Communications in Mathematics **30** (2022), no. 2, 1–12
DOI: https://doi.org/10.46298/cm.9274

Local derivations of semisimple Leibniz algebras

Ivan Kaygorodov, Karimbergen Kudaybergenov and Inomjon Yuldashev

Abstract. We prove that every local derivation on a complex semisimple finite-dimensional Leibniz algebra is a derivation.

The study of local derivations started with Kadison's article [22]. After this work, appear numerous new results related to the description of numerous local mappings (such that local derivations, 2-local derivations, bilocal derivations, bilocal Lie derivations, weak-2-local derivations, local automorphisms, 2-local Lie $*$-automorphisms, 2-local $*$-Lie isomorphisms and so on) of associative algebras (see, for example, [1], [3], [4], [5], [7], [8], [24]). The study of local and 2-local derivations of non-associative algebras was initiated in some papers of Ayupov and Kudaybergenov (for the case of Lie algebras, see [6], [7]). In particular, they proved that there are no non-trivial local and 2-local derivations on complex semisimple finite-dimensional Lie algebras. In [7] it is also given examples of 2-local derivations on nilpotent Lie algebras which are not derivations. After the cited works, the study of local and 2-local derivations was continued for many types of algebras, such that Leibniz algebras [8], Jordan algebras [3], n-ary algebras [19] and so on. The first example of a simple (ternary) algebra with non-trivial local derivations is constructed by Ferreira, Kaygorodov and Kudaybergenov in [19]. After that, the first example of a simple (binary) algebra with non-trivial local derivations/automorphisms was constructed by Ayupov, Elduque and Kudaybergenov in [4],[5]. The present paper is devoted to the study of local derivations of semisimple finite-dimensional Leibniz algebras.

This work is supported by the Russian Science Foundation under grant 18-71-10007.
MSC 2020: 17A32, 17A60, 17B10, 17B20.
Keywords: Lie algebra, Leibniz algebra, simple algebra, semisimple algebra, local derivation.
Affiliation:
Ivan Kaygorodov – Centro de Matemática e Aplicações, Universidade da Beira Interior, Covilhã, Portugal
Siberian Federal University, Krasnoyarsk, Russia
E-mail: kaygorodov.ivan@gmail.com
Karimbergen Kudaybergenov – V. I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan
Karakalpak State University, Nukus, Uzbekistan
E-mail: karim2006@mail.ru
Inomjon Yuldashev – Nukus State Pedagogical Institute, Nukus, Uzbekistan
E-mail: i.yuldashev1990@mail.ru

Definition 0.1. Let $\mathbf{A}$ be an algebra. A linear map $\Delta : \mathbf{A} \to \mathbf{A}$ is called a local derivation, if for any element $x \in \mathbf{A}$ there exists a derivation $\mathfrak{D}_x : \mathbf{A} \to \mathbf{A}$ such that $\Delta(x) = \mathfrak{D}_x(x)$.

1 Structure of semisimple Leibniz algebras and their derivations

1.1 Leibniz algebras

Leibniz algebras present a "non antisymmetric" generalization of Lie algebras. It appeared in some papers of Bloh [in 1960s] and Loday [in 1990s]. Recently, they appeared in many geometric and physics applications (see, for example, [12], [14], [16], [23], [29] and references therein). A systematic study of algebraic properties of Leibniz algebras is started from the Loday paper [26]. So, several classical theorems from Lie algebras theory have been extended to the Leibniz algebras case; many classification results regarding nilpotent, solvable, simple, and semisimple Leibniz algebras are obtained (see, for example, [2], [8], [9], [10], [11], [13], [15], [17], [18], [25], [27], [28], [30], and references therein). Leibniz algebras is a particular case of terminal algebras and, on the other hand, symmetric Leibniz algebras are Poisson admissible algebras.

An algebra $(\mathcal{L}, [\cdot, \cdot])$ over a field $\mathbb{F}$ is called a (right) Leibniz algebra if it satisfies the property

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

which is called Leibniz identity. For a Leibniz algebra $\mathcal{L}$, a subspace generated by its squares $\mathcal{I} = \operatorname{span}\{[x, x] : x \in \mathcal{L}\}$ due to Leibniz identity becomes an ideal, and the quotient $\mathcal{G}_{\mathcal{L}} = \mathcal{L}/\mathcal{I}$ is a Lie algebra called liezation of $\mathcal{L}$. Moreover, $[\mathcal{L}, \mathcal{I}] = 0$. Following ideas of Dzhumadildaev [17], a Leibniz algebra $\mathcal{L}$ is called simple if its liezation is a simple Lie algebra and the ideal $\mathcal{I}$ is a simple ideal. Equivalently, $\mathcal{L}$ is simple iff $\mathcal{I}$ is the only non-trivial ideal of $\mathcal{L}$. A Leibniz algebra $\mathcal{L}$ is called semisimple if its liezation $\mathcal{G}_{\mathcal{L}}$ is a semisimple Lie algebra. Simple and semisimple Leibniz algebras are under certain interest now [8], [9], [14], [17], [18], [27], [28].

