic} \overline{X}$). This proves that $\{V\} \neq \{V'\}$, thereby completing the proof of Lemma 1. The same proof works for Lemma 4; the only difference being that X is of degree 6.

It would also be useful for the reader if you include more modern references. It would be valuable to have a reference to the recent survey article by Manin and Tsfasman [10], so that the reader can compare with other papers about birational automorphisms on rational varieties like the ones by Iskovskikh and Manin (see also more recent papers [4], [6]).

It would also be useful if you include a reference to the excellent article by M. Artin [3] (and also to the one by Amitsur [2]) about Severi–Brauer varieties in Springer LNM vol. 917. Finally, I recommend you to make a fuller use of étale cohomology (cf. Milne's book [11]) which is the natural language for many of the results and arguments of the paper.

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