Communications in Mathematics 30 (2022) 93–99
DOI: https://doi.org/10.46298/cm.9294

On existence of normal p-complement of finite groups with restrictions on the conjugacy class sizes

Ilya B. Gorshkov

Abstract. The greatest power of a prime p dividing the natural number n will be denoted by n_p. Let $Ind_G(g) = |G : C_G(g)|$. Suppose that G is a finite group and p is a prime. We prove that if there exists an integer $\alpha > 0$ such that $Ind_G(a)_p \in \{1, p^\alpha\}$ for every a of G and a p-element $x \in G$ such that $Ind_G(x)_p > 1$, then G includes a normal p-complement.

1 Introduction

In this paper, all groups are finite. Denote the set of prime divisors of positive integer n by $\pi(n)$, and by the set $\pi(|G|)$ for a group G by $\pi(G)$. For a set of primes π and a positive integer n we will denote $n_\pi = \prod_{p\in\pi} |n|_p$. Let G be a group and take $a \in G$. With a^G standing for the conjugacy class in G containing a, put $N(G) = \{|x^G|, x \in G\} \setminus \{1\}$. Denote by $|G||_p$ the number p^n such that $N(G)$ contains a multiple of p^n and avoids multiples of p^{n+1}. For $\pi \subseteq \pi(G)$ put $|G||_\pi = \prod_{p\in\pi} |G||_p$. For brevity, $|G||$ is meaning $|G||_{\pi(G)}$. Observe that $|G||_p$ divides $|G|_p$ for each $p \in \pi(G)$. However, $|G||_p$ can be less than $|G|_p$. Take a set of primes π, denote $\Theta_\pi = \{\tau \subseteq \pi \mid \tau \neq \varnothing, |\tau| \geq |\pi| - 1\}$.

Definition 1.1. Let π be a set of primes. We say that a group G satisfies the condition π^* or G is a π^*-group and write $G \in \pi^*$ if for every $a \in N(G)$ there exists $\tau_a \in \Theta_\pi$ such that $a_\pi = |G||_{\tau_a}$.

Ishikawa [8] proved that a group G with $N(G) = \{p^\alpha\}$ is nilpotent class at most 3. Casolo, Dolfi and Jabara [2] described the set of $\{p\}^*$-groups. In particular, they proved that any group of $\{p\}^*$ is solvable and includes a normal p-compliment. Camina [2.10] proved that a group G with $\{p, q\}^*$-property is nilpotent if $N(G) = \{p^n, q^m, p^n q^m\}$. Beltram and Filipe [1] extended Camina's theorem in the following way. Let G be a group whose set of conjugacy class sizes is $\{1, n, m, nm\}$, where n and m are coprime positive integers; then G is nilpotent and the integers n and m are prime-power numbers; in particular $G \in \{n, m\}^*$. The author of [7] investigate $\{p, q\}^*$-groups with trivial center.

In the present paper we will investigate some generalisations of the property $\{p\}^*$.

Definition 1.2. We say that a group G satisfies the condition $R(p)$ or G is a $R(p)$-group and write $G \in R(p)$ if there exists an integer $\alpha > 0$ such that $a_p \in \{1, p^\alpha\}$ for each $a \in N(G)$.

Note that, if $G \in \pi^*$, then $G \in R(p)$ for each $p \in \pi$. The set of $R(p)$-group disjoins on two subsets $R(p)^*$ and $R(p)^{**}$:

(*i*) $G \in R(p)^*$ if G contains a p-element h such that $Ind_G(h)_p > 1$;

MSC 2020: 20DXX,20E45
Keywords: finite group, conjugacy classes, normal p-complement
The work is supported by Russian Science Foundation (project 18-71-10007).
Affiliation:
Sobolev Institute of Mathematics SB RAS Novosibirsk, Russia Siberian Federal University, Krasnoyarks, Russia
E-mail: ilygor8@gmail.com

(ii) $G \in R(p)^{**}$ if $Ind_G(h)_p = 1$ for each p-element $h \in G$.

We prove the following theorem.

Theorem 1.3. *If $G \in R(p)^*$, then G has a normal p-complement.*

It follows from the theorem that the center of a $R(p)^*$-group is not trivial.

Corollary 1.4. *If $G \in R(p)^*$ and $P \in Syl_p(G)$, then $Z(P) \leq Z(G)$.*

Vasil'ev [15] proved that if G is a $R(p)$-group with trivial center and $|G||_p = p$, then Sylow p-subgroups of G are abelian. This assertion is true in the general case.

Corollary 1.5. *If $G \in R(p)$ and $Z(G) = 1$, then Sylow p-subgroups of G are abelian.*

2 Preliminary results

Lemma 2.1 ([5, Lemma 1.4]). *For a finite group G, take $K \trianglelefteq G$ and put $\overline{G} = G/K$. Take $x \in G$ and $\overline{x} = xK \in G/K$. The following claims hold:*

(i) $|x^K|$ *and* $|\overline{x}^{\overline{G}}|$ *divide* $|x^G|$.

(ii) *For neighboring members L and M of a composition series of G, with $L < M$, take $x \in M$ and the image $\widetilde{x} = xL$ of x. Then $|\widetilde{x}^S|$ divides $|x^G|$, where $S = M/L$.*

(iii) *If $y \in G$ with $xy = yx$ and $(|x|, |y|) = 1$, then $C_G(xy) = C_G(x) \cap C_G(y)$.*

(iv) *If $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.*

(v) $\overline{C_G(x)} \leq C_{\overline{G}}(\overline{x})$.

Lemma 2.2 ([2, Lemma 2.1]). *Let x, y be elements of a group G and assume at least one of the following conditions:*

(i) *x and y commute and have coprime orders;*

(ii) *$x \in N, y \in M$ with $N, M \trianglelefteq G$ and $N \cap M = 1$.*

Then $C_G(xy) = C_G(x) \cap C_G(y)$.

Lemma 2.3 ([2, Lemma 2.7]). *Let A be a group acting via automorphisms on a group G and N be a normal A-invariant subgroup of G. If $(|A|, |N|) = 1$, then:*

(i) $C_{G/N}(A) = C_G(A)N/N$;

(ii) $|C_G(A)| = |C_N(A)||C_{G/N}(A)|$.

Lemma 2.4 ([5, Lemma 4]). *Take $g \in G$. If each conjugacy class of G contains an element h such that $g \in C_G(h)$, then $g \in Z(G)$.*

Lemma 2.5. *If $G \in R(p)$ and $N \trianglelefteq G$ such that $|N|_p = |G|_p$, then $N \in R(p)$ or $|N||_p = 1$.*

Proof. Since N is a normal subgroup and N includes every Sylow p-subgroup of G, we have N includes every Sylow p-subgroup of $C_G(x)$ for any $x \in G$. Therefore, $Ind_G(x)_p = Ind_N(x)_p$ and the lemma is proved. ■

Lemma 2.6. *If $G \in R(p)$, $N \trianglelefteq G$ is a p'-group, then $G/N \in R(p)$ or $|G/N||_p = 1$.*

Proof. Let $^- : G \to G/N$ be a natural homomorphism. We have $Ind_G(h)$ is a multiple of $Ind_{\overline{G}}(\overline{h})$ for any $h \in G$. Therefore, $|G||_p \geq |\overline{G}||_p$. Assume that there exists $\overline{x} \in \overline{G}$ such that $1 < Ind_{\overline{G}}(\overline{x})_p < |\overline{G}||_p$. Let H be a Sylow p-subgroup of $C_{\overline{G}}(\overline{x})$. Therefore, $|G|_p > |H| > |G|_p/|G||_p$. Put $T < G$ is a p-group such that $\overline{T} = H$. From Lemma 2.3 follows that $C_G(T)$ contains y such that $yN = \overline{x}$. Since $C_G(y) \geq T$, we obtain $Ind_G(y)_p \leq |G|_p/|T| < |G||_p$. Therefore, $Ind_G(y)_p = 1$ and consequently $Ind_{\overline{G}}(yN)_p = 1 = Ind_{\overline{G}}(\overline{x})_p$; a contradiction. ■

The prime graph $GK(G)$ of a finite group G is defined as follows. The vertex set of $GK(G)$ is the set $\pi(G)$. Two distinct primes $p, q \in \pi(G)$ considered as vertices of the graph are adjacent by the edge if and only if there is an element of order pq in G. Denote by $s(G)$ the number of connected components of $GK(G)$ and by $\pi_i(G), i = 1, ..., s(G)$, its i-th connected component. If G has even order, then put $2 \in \pi_1(G)$.

Lemma 2.7. *[18, Theorem A] If a finite group G has disconnected prime graph, then one of the following conditions holds:*

(a) $s(G) = 2$ and G is a Frobenius or 2-Frobenius group;

(b) there is a nonabelian simple group S such that $S \leq G = G/F(G) \leq Aut(S)$, where $F(G)$ is the maximal normal nilpotent subgroup of G; moreover, $F(G)$ and G/S are $\pi_1(G)$-subgroups, $s(S) \geq s(G)$, and for every i with $2 \leq i \leq s(G)$ there is j with $2 \leq j \leq s(S)$ such that $\pi_i(G) = \pi_j(S)$.

Lemma 2.8 ([4, Lemma 5.3.4])**.** *Let $A \times B$ be a group of automorphisms of the p-group P with A a p'-group and B a p-group. If A acts trivially on $C_P(B)$, then $A = 1$.*

Lemma 2.9 ([4, Lemma 5.2.3])**.** *Let A be a p'-group of automorphisms of the abelian group P. Then we have $P = C_P(A) \times [P, A]$*

Lemma 2.10. *[2.10, Lemma 1] If, for some prime p, every p'-element of a group G has index prime to p, then the Sylow p-subgroup of G is a direct factor of G.*

Lemma 2.11. [16, Lemma 3.6] *For distinct primes s and r, consider a semidirect product H of a normal r-subgroup T and a cyclic subgroup $C = \langle g \rangle$ of order s with $[T, g] \neq 1$. Suppose that H acts faithfully on a vector space V of positive characteristic t not equal to r. If the minimal polynomial of g on V does not equal $x^s - 1$, then*

(i) $C_T(g) \neq 1$;

(ii) T is nonabelian;

(iii) $r = 2$ and s is a Fermat prime.

Lemma 2.12. *[6, Lemma 11] If $S \leq A \leq Aut(S)$, where S is a nonabelian simple group, then $|A| = |A||$.*

Lemma 2.13. *[11, Theorem B] Let G be a finite group and p a prime. Suppose that for every p-element x the number $|x^G|$ is a p'-number. Then,*

$$O^{p'}(G/O_{p'}(G)) = S_1 \times \cdots \times S_r \times H,$$

where H has an abelian Sylow p-subgroup, $r \geq 0$, and S_i is a nonabelian simple group with either

(i) $p = 3$ and $S_i \simeq Ru$ or J_4 or $S_i \simeq {}^2F_4(q)'$, $9 \nmid (q + 1)$; or

(ii) $p = 5$ and $S_i \simeq Th$ for all i.

3 Proof

Let G be a counterexample for assertion of the theorem of minimal order.

Lemma 3.1. $O_{p'}(G) = 1$

Proof. From Lemma 2.6 it follows that $G/O_{p'}(G) \in R(p)$ or $|G/O_{p'}(G)||_p = 1$. We can think that $G/O_{p'}(G)$ does not include a normal p-complement, else G contains a normal p-complement. Therefore, $G/O_{p'}$ a counterexample for assertion of theorem; a contradiction with minimality G. If $|G/O_{p'}(G)||_p = 1$, then Lemma 2.10 implies that $G/O_{p'}(G)$ is a p-group. Therefore, $O_{p'}(G)$ is a normal p-compliment of G; a contradiction. ■

Lemma 3.2. *Orders of minimal normal subgroups of G are multiples of p.*

Proof. It follows from Lemma 3.1. ∎

Lemma 3.3. *Each minimal normal subgroup of G is a p-group.*

Proof. Let H be the socle of G. Then H has expression in form $S \times X$, where $S = S_1 \times S_2 \times ... \times S_n$, for nonabelian simple groups $S_1, ..., S_n$ and a p-group X. It follows from Lemma 3.2 that p divides the order of S_i for all $1 \leq i \leq n$. Assume that G contains a p-element x such that $S_1^x \neq S_1$. Let $D = \langle aa^x a^{x^2} ... a^{|x|-1} | a \in S_1 \rangle$. We have $D = C_{S_1 \times S_1^x ...}(x)$ and $D \simeq S_1$. Since $|S_1|$ is a multiple of p, we see that $Ind_G(x)_p > 1$. By Lemma 2.12, we obtain D contains a p'-element y such that $Ind_D(y)_p = |D|_p$. Thus, $Ind_G(xy)_p > Ind_G(x)_p = |G||_p$; a contradiction. It follows that $S_i^y = S_i$, for any $1 \leq i \leq n$ and for any p-element y. Take $h_1 \in S_1$ and $h_2 \in S_2$. We have a p-element $y \in C_G(h_1 h_2)$ iff $y \in C_G(h_1) \cap C_G(h_2)$. Assume that $n > 1$. Since $|S_i| = |S_i||$, we see that S_i contains an element h_i such that $|h_i^{S_i}|_p = |S_i|_p$, in particular $|h_i^G|_p > 1$. Let A be a Sylow p-subgroup of $C_G(h_1 h_2)$ and B be a Sylow p-subgroup of $C_G(h_1)$. Then $A < C_G(h_1) \cap C_G(h_2)$ and $|G|_p > |B| > |A|$; a contradiction. Therefore, $n = 1$.

From Lemma 2.12, it follows that S contains an element h such that a p-element $y \in C_G(h)$ iff $y \in C_G(S)$. If $C_G(S)$ contains y such that $|y^{C_G(S)}|_p > 1$, then $|(yh)^G|_p > |h^G|_p$; a contradiction. From Lemma 2.10 it follows that $C_G(S) = O_p(G)$. Moreover $O = O_p(G)$ is abelian. We have $|h^G|_p = |G|_p / |O|_p$. Take $a \in S$ such that $|a^S|_p < |S|_p$. Hence $|a^G|_p < |h^G|_p$. This implies that $|a^S|_p = 1$. From Lemma 2.13 it follows that a Sylow p-subgroup of S is abelian or S is isomorphic to one of groups $J_4, Ru,\ {}^2F_4(q)', Th$. Also it signifies that $S \in R(p)$.

Assume that there exists a p-element $x \in G \setminus H$ such that $Ind_G(x)_p > 1$ and x acts on S as an outer automorphism. From Lemma 2.10 and the equation $Ind_{C_G(x)}(y)_p = 1$ for each p'-element $y \in C_G(x)$ it follows that $C_G(x) = L \times T$, where L is a Sylow p-subgroup of $C_G(x)$. Therefore, $C_S(x) = \widetilde{P} \times \widetilde{L}$, where $\widetilde{P}$ is a Sylow p-subgroup of $C_S(x)$.

Assume that $S \simeq Alt_n$, where $n \geq 5$. Since $\pi(Out(S)) = \{2\}$, we obtain $p = 2$. If $n \neq 6$, then $C_{S_1}(x) \simeq Alt_{n-2}$; a contradiction. If $n = 6$, then $C_S(x) \simeq Alt_4$ or $C_{S_1}(x) \simeq Sym_3$; a contradiction.

From [3] it follows that S is not isomorphic to a sporadic simple group or the Tits group.

Therefore, S is a group of Lie type. Assume that a Sylow p-subgroup of S is nonabelian. Since S is not isomorphic to one of a sporadic groups, it follows that $S \simeq\ {}^2F_4(q)', 9 \nmid q+1$ and $p = 3$. Therefore, x acts on S as a field automorphism. Thus, $q = 2^{3(2m+1)}$; this contradicts with that $9 \nmid q+1$. Thus, Sylow p-subgroups of S are abelian.

Assume that $p = 2$. From description of simple groups with abelian Sylow 2-subgroup [17] it follows that S is isomorphic to one of a groups $L_2(q)$ where $q = 2^f$ or $q \equiv 3, 5 (mod\ 8)$, J_1 or ${}^2G_2(q)$ where $q = 3^{2m+1}$ and $m \geq 1$. Put $P \in Syl_2(S)$. From [9, Theorems 1 and 7] it follows that $C_S(P) = Z(P)$ or S is isomorphic to $L_2(q)$ where q odd. Let S be isomorphic to $L_2(q)$ for some odd q. If x acts on S as a field automorphism, then $C_S(x) \simeq L_2(d)$, where d divides q, in particular $C_S(x)$ is not a direct product of a Sylow p-subgroup and a p-complement; a contradiction. Assume that x acts on S as diagonal automorphism or a diagonal-field automorphism. Therefore, S contains a 2-element z such that $C_G(z) \cap x^G = \varnothing$. Consequently $1 < |z^G|_p < |h^G|_p$; a contradiction. Hence, S does not isomorphic $L_2(q)$ for odd q. It follows that $C_S(P) = Z(P)$. Since $|S| = |S||$ and $S \in R(p)$, we see that $\{p\}$ is a connected component of $GK(S)$. The group of outer automorphisms of J_1 is trivial, therefore S does not isomorphic J_1. If $S \simeq^2 G_2(q)$, then from [14] it follows that 2 is not a connected components of $GK(S)$. If $S \simeq L_2(q)$ for even q, then $Out(S)$ is isomorphic to a group of field automorphisms. By analogy as before we can assume that $C_S(x)$ is not a direct product of a Sylow p-subgroup and a p-complement; a contradiction. Thus, $p > 2$.

From description of finite simple groups with an abelian Sylow p-subgroup [12] it follows that p does not divide the orders of graph and diagonal automorphism groups. Lemma 2.5 and fact that subgroup of field automorphisms is a normal subgroup of $Out(S)$, implies $G \simeq (S \times X).F$, where F is a some cyclic p-group. In particular, we get that $X.F$ is a p-group. We can assume that $F = \langle xX \rangle$. As noted above $C_X(x) < X$; a contradiction with fact that $|a^G|_p = 1$ for each $a \in X$. Let $x \in G$ be a p-element such that $Ind_G(x)_p > 1$. We have $x \in SC_G(S)$. Since S and $C_G(S)$ are normal subgroups of G with trivial intersection, we obtain x has unique expression in form $x = x_S x_C$ where $x_S \in S, x_C \in C_G(S)$. Moreover from Lemma 2.2 it follows that $C_G(x) = C_G(x_S) \cap C_G(x_C)$. From Lemma 2.12 it follows that S contains h such that a p-element $y \in C_G(h)$

iff $y \in C_G(S)$. Therefore, for each $A \in Syl_p(C_G(h))$ there exists $B \in Syl_p(C_G(x_S))$ such that $A < B$. Since $Ind_S(x_S)_p < Ind_S(h)_p$ and $x_C \in C_G(h)$, we obtain $Ind_G(x)_p < In_G(hx_C)$; a contradiction. ■

Let $O = O_p(G)$. Lemma 3.3 implies that O includes the socle of G. Therefore, $C_G(O) = Z(O)$, and for each $h \in G \setminus O$ we have $Ind_G(h)_p = |G||_p$.

Lemma 3.4. $|G|_p > |O|$

Proof. Assume that $|G|_p = |O|$. Let $x \in O$ such that $Ind_O(x) > 1$ and $h \in G$ be a p'-element. We have $|C_O(x)| > |Z(O)|$. Therefore, $|C_O(h)| > |Z(O)|$, consequently $C_O(h)$ contains an element y such that $Ind_O(y) > 1$. Hence $C_O(y) = C_O(h)$. From Lemma 2.8 it follows that $C_O(h) = O$; a contradiction. ■

From Lemma 3.4 if follows that $|G|_p > |O|$. Let $h \in G$ be a p'-element and $x \in C_G(h) \setminus O$ be a p-element. Using Lemma 2.8 we can show that $Ind_G(x)_p = 1$, so $x \in C_G(O) = Z(O)$, which is a contradiction. In particular p is a connected component of $GK(G/O)$.

Lemma 3.5. *The group O is abelian.*

Proof. Assume that there exists $x \in O \setminus Z(O)$. Put $h \in G$ is a p'-element. We have $|C_G(h)|_p = |C_G(x)|_p$, in particular $C_G(h)$ contains a p-element y such that $Ind_G(y)_p > 1$. From Lemma 2.8 it follows that $O < C_G(h)$; a contradiction. ■

Lemma 3.6. $Ind_G(x)_p = 1$ *for each* $x \in O$.

Proof. Assume that there exists $x \in O$ such that $Ind_G(x)_p > 1$. Lemma 3.5 implies that O is abelian. Therefore, $C_G(x) \geq O$. Put $h \in G$ is a p'-element. We know that p is a connected component of $GK(G/O)$, hence for each Sylow p-subgroup P of $C_G(h)$ we have $P \leq O$. Since $C_G(O) = Z(O)$, we see that $P \neq O$. Consequently $|G|_p/|C_O(x)| \geq |G|_p/|C_G(x)|_p = |G||_p$. Therefore, $|P| \geq |O|$; a contradiction. ■

Lemma 3.7. *The group G is non solvable.*

Proof. Assume that G is solvable. From Lemma 2.7 it follows that G/O is a Frobenius or 2-Frobenius group. Since kernel of G/O is a p'-group and Lemma 2.5, we obtain G/O is a Frobenius group with p'-kernel $\overline{K}$ and complement $\overline{F}$, else G is not minimal. Put $K < G$ be a minimal subgroup such that $KO/O = \overline{K}$. From Frattini argument we have $N(K)O/O \simeq \overline{K}$. Let $F \leq N_G(K)$ be a minimal subgroup such that $FO/O = \overline{F}$. Since G/O is a Frobenius group with the complement $\overline{F}$, we see that $N_G(F) < OF$; in particular $\pi(N_G(F)) = \{p\}$.

Let $H < K$ be maximal with respect to inclusion subgroup of K such that $C_O(H) > Z(G)$. We show that H is not trivial. For each $h \in K$ and $y \in F \setminus O$ we have $|h^G|_p = |y^G|_p$ and $|C_G(y)|_p > |Z(G)|_p$. Therefore, $C_O(h) > Z(G)$, in particular $H > 1$.

Let $x \in C_O(H) \setminus Z(G)$. Lemma 3.6 implies that $x \in Z(P)$ for some Sylow p-subgroup P of G. Hence $C_G(x)$ includes a subgroup V which is conjugated with F. We have $H < C_G(x)$. Since H is a maximal p'-subgroup with a non trivial centralizer in O, we see that H is a Hall p'-subgroup of $C_G(x)$. In particular $C_G(x) = O.\overline{H}.\overline{F}$, where $\overline{H} = HO/O$. Let $y \in C_O(H) \setminus Z(G)$. We can show that $C_G(y) = O.\overline{H} \leftthreetimes \overline{R}$. Since $\overline{R} < N_{G/O}(\overline{H})$ and $\overline{F} < N_{G/O}(\overline{H})$, we have $\overline{HRF}$ is a Frobenius group with the kernel $\overline{H}$ and the complement $\overline{F}$. Therefore, $\overline{HRF} = \overline{HR} = \overline{HF}$ and $C_G(y) = C_G(x)$. Thus, $C_G(x) = C_G(C_O(H))$.

Let $N = N_K(H)$. Since K is nilpotent, we get that $H = K$ or $N > H$. We have $N < N_G(C_G(H))$. Therefore, $N < N_G(C_G(C_O(H)))$. If $N > H$, then $N \not\leq C_G(C_O(H))$. Thus, $N_G(F)$ includes a subgroup X such that $(X/H) \simeq N/H$; a contradiction with fact that $N_G(F)$ is a p-group. It follows that $H = K$. We have $F < C_G(C_O(H))$ and O is abelian. Therefore, $C_O(H) \leq Z(G)$; a contradiction. ■

From Lemmas 3.7 and 2.7 it follows that $G/O \simeq F.S$, where F is a nilpotent $\pi_1(G/O)$-group, and S is a nonabelian simple group. Let $g \in G$ be a p'-element. Put $H_g \leq C_G(g)$ a subgroup generated by all $(\pi(|g|) \cup \{p\})'$-elements. We have $H_g \leq C_G(C_O(g))$. Therefore, H_g acts regularly on $O/C_O(g)$. Hence Sylow subgroups of H_g are cyclic or quaternion groups.

Lemma 3.8. $F = 1$.

Proof. Assume that $|\pi(F)| > 1$. Let $h \in F$ be a t-element for some $t \in \pi(F)$. We have $C_F(h)$ includes some Hall t'-subgroup T of F. Since Sylow subgroups of H_h are cyclic or quaternion, we get S acts trivially on T. Therefore, F is a t-group for some prime $t \neq p$.

Assume that there exists $h \in Z(F)$ such that $|\pi(C_{F.S}(h))| > 1$. Let $g \in C_{F.S}(h)$ be a t'-element. Assume that $Z(F) < C_{F.S}(g)$. Since $Z(F)$ is a normal subgroup of $F.S$, we have $\langle g^G \rangle < C_{F.S}(Z(F))$. The group S is a simple group, therefore, $\langle g^G \rangle F/F = S$. In particular $F.S$ contains an element of order pt; a contradiction. Hence $[Z(F), g] > 1$. We have H_h acts regularly on the $O/C_O(h)$ and $\langle g^{C_{F.S}(h)} \rangle \leq H_h$. This assertion contradicts with Lemma 2.11. Therefore, $\pi(C_{F.S}(h)) = \{t\}$ for each $h \in Z(F)$.

Assume that there exists $a \in F.S$ a t-element such that $|\pi(C_{F.S}(a))| > 1$. Since $C_{F.S}(a) \cap Z(F) > 1$, we get a contradiction with Lemma 2.11.

Therefore, t is a connected component of $GK(F.S)$. Since $t \in \pi_1(GK(F.S))$, we get $t = 2$. From a description of the prime graph of finite simple groups [14] it follows that $S \simeq Alt_5$. From Brauer 2-character tables [3] it follows that $F.S$ contains an element x such that $|x| \in \{6, 10\}$; a contradiction. ■

From Lemmas 3.7 and 3.8 it follows that G/O is a simple group and p is a connected component of $GK(G/O)$.

Lemma 3.9. $G \simeq O$

Proof. Assume that S is an alternating simple group of degree n. An alternating group has disconnected prime graph iff one of the numbers $n, n-1, n-2$ is prime, and this number is a connected component of the prime graph. In particular, if $n > 6$, then $3 \in \pi_1(S)$. If $n > 6$, then S contains an element g of order 3 such that $C_S(g) \simeq \langle g \rangle \times Alt_{n-3}$. Therefore, in this case H_g includes a Frobenius group; a contradiction with assertion that H_g acts regular on $O/C_O(g)$. Let $n \in \{5, 6\}$. Therefore, S contains an element g such that for each p-element $h \in S$ we have $\langle h, g \rangle = S$. We have $C_O(g) > Z(G)$. Put $x \in C_O(g) \setminus Z(G)$. It follows from Lemma 3.5 that $|x^G|_p = 1$. Hence, $C_G(x)$ includes a Sylow p-subgroup P of G. In particular $\langle g, PO/O \rangle = S$. That signifies that $C_G(x) = G$; a contradiction.

Assume that S is a group of Lie type. If Lie rank of S is more then 2, then S contains an element g such that H_g includes a Frobenius group. Therefore, we can assume that Lie rank of S is 1 or 2. Assume that $S \simeq L_2(q)$. We have that S is generated by a pair a, b where $|a| = (q+1)/2, |b| = (q-1)/2$. Since $p = (q-1)/2$ or $p = (q+1)/2$, we can assume that $C_O(g) \leq Z(G)$; a contradiction. Groups $L_3(q)$ and $U_3(q)$ contain an element g such that $C_S(g)$ includes $L_2(q)$. Therefore, S is not isomorphic to one of a $L_3(q)$ or $U_3(q)$. Similarly, it can be shown that S is not isomorphic to $B_2(q),^2 B_2(q), G_2(q),^2 G_2(q)$ and sporadic groups. ■

Lemma 3.9 completes proof of the theorem.

4 Proof of Corollaries

Proof of Corollary 1.

Proof. If G is a p-group, then the corollary is satisfied. Let $h \in G$ be a p'-element. We have $|C_G(h)|_p > |Z(P)|$, where $P \in Syl_p(G)$. Let $H \in Syl_p(C_G(h))$. It follows from Lemma 2.6 that $G/O_{p'}(G) \in R^*(p)$. From Theorem 1 we get that $G/O_{p'}(G)$ is a p-group. Since $|H| > |Z(P)|$, we get that H contains x such that $Ind_{G/O_{p'}(G)}(xO_{p'}(G)) = p^e$, for some $e > 0$. Therefore, $Ind_{G/O_{p'}(G)}(xO_{p'}(G))_p = Ind_G(x)_p = p^e$. Hence $C_G(h)$ includes some Sylow p-subgroup of $C_G(x)$. From Lemma 2.4 it follows that $Z(P) \leq Z(G)$. ■

Proof of Corollary 2.

Proof. If $G \in R^*(p)$, then from Corollary 1 it follows that $Z(G) > 1$; a contradiction. Therefore, $Ind_G(x)_p = 1$ for each p-element x of G. From Lemma 2.13 it follows that $O^{p'}(G/O_{p'}(G)) = S_1 \times \cdots \times S_r \times H$. From Lemmas 2.5 and 2.6 it follows that $O^{p'}(G/O_{p'}(G)) \in R(p)$. Since S_1 is a subnormal subgroup of $G/O_{p'}(G)$, we get $|x^{G/O_{p'}(G)}|$ is a multiple of $|x^{S_1}|$ for each $x \in S_1$. Assume that $p = 3$ and $S_1 \simeq Ru$. We have S_1 contains x of order 12 such that $|C_{S_1}(x)| = 24$. Therefore, $1 < |x^{S_1}|_3 < |G||_3$; a contradiction. If $S_1 \simeq J_4$, then S_1 contains x of order 21 such that $|C_{S_1}(x)| = 42$; a contradiction with definition of $R^*(p)$-groups. Assume that $S_1 \simeq {}^2F_4(q)'$. According to the results of [13], there is just one conjugacy class of elements of order 3 in S_1. Therefore, $N_{S_1}(\langle x\rangle) \simeq C_{S_1}(x) : 2 = 3.U_3(q) : 2$ where $x \in S_1$ is an element of order 3 [10]. Thus, S_1 contains an element y of order 6 such that $1 < |y^{S_1}|_3 < |G||_3$; a contradiction. Assume that $p = 5$ and $S_1 \simeq Th$. In this case S_1 contains an element x of order 8 such that $|C_{S_1}(x)| = 96$; a contradiction. The assertion of Corollary 2 follows from Lemma 2.13. ■

References

[1] A. Beltran, M. Felipe: On the solvability of groups with four class sizes. *J. Algebra Appl.* 11 (9) (2012) 1–7.

[2] C. Casolo, S. Dolfi, E. Jabara: Finite groups whose noncentral class sizes have the same p-part for some prime p. *Israel J. Math.* 192 (2012) 197–219.

[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson: *Atlas of finite groups*. Oxford: Clarendon Press (1985).

[4] D. Gorenstein: *Finite groups*. New York-London (1968).

[5] I. B. Gorshkov,: On Thompson's conjecture for alternating and symmetric groups of degree more then 1361. *Proceedings of the Steklov Institute of Mathematics* 293 (1) (2016) 58–65.

[6] I. B. Gorshkov: On Thompson's conjecture for finite simple groups. *Communications in Algebra* 47 (2019) 5192–5206.

[7] I. B. Gorshkov: On a connection between the order of a finite group and the set of conjugacy classes size. arXiv:1804.04594.

[8] K. Ishikawa: On finite p-groups which have only two conjugacy length. *Israel J. Math.* 129 (2002) 119–123.

[9] A. S. Kondrat'ev, V. D. Mazurovv: 2-signalizers of finite simple groups. *Algebra Logic* 42 (5) (2003) 333–348.

[10] G. Malle: The Maximal Subgroups of ${}^2F_4(q)$. *J. Algebra* 139 (1991) 52–69.

[11] G. Navarro, P. H. Tiep: Abelian Sylow subgroups in a finite group. *J. Algebra* 398 (2014) 519–526.

[12] R. Shen, Y. Zhou: Finite simple groups with some abelian Sylow subgroups. *Kuwait J. Sci.* 43 (2) (2016) 1–15.

[13] K. Shinoda,: The conjugacy classes of the finite Ree groups of type ${}^2F_4(q)$. *J. Fat. Sci. Uniu. Tokyo* 22 (1975) 1–15.

[14] A. V. Vasilev, E. P. Vdovin: An Adjacency Criterion for the Prime Graph of a Finite Simple Group. *Algebra Logic* 44 (6) (2005) 381–406.

[15] A. V. Vasilev, E. P. Vdovin: On Thompson's conjecture. *Siberian electronic mathematical reports* 6 (2009) 457–464.

[16] A. V. Vasilev: On finite groups isospectral to simple classical groups. *J. Algebra* 423 (2015) 318–374.

[17] J. H. Walter: The characterization of finite groups with abelian Sylow 2-subgroups. *Ann. of Math.* 89 (2) (1969) 405–514.

[18] J. Williams: Prime graph components of finite groups. *J. Algebra* 69 (1981) 487–513.

Received: October 03, 2019
Accepted for publication: March 04, 2020
Communicated by: Ivan Kaygorodov