Let $\mathcal{G}$ be a Lie algebra and $\mathcal{V}$ a (right) $\mathcal{G}$-module. Endow on vector space $\mathcal{L} = \mathcal{G} \oplus \mathcal{V}$ the bracket product as follows:

$$[(g_1, v_1), (g_2, v_2)] := ([g_1, g_2], v_1 \cdot g_2),$$

where $v \cdot g$ (sometimes denoted as $[v, g]$) is an action of an element g of $\mathcal{G}$ on $v \in \mathcal{V}$. Then $\mathcal{L}$ is a Leibniz algebra, denoted as $\mathcal{G} \ltimes \mathcal{V}$. The following theorem proved by Barnes [11] presents an analog of Levi-Malcev's theorem for Leibniz algebras.

Theorem 1.1. *If $\mathcal{L}$ is a finite-dimensional Leibniz algebra over a field of characteristic zero, then $\mathcal{L} = \mathcal{S} \ltimes \mathcal{I}$, where $\mathcal{S}$ is a semisimple Lie subalgebra of $\mathcal{L}$.*

It should be noted that $\mathcal{I}$ is a non-trivial module over the Lie algebra $\mathcal{S}$. We say that a semisimple Leibniz algebra $\mathcal{L}$ is decomposable, if $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are non-trivial semisimple Leibniz algebras. Otherwise, we say that $\mathcal{L}$ is indecomposable. Now we recall the structure of semisimple Leibniz algebras (see [9]). Any complex semisimple finite-dimensional Leibniz algebra $\mathcal{L}$ represented as

$$\mathcal{L} = \bigoplus_{i=1}^{n} \left(\mathcal{S}_i \ltimes \mathcal{I}_i\right), \tag{1}$$

where each $\mathcal{S}_i \ltimes \mathcal{I}_i$ is an indecomposable Leibniz algebra (see [9, Lemma 3.2]).

1.2 Derivations of semisimple Leibniz algebras

Let $\mathcal{L}$ be a semisimple Leibniz algebra of the form (1). It is clear that

$$\mathfrak{Der}\left(\mathcal{L}\right) = \bigoplus_{i=1}^{n} \mathfrak{Der}\left(\mathcal{S}_i \ltimes \mathcal{I}_i\right).$$

Hence

$$\mathfrak{LDer}\left(\mathcal{L}\right) = \bigoplus_{i=1}^{n} \mathfrak{LDer}\left(\mathcal{S}_i \ltimes \mathcal{I}_i\right).$$

So, it suffices to consider local derivations indecomposable semisimple Leibniz algebras.

Any indecomposable semisimple Leibniz algebra $\mathcal{L}$ represented as

$$\mathcal{L} = \mathcal{S} \ltimes \left(\bigoplus_{k=1}^{m} \mathcal{V}_k\right),$$

where each $\mathcal{S} \ltimes \mathcal{V}_k$ is also indecomposable semisimple Leibniz algebra.

Let $\mathcal{S} \ltimes \left(\bigoplus\limits_{k=1}^{m} \mathcal{V}_k\right)$ be an indecomposable semisimple Leibniz algebra. Denote by $\Gamma_{\mathcal{S}}$ the set of all $k = 1, \ldots, m$ such that $\mathcal{S}$ and $\mathcal{V}_k$ are isomorphic as $\mathcal{S}$-modules and denote by $\Gamma_{\mathcal{V}}$ the set of all pairs $\{i, j\}$ such that $\mathcal{V}_i$ and $\mathcal{V}_j$ are isomorphic as $\mathcal{S}$-modules.

Any $\mathfrak{D} \in \mathfrak{Der}\left(\mathcal{S} \ltimes \left(\bigoplus\limits_{k=1}^{m} \mathcal{V}_k\right)\right)$ is of the form

$$\mathfrak{D} = R_a + \sum_{k \in \Gamma_{\mathcal{S}}} \varpi_k \theta_k + \sum_{\{i,j\} \in \Gamma_{\mathcal{V}}} \lambda_{i,j} \pi_{i,j}, \tag{2}$$

where $\pi_{i,j} \in \operatorname{Hom}_{\mathcal{S}}\left(\mathcal{V}_i, \mathcal{V}_j\right)$, $\theta_k \in \operatorname{Hom}_{\mathcal{S}}\left(\mathcal{S}, \mathcal{V}_k\right)$ and R_a is the standard right multiplication on $a \in \mathcal{S}$ (see [9, Theorem 4.5]).

2 Local derivations on semisimple Leibniz algebras

The present part of the paper is dedicated to the proof of the following theorem.

Theorem 2.1. *Let $\mathcal{L} = \mathcal{S} \ltimes \left(\bigoplus\limits_{k=1}^{m} \mathcal{V}_k\right)$ be a complex semisimple finite-dimensional Leibniz algebra. Then any local derivation Δ on $\mathcal{L}$ is a derivation.*

As we have mentioned above it suffices to consider local derivations of indecomposable semisimple Leibniz algebras. From now on $\mathcal{L} = \mathcal{S} \ltimes \left(\bigoplus\limits_{k=1}^{m} \mathcal{V}_k\right)$ is a complex finite-dimensional indecomposable semisimple Leibniz algebra.

Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{S}$. Consider a root space decomposition of $\mathcal{S}$:

$$\mathcal{S} = \mathcal{H} \oplus \bigoplus_{\alpha\in\Gamma} \mathcal{S}_\alpha,$$

where Γ is the set of all nonzero linear functionals α of $\mathcal{H}$ such that

$$\mathcal{S}_\alpha = \{x \in \mathcal{S} : [h, x] = \alpha(h)x, \forall\, h \in \mathcal{H}\} \neq \{0\}.$$

Let

$$\mathcal{V}_k = \bigoplus_{\beta\in\Phi_k} \mathcal{V}_k^\beta$$

be a weight decomposition of $\mathcal{V}_k$, where Φ_k is the set of all weights.

For $q = 1, \ldots, m$ denote by $\mathrm{pr}_q : \bigoplus\limits_{k=1}^{m} \mathcal{V}_k \to \mathcal{V}_q$ a projection mapping defined as follows

$$\mathrm{pr}_q\left(\sum_{k=1}^{m} v_k\right) = v_q.$$

Let us define a mapping $\Delta_{p,q} : \mathcal{S} \ltimes \mathcal{V}_p \to \mathcal{V}_q$ as follows

$$\Delta_{p,q}(x + v) = \mathrm{pr}_q\left(\Delta(x + v)\right),\; x + v \in \mathcal{S} \ltimes \mathcal{V}_p.$$

By (2) for any $x + v \in \mathcal{S} \ltimes \mathcal{V}_p^\beta$ there exist $a_{x+v} \in \mathcal{S}$ and complex numbers $\omega_k^{(x+v)}, \lambda_{i,j}^{(x+v)}$ such that

$$\Delta(x + v) = [x + v, a_{x+v}] + \sum_{k\in\Gamma_{\mathcal{S}}} \omega_k^{(x+v)}\theta_k(x) + \sum_{\{i,j\}\in\Gamma_{\mathcal{V}}} \lambda_{i,j}^{(x+v)}\pi_{i,j}(v).$$

Let $a_{x+v} = h_{x+v} + \sum\limits_{\alpha\in\Gamma} c_\alpha^{(x+v)} e_\alpha \in \mathcal{H} \oplus \oplus_{\alpha\in\Gamma}\mathcal{S}_\alpha$, and denote $\Gamma_q = \{\alpha : [v, e_\alpha] \in \mathcal{V}_q\}$. Then

$$\Delta_{p,q}(x + v) = \quad \mathrm{pr}_q\left([x + v, a_{x+v}] + \sum_{k\in\Gamma_{\mathcal{S}}} \omega_k^{(x+v)}\theta_k(x) + \sum_{\{i,j\}\in\Gamma_{\mathcal{V}}} \lambda_{i,j}^{(x+v)}\pi_{i,j}(v)\right),$$

that is

$$\Delta_{p,q}(x + v) = \quad \delta_{p,q}[v, h_{x+v}] + \left[v, \sum_{\alpha\in\Gamma_q} c_\alpha^{(x+v)} e_\alpha\right] + \omega_q^{(x+v)}\theta_q(x) + \lambda_{p,q}^{(x+v)}\pi_{p,q}(v), \qquad (3)$$

where $\delta_{p,q}$ is the Kronecker delta.

If necessary, after renumbering we can assume that there is a number $k \in \{1, \ldots, m\}$ such that $\mathcal{S}$ and $\mathcal{V}_i$ are isomorphic $\mathcal{S}$-modules for all $i = 1, \ldots, k$, and $\mathcal{S}$ and $\mathcal{V}_i$ are not isomorphic $\mathcal{S}$-modules for all $i = k+1, \ldots, m$. According (2), we can consider two possible cases separately.

Let us first consider an indecomposable semisimple Leibniz algebra $\mathcal{L} = \mathcal{S} \ltimes \left(\bigoplus_{i=1}^{m} \mathcal{V}_i\right)$ such that $\mathcal{S}$ and $\mathcal{V}_i \, (i = 1, \ldots, m)$ are isomorphic $\mathcal{S}$-modules.

Lemma 2.2. *Let Δ be a local derivation on $\mathcal{L}$ such that $\Delta(\mathcal{L}) \subseteq \mathcal{V}$. Then Δ is a derivation.*

Proof. Fix the indices p, q. Let us show that there are complex numbers ω_q and $\lambda_{p,q}$ such that

$$\Delta_{p,q} = \omega_q \theta_q + \lambda_{p,q} \pi_{p,q}.$$

Fix a basis $\{x_1, \ldots, x_m\}$ in $\mathcal{S}$. The system of vectors $\{\theta_s(x_i)\}_{1 \leq i \leq m}$ is a basis in $\mathcal{V}_s$ (for $s = p, q$). Here, θ_s is an $\mathcal{S}$-module isomorphism from $\mathcal{S}$ onto $\mathcal{V}_s$, in particular,

$$\theta_s([x, y]) = [\theta_s(x), y].$$

Using (3) for $x = x_i$ and take a complex number $\omega_q^{(i)}$ such that

$$\Delta_{p,q}(x_i) = \omega_q^{(i)} \theta_q(x_i).$$

Now for the element $x = x_i + x_j$, where $i \neq j$, take a complex number $\omega_q^{(i,j)}$ such that

$$\Delta_{p,q}(x_i + x_j) = \omega_q^{(i,j)} \theta_q(x_i + x_j) = \omega_q^{(i,j)} \theta_q(x_i) + \omega_q^{(i,j)} \theta_q(x_j).$$

On the other hand,

$$\Delta_{p,q}(x_i + x_j) = \omega_q^{(i)} \theta_q(x_i) + \omega_q^{(j)} \theta_q(x_j).$$

Comparing the last two equalities we obtain $\omega_q^{(i)} = \omega_q^{(j)}$ for all i, j. This means that there exists a complex number ϖ_q such that

$$\Delta_{p,q}(x_i) = \varpi_q \theta_q(x_i). \tag{4}$$

Now by (3) for $x = x_i + \theta_p(x_i) \in \mathcal{S} \ltimes \mathcal{V}_p$ take complex numbers ω_i and λ_i such that

$$\Delta(x_i + \theta_p(x_i)) = \omega_i \theta_q(x_i) + \lambda_i \pi_{p,q}(\theta_p(x_i)) = (\omega_i + \lambda_i)\, \theta_q(x_i).$$

Taking into account (4) we obtain that

$$\begin{array}{rcl} \Delta_{p,q}(\theta_p(x_i)) & = & \Delta_{p,q}(x_i + \theta_p(x_i)) - \Delta_{p,q}(x_i) \\ & = & (\omega_i + \lambda_i)\theta_q(x_i) - \varpi_q \theta_q(x_i) \quad = \quad (\omega_i - \varpi_q + \lambda_i)\theta_q(x_i). \end{array}$$

This means that for every $i \in \{1,\dots,m\}$ there exists a complex number Λ_i such that

$$\Delta(\theta_p(x_i)) = \Lambda_i\theta_q(x_i). \tag{5}$$

Take an element $x = x_i + x_j + \theta_p(x_i + x_j) \in \mathcal{S} \ltimes \mathcal{V}_p$, where $i \neq j$. By (3), we get that

$$\Delta_{p,q}(x_i + x_j + \theta_p(x_i + x_j)) = \omega_{i,j}\theta_q(x_i + x_j) + \lambda_{i,j}\theta_q(x_i + x_j).$$

Taking into account (4) we obtain that

$$\begin{aligned}\Delta_{p,q}\left(\theta_q(x_i + x_j)\right) &= \Delta_{p,q}(x_i + x_j + \theta_q(x_i + x_j)) - \Delta_{p,q}(x_i + x_j)\\ &= (\omega_{i,j} - \varpi_q + \lambda_{i,j})\theta_q(x_i + x_j).\end{aligned}$$

On the other hand, by (5),

$$\Delta_{p,q}(\theta_q(x_i + x_j)) = \Delta_{p,q}(\theta_q(x_i)) + \Delta_{p,q}(\theta_q(x_j)) = \lambda_i\theta_q(x_i) + \lambda_j\theta_q(x_j).$$

Comparing the last two equalities we obtain that $\lambda_i = \lambda_j$ for all i and j. This means that there exist a complex number $\lambda_{p,q}$ such that

$$\Delta_{p,q}(\theta_q(x_i)) = \lambda_{p,q}\theta_q(x_i). \tag{6}$$

Combining (4) and (6) we obtain that $\Delta_{p,q} = \varpi_q\theta_q + \lambda_{p,q}\pi_{p,q}$. This means that Δ is a derivation. The proof is completed. □

In the next lemma we consider $\mathcal{L} = \mathcal{S} \ltimes \left(\bigoplus\limits_{k=1}^{m} \mathcal{V}_k\right)$, an indecomposable semisimple Leibniz algebra, such that $\mathcal{S}$ and $\mathcal{V}_k$ are not isomorphic $\mathcal{S}$-modules for all $k = 1,\dots,m$.

Lemma 2.3. *Let Δ be a local derivation on $\mathcal{L}$ such that $\Delta(\mathcal{L}) \subseteq \mathcal{V}$. Then Δ is a derivation.*

Proof. Let $\left\{v_1^{(1)},\dots,v_n^{(1)}\right\}$ be a basis of $\mathcal{V}_1$. Since $\mathcal{V}_1$ and $\mathcal{V}_k$ are isomorphic, it follows that $\left\{v_i^{(q)} = \pi_{1,q}(v_i^{(1)}) : i = 1,\dots n\right\}$ is a basis of $\mathcal{V}_q$.

Without lost of generality we can assume that for any $v_i^{(1)}$ there exists a weight β_i such that $v_i^{(1)} \in \mathcal{V}_{\beta_i}$. Let h_0 be a strongly regular element in $\mathcal{H}$, that is, $\alpha(h_0) \neq \beta(h_0)$ for any $\alpha,\beta \in \Gamma, \alpha \neq \beta$. For $x = h_0 + v_i^{(1)} \in \mathcal{S} \ltimes \mathcal{V}_1$ take an element $a_x \in \mathcal{S}$ and complex numbers $\lambda_k^{(x)}$ such that

$$\Delta\left(h_0 + v_i^{(1)}\right) = [h_0, a_x] + \left[v_i^{(1)}, a_x\right] + \sum_{k\in\Gamma_{1,k}} \lambda_k^{(x)} v_i^{(k)}.$$

Taking into account that h_0 is strongly regular, from $[h_0, a_x] = 0$, we have that $a_x \in \mathcal{H}$. Further

$$\begin{aligned}\Delta\left(v_i^{(1)}\right) &= \Delta\left(h_0 + v_i^{(1)}\right)\\ &= \left[v_i^{(1)}, a_x\right] + \sum_{k\in\Gamma_{1,k}} \lambda_k^{(x)} v_i^{(k)} = (\beta_i(a_x) + \lambda_1^{(x)})v_i^{(1)} + \sum_{1<k\in\Gamma_{1,k}} \lambda_k^{(x)} v_i^{(k)}.\end{aligned}$$

Now we change the element $x = h_0 + v_i^{(1)}$ to the element $\overline{x} = h_0 + v_i^{(1)} + v_j^{(1)}$ $(i \neq j)$, then similar as above we get that

$$\Delta\left(v_i^{(1)} + v_j^{(1)}\right) = (*)v_i^{(1)} + (*)v_j^{(1)} + \sum_{1<k\in\Gamma_{1,k}} \lambda_k^{(\overline{x})}\left(v_i^{(k)} + v_j^{(k)}\right).$$

Comparing the last two equalities we can see that there are $\lambda_2, \ldots, \lambda_n \in \mathbb{C}$ such that

$$\Delta\left(v_i^{(1)}\right) = (*)v_i^{(1)} + \sum_{1<k\in\Gamma_{1,k}} \lambda_k v_i^{(k)}.$$

Replacing Δ with $\Delta - \sum\limits_{1<k\in\Gamma_{1,k}} \lambda_k \pi_{1,k}$ we obtain a new local derivation which maps $\mathcal{V}_1$ into itself. Due to (2), there exist complex numbers $\lambda_i, i = 1, \ldots, n$ such that

$$\Delta\left(v_i^{(1)}\right) = \lambda_i v_i^{(1)}. \tag{7}$$

We shall show that $\lambda_1 = \ldots = \lambda_n$. For a fixed $v_i^{(1)}$ $(i \neq 1)$ we have that

$$\Delta\left(v_1^{(1)} + v_i^{(1)}\right) = \lambda_1 v_1^{(1)} + \lambda_i v_i^{(1)}. \tag{8}$$

Without loss of generality we can assume that β_1 is a fixed highest weight of $\mathcal{V}_1$. It is known [20, Page 108] that the weight $\beta_1 - \beta_i$ can be represented as

$$\beta_1 - \beta_i = n_1\alpha_1 + \ldots + n_l\alpha_l,$$

where $\alpha_1, \ldots, \alpha_l$ are simple roots of $\mathcal{S}$, $n_1, \ldots, n_l$ are non negative integers.

Below we shall consider two separated cases.

Case 1. $\alpha_0 = n_1\alpha_1 + \ldots + n_l\alpha_l$ is not a root. Take the following element

$$y = n_1 e_{\alpha_1} + \ldots + n_l e_{\alpha_l} + v_1^{(1)} + v_i^{(1)}.$$

By the definition of local derivation we can find an element $a_y = h + \sum\limits_{\alpha\in\Gamma} c_\alpha e_\alpha \in \mathcal{S}$ and a number $\lambda^{(y)}$ such that

$$\Delta(y) = [y, a_y] + \lambda^{(y)}\left(v_1^{(1)} + v_i^{(1)}\right).$$

Taking into account (7) and $\Delta(y) \in \mathcal{V}$, we obtain that

$$\left[\sum_{s=1}^{l} n_s e_{\alpha_s}, h + \sum_{\alpha\in\Gamma} c_\alpha e_\alpha\right] = 0.$$

Thus

$$\sum_{s=1}^{l} n_s \alpha_s(h) e_{\alpha_s} + \sum_{t=1}^{l}\sum_{\alpha\in\Gamma} (*) e_{\alpha+\alpha_t} = 0,$$

where the symbols $(*)$ denote appropriate coefficients. The second summand does not contain any element of the form e_{α_s}. Indeed, if we assume that $\alpha_s = \alpha + \alpha_t$, we have that $\alpha = \alpha_s - \alpha_t$. But $\alpha_s - \alpha_t$ is not a root, because α_s, α_t are simple roots. Hence all coefficients of the first summand are zero, i.e.,

$$n_1\alpha_1(h) = \ldots = n_l\alpha_l(h) = 0.$$

Further

$$\Delta\left(v_1^{(1)} + v_i^{(1)}\right) = \Delta(y) = \left[v_1^{(1)} + v_i^{(1)}, a_x\right] + \lambda^{(y)}\left(v_1^{(1)} + v_i^{(1)}\right).$$

Let us calculate the product $\left[v_1^{(1)} + v_i^{(1)}, a_x\right]$. We have

$$\begin{aligned}\left[v_1^{(1)} + v_i^{(1)}, a_x\right] &= \left[v_1^{(1)} + v_i^{(1)}, h + \sum_{\alpha\in\Phi} c_\alpha e_\alpha\right] \\ &= \beta_1(h)v_{\beta_1}^{(1)} + \beta_2(h)v_{\beta_2}^{(1)} + \sum_{t=1}^{2}\sum_{\alpha\in\Phi}(*)v_{\beta_t+\alpha}^{(1)}.\end{aligned}$$

The last summand does not contain $v_{\beta_1}^{(1)}$ and $v_{\beta_i}^{(1)}$, because $\beta_1 - \beta_i$ is not a root by the assumption. This means that

$$\Delta\left(v_{\beta_1}^{(1)} + v_{\beta_i}^{(1)}\right) = \left(\beta_1(h) + \lambda^{(y)}\right)v_{\beta_1}^{(1)} + \left(\beta_i(h) + \lambda^{(y)}\right)v_{\beta_i}^{(1)}. \tag{9}$$

The difference of the coefficients of the right side is

$$\beta_1(h) - \beta_i(h) = \sum_{s=1}^{l} n_s\alpha_s(h) = 0,$$

because of $n_1\alpha_1(h) = \ldots = n_l\alpha_l(h) = 0$. Finally, comparing coefficients in (8) and (9) we get

$$\lambda_1 = \beta_1(h) + \lambda^{(y)} = \beta_i(h) + \lambda^{(y)} = \lambda_i.$$

Case 2. $\alpha_0 = n_1\alpha_1 + \ldots + n_l\alpha_l$ is a root. Note that $\dim\mathcal{V}_{\beta_1} = 1$, because β_1 is a highest weight. Since $\beta_1 - \beta_i$ is a root, [21, Lemma 3.2.9] implies that $\dim\mathcal{V}_{1,\beta_i} = \dim\mathcal{V}_{1,\beta_1}$, and hence there exist numbers $t_{-\alpha_0} \neq 0$ and t_{α_0} such that

$$\left[v_{\beta_1}^{(1)}, e_{-\alpha_0}\right] = t_{-\alpha_0}v_{\beta_i}^{(1)}, \ \left[v_{\beta_i}^{(1)}, e_{\alpha_0}\right] = t_{\alpha_0}v_{\beta_1}^{(1)}.$$

Take the following element

$$z = t_{-\alpha_0}e_{\alpha_0} + t_{\alpha_0}e_{-\alpha_0} + v_{\beta_1}^{(1)} + v_{\beta_i}^{(1)},$$

and choose an element $a_z = h + \sum\limits_{\alpha\in\Phi} c_\alpha e_\alpha \in \mathcal{S}$ and a number λ_z such that

$$\Delta(z) = [z, a_z] + \lambda_z \left(v^{(1)}_{\beta_1} + v^{(1)}_{\beta_i}\right).$$

Since $\Delta(z) \in \mathcal{V}$, we obtain that

$$\left[t_{-\alpha_0} e_{\alpha_0} + t_{\alpha_0} e_{-\alpha_0}, h + \sum_{\alpha\in\Phi} c_\alpha e_\alpha\right] = 0.$$

Now rewrite the last equality as

$$\alpha_0(h) t_{-\alpha_0} e_{\alpha_0} - \alpha_0(h) t_{\alpha_0} e_{-\alpha_0} + (t_{-\alpha_0} c_{-\alpha_0} - t_{\alpha_0} c_{\alpha_0}) h_{\alpha_0} + \sum_{\alpha\neq\pm\alpha_0} (*) e_{\alpha\pm\alpha_0} = 0,$$

where $h_{\alpha_0} = [e_{\alpha_0}, e_{-\alpha_0}] \in \mathcal{H}$. The last summand in the sum does not contain elements e_{α_0} and $e_{-\alpha_0}$. Indeed, if we assume that $\alpha_0 = \alpha - \alpha_0$, we have that $\alpha = 2\alpha_0$. But $2\alpha_0$ is not a root. Hence the first three coefficients of this sum are zero, i.e.,

$$\alpha_0(h) = 0,\ t_{\alpha_0} c_{\alpha_0} = t_{-\alpha_0} c_{-\alpha_0}. \tag{10}$$

Further

$$\Delta\left(v^{(1)}_{\beta_1} + v^{(1)}_{\beta_i}\right) = \Delta(z) = \left[v^{(1)}_{\beta_1} + v^{(1)}_{\beta_i}, a_z\right] + \lambda_z \left(v^{(1)}_{\beta_1} + v^{(1)}_{\beta_i}\right).$$

Let us consider the element $\left[v^{(1)}_{\beta_1} + v^{(1)}_{\beta_i}, a_z\right]$. We have

$$\begin{aligned}
\left[v^{(1)}_{\beta_1} + v^{(1)}_{\beta_i}, a_z\right] &= \left[v^{(1)}_{\beta_1} + v^{(1)}_{\beta_i}, h + \sum_{\alpha\in\Phi} c_\alpha e_\alpha\right] \\
&= \left[v^{(1)}_{\beta_1}, h\right] + c_{\alpha_0}\left[v^{(1)}_{\beta_i}, e_{\alpha_0}\right] + \left[v^{(1)}_{\beta_i}, h\right] \\
&\quad + c_{-\alpha_0}\left[v^{(1)}_{\beta_1}, e_{-\alpha_0}\right] + c_{\alpha_0}\left[v^{(1)}_{\beta_1}, e_{\alpha_0}\right] + c_{-\alpha_0}\left[v^{(1)}_{\beta_i}, e_{-\alpha_0}\right] \\
&\quad + \sum_{\alpha\neq\pm\alpha_0} c_\alpha \left[v^{(1)}_{\beta_1}, e_\alpha\right] + \sum_{\alpha\neq\pm\alpha_0} c_\alpha \left[v^{(1)}_{\beta_i}, e_\alpha\right] \\
&= (\beta_1(h) + t_{\alpha_0} c_{\alpha_0}) v^{(1)}_{\beta_1} + (\beta_i(h) + t_{-\alpha_0} c_{-\alpha_0}) v^{(1)}_{\beta_i} \\
&\quad + (*) v^{(1)}_{2\beta_1 - \beta_i} + (*) v^{(1)}_{2\beta_i - \beta_1} \\
&\quad + \sum_{\alpha\neq\pm\alpha_0} (*) v^{(1)}_{\beta_1+\alpha} + \sum_{\alpha\neq\pm\alpha_0} (*) v^{(1)}_{\beta_i+\alpha}.
\end{aligned}$$

The last three summands do not contain $v^{(1)}_{\beta_1}$ and $v^{(1)}_{\beta_i}$, because $\beta_1 - \beta_i = \alpha_0$ and $\alpha \neq \pm\alpha_0$. This means that

$$\begin{aligned}
\Delta\left(v^{(1)}_{\beta_1} + v^{(1)}_{\beta_i}\right) &= (\beta_1(h) + t_{\alpha_0} c_{\alpha_0} + \lambda_z) v^{(1)}_{\beta_1} \\
&\quad + (\beta_i(h) + t_{-\alpha_0} c_{-\alpha_0} + \lambda_z) v^{(1)}_{\beta_i}.
\end{aligned} \tag{11}$$

Taking into account (10) we find the difference of coefficients on the right side:

$$(\beta_1(h) + t_{\alpha_0}c_{\alpha_0}) - (\beta_i(h) + t_{-\alpha_0}c_{-\alpha_0}) = \alpha_0(h) + t_{\alpha_0}c_{\alpha_0} - t_{-\alpha_0}c_{-\alpha_0} = 0.$$

Combining (8) and (11) we obtain that

$$\lambda_1 = \beta_1(h) + t_{\alpha_0}c_{\alpha_0} + \lambda_z = \beta_i(h) + t_{-\alpha_0}c_{-\alpha_0} + \lambda_z = \lambda_i.$$

So, we have proved that $\Delta\left(v_i^{(1)}\right) = \lambda_1 v_i^{(1)}$ for all $i = 1, \ldots, n$. By a similar way we obtain that $\Delta\left(v_i^{(k)}\right) = \lambda_k v_i^{(k)}$ for all $i = 1, \ldots, n_k$. Thus $\Delta = \sum\limits_{k=1}^{m} \lambda_k \pi_{k,k}$, and therefore Δ is a derivation.

The proof is completed. □

Proof of Theorem 2.1. Let Δ be an arbitrary local derivation on $\mathcal{L}$. For an arbitrary element $x \in \mathcal{S}$ take a derivation $\mathfrak{D} = R_{a_x} + \sum\limits_{k\in\Gamma_{\mathcal{S}}} \varpi_k^{(x)}\theta_k^{(x)} + \sum\limits_{\{i,j\}\in\Gamma_{\mathcal{V}}} \lambda_{i,j}^{(x)}\pi_{i,j}^{(x)}$ of the form (2) such that

$$\Delta(x) = [x, a_x] + \sum_{k\in\Gamma_{\mathcal{S}}} \omega_{x,k}^{(x)}\theta_k^{(x)}(x).$$

Then the mapping

$$x \in \mathcal{S} \to [x, a_x] \in \mathcal{S}$$

is a well-defined local derivation on $\mathcal{S}$, and by [6, Theorem 3.1] it is a derivation generated by an element $a \in \mathcal{S}$. Then the local derivation $\Delta - R_{a_x}$ maps $\mathcal{L}$ into $\mathcal{V}$. By Lemmas 2.2 and 2.3 we get that $\Delta - R_{a_x}$ is a derivation and therefore Δ is also a derivation. The proof is completed. □

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Received: January 03, 2022
Accepted for publication: January 28, 2022
Communicated by: Friedrich Wagemann