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Galois cohomology of reductive algebraic groups over the field of real numbers

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Abstract. We describe functorially the first Galois cohomology set $H^1(\mathbb{R}, G)$ of a connected reductive algebraic group G over the field $\mathbb{R}$ of real numbers in terms of a certain action of the Weyl group on the real points of order dividing 2 of the maximal torus containing a maximal compact torus.

This result was announced with a sketch of proof in the author's 1988 note [3]. Here we give a detailed proof and a few examples.

To the memory of Arkady L'vovich Onishchik

1 Introduction

Let G be a connected reductive algebraic group over the field $\mathbb{R}$ of real numbers. We wish to compute the first Galois cohomology set $H^1(\mathbb{R}, G) = H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G(\mathbb{C}))$. In terms of Galois cohomology one can state answers to many natural questions; see Serre [19], Section III.1, and Berhuy [1].

The Galois cohomology of classical groups and adjoint groups is well known. The Galois cohomology of compact groups was computed by Borel and Serre [2], Theorem 6.8; see also Serre's book [19], Section III.4.5. Here we consider the case of a general connected reductive group over $\mathbb{R}$. We describe $H^1(\mathbb{R}, G)$ in terms of a certain action of the Weyl group on the first Galois cohomology of the maximal torus containing a maximal compact torus. Our main result is Theorem 3.1.

Our description of $H^1(\mathbb{R}, G)$ is inspired by Borel and Serre [2]. Our result was announced in [3]; here we give a detailed proof and a few examples.

Since it was announced in [3], our Theorem 3.1 has been used in a few articles, in particular, in [17], [10], and [15]. In [4], Gornitskii, Rosengarten, and the author described, using Theorem 3.1, the Galois cohomology of *quasi-connected* reductive $\mathbb{R}$-groups (normal subgroups of connected reductive $\mathbb{R}$-groups). Our description in [4] is similar to that of Theorem 3.1. In [5] Evenor and the author used Theorem 3.1 to describe *explicitly* the

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Galois cohomology of simply connected semisimple $\mathbb{R}$-groups. In [6] and [7], Timashev and the author used Theorem 3.1 to describe explicitly the Galois cohomology of connected reductive $\mathbb{R}$-groups.

Note that cited articles refer to Theorem 9 of an early preprint version of this note. In this published version, Theorem 9 became Theorem 3.1.

2 Preliminaries

We recall the definition of the first Galois cohomology set $H^1(\mathbb{R}, G)$ of an algebraic group G defined over $\mathbb{R}$. The set of 1-cocycles is defined by $Z^1(\mathbb{R}, G) = \{z \in G(\mathbb{C}) \mid z\bar{z} = 1\}$ where the bar denotes complex conjugation. The group $G(\mathbb{C})$ acts on the right on $Z^1(\mathbb{R}, G)$ by

$$z * x = x^{-1} z \bar{x},$$

where $z \in Z^1(\mathbb{R}, G)$ and $x \in G(\mathbb{C})$. By definition $H^1(\mathbb{R}, G) = Z^1(\mathbb{R}, G)/G(\mathbb{C})$. Let $G(\mathbb{R})_2$ denote the subset of elements of $G(\mathbb{R})$ of order 2 or 1. Then $G(\mathbb{R})_2 \subset Z^1(\mathbb{R}, G)$, and we obtain a canonical map $G(\mathbb{R})_2 \to H^1(\mathbb{R}, G)$.

2.1. Lemma. *Let S be an algebraic $\mathbb{R}$-torus. Let S_0 denote the largest compact (that is, anisotropic) $\mathbb{R}$-subtorus in S, and let S_1 denote the largest split subtorus in S. Then :*

(a) *The map $\lambda\colon S(\mathbb{R})_2 \to H^1(\mathbb{R}, S)$ induces a canonical isomorphism*

$$S(\mathbb{R})_2/S_1(\mathbb{R})_2 \xrightarrow{\sim} H^1(\mathbb{R}, S).$$

(b) *The composite map $\mu\colon S_0(\mathbb{R})_2 \to H^1(\mathbb{R}, S_0) \to H^1(\mathbb{R}, S)$ is surjective.*

(c) *$(S_0 \cap S_1)(\mathbb{R}) = S_0(\mathbb{R})_2 \cap S_1(\mathbb{R})_2$, and the surjective map μ of (b) induces an isomorphism*

$$S_0(\mathbb{R})_2/(S_0 \cap S_1)(\mathbb{R}) \xrightarrow{\sim} H^1(\mathbb{R}, S).$$

Proof. Any $\mathbb{R}$-torus is isomorphic to a direct product of tori of three types, see Casselman [9], Section 2:

(1) $\mathbb{G}_{\mathrm{m},\mathbb{R}}$,

(2) $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{\mathrm{m},\mathbb{C}}$,

(3) $\mathrm{R}^1_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{\mathrm{m},\mathbb{C}}$. Here $\mathbb{G}_\mathrm{m}$ denotes the multiplicative group, $\mathrm{R}_{\mathbb{C}/\mathbb{R}}$ denotes the Weil restriction of scalars, and

$$\mathrm{R}^1_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{\mathrm{m},\mathbb{C}} = \ker\left[\mathrm{Nm}_{\mathbb{C}/\mathbb{R}}\colon\ \mathrm{R}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathbb{G}_{\mathrm{m},\mathbb{R}}\right],$$

where $\mathrm{Nm}_{\mathbb{C}/\mathbb{R}}$ is the norm map.

We prove (a). The composite homomorphism $S_1(\mathbb{R})_2 \hookrightarrow S(\mathbb{R})_2 \to H^1(\mathbb{R}, S)$ factors via $H^1(\mathbb{R}, S_1) = 1$, and hence it is trivial. We obtain an induced homomorphism

$$S(\mathbb{R})_2/S_1(\mathbb{R})_2 \to H^1(\mathbb{R}, S);$$

we must prove that it is bijective. It suffices to consider the three cases:

(1) $S = \mathbb{G}_{\mathrm{m},\mathbb{R}}$, that is, $S(\mathbb{R}) = \mathbb{R}^\times$. Then $H^1(\mathbb{R}, S) = 1$. We have $S_1 = S$, so $S(\mathbb{R})_2/S_1(\mathbb{R})_2 = 1$. This proves (a) in case (1).

(2) $S = \mathrm{R}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{\mathrm{m},\mathbb{C}}$, that is $S(\mathbb{R}) = \mathbb{C}^\times$. Then $H^1(\mathbb{R}, S) = 1$. We have $S_1 = \mathbb{G}_{\mathrm{m},\mathbb{R}}$, $S_1(\mathbb{R}) = \mathbb{R}^\times$, $S_1(\mathbb{R})_2 = \{1, -1\} = S(\mathbb{R})_2$, so $S(\mathbb{R})_2/S_1(\mathbb{R})_2 = 1$. This proves (a) in case (2).

(3) $S = \mathrm{R}^1_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{\mathrm{m},\mathbb{C}}$, that is $S(\mathbb{R}) = \{x \in \mathbb{C}^\times \mid \mathrm{Nm}(x) = 1\}$, where $\mathrm{Nm}(x) = x\bar{x}$. Then by the definition of Galois cohomology $H^1(\mathbb{R}, S) = \mathbb{R}^\times/\,\mathrm{Nm}(\mathbb{C}^\times) \simeq \{-1, 1\}$. The homomorphism $S(\mathbb{R})_2 = \{-1, 1\} \to H^1(\mathbb{R}, S)$ is an isomorphism. This proves (a) in case (3).

Assertion (b) reduces to the cases (1), (2), (3), where it is obvious (note that only in case (3) we have $H^1(\mathbb{R}, S) \neq 1$).

Concerning (c), we have a commutative diagram

$$\begin{array}{ccc} S_0(\mathbb{R})_2 & \hookrightarrow & S(\mathbb{R})_2 \\ \downarrow & \overset{\mu}{\searrow} & \downarrow \lambda \\ H^1(\mathbb{R}, S_0) & \longrightarrow & H^1(\mathbb{R}, S). \end{array}$$

We see from (a) that $\ker \mu = S_0(\mathbb{R})_2 \cap S_1(\mathbb{R})_2$, and we know from (b) that μ is surjective. Thus we obtain a canonical isomorphism

$$S_0(\mathbb{R})_2/(S_0(\mathbb{R})_2 \cap S_1(\mathbb{R})_2) \xrightarrow{\sim} H^1(\mathbb{R}, S).$$

It remains only to check that $S_0(\mathbb{R})_2 \cap S_1(\mathbb{R})_2 = (S_0 \cap S_1)(\mathbb{R})$. This can be easily checked in each of the cases (1), (2), (3) (note that only in case (2) this group is nontrivial). □

2.2. Corollary. *Assume that S is an $\mathbb{R}$-torus such that $S = S' \times S''$, where S' is a compact torus and $S'' = \mathrm{R}_{\mathbb{C}/\mathbb{R}}\, T$, where T is a $\mathbb{C}$-torus. Then $H^1(\mathbb{R}, S) = H^1(\mathbb{R}, S') = S'(\mathbb{R})_2$.*

Proof. The assertion follows from the proof of Lemma 2.1(a), because S' is a direct product of tori of type (3); hence $H^1(\mathbb{R}, S') = S'(\mathbb{R})_2$, and S'' is a direct product of tori of type (2), whence $H^1(\mathbb{R}, S'') = 1$. □

We say that a connected real algebraic group H is *compact*, if the group $H(\mathbb{R})$ is compact, that is, H is reductive and anisotropic. We shall need the following two standard facts.

2.3 Lemma (well-known). *Any nontrivial semisimple algebraic group H over $\mathbb{R}$ contains a nontrivial connected compact subgroup.*

Proof. This assertion follows from the classification, (see, for instance, Helgason [12], Section X.6.2, Table V). We prove it without using the classification.

Let $\kappa\colon \mathfrak{h}\times\mathfrak{h}\to\mathbb{R}$ denote the Killing form on $\mathfrak{h}$. Let $\mathfrak{h}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of the real semisimple Lie algebra $\mathfrak{h}=\mathrm{Lie}\ H$. This means that the linear transformation

$$\theta\colon \mathfrak{h}\to\mathfrak{h},\quad k+p\mapsto k-p\quad \text{for } k\in\mathfrak{k},\, p\in\mathfrak{p}$$

is an automorphism of $\mathfrak{h}$, and that the bilinear form

$$b_\theta(x,y)=-\kappa\big(x,\theta(y)\big)$$

is positive definite on $\mathfrak{h}$. Set

$$K=\big\{h\in H \mid \mathrm{Ad}\ h\in O(\mathfrak{h},b_\theta)\big\}.$$

Then K is a real algebraic subgroup of H. We have $\mathrm{Lie}\ K=\mathfrak{k}$; see Gorbatsevich, Onishchik, and Vinberg [11], Section 4.3.2. Since $H(\mathbb{R})$ has finitely many connected components and the center of $H(\mathbb{R})^0$ is finite, by [11], Corollary 5 of Theorem 4.3.2, the group $K(\mathbb{R})$ is compact.

Since $\mathfrak{p}$ and $\mathfrak{k}$ are the eigenspaces of θ with eigenvalues -1, and $+1$, respectively, we have $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$. If $\mathfrak{k}=0$, then $\mathfrak{h}=\mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}]=0$, whence $\mathfrak{h}$ is commutative, which is clearly impossible. Thus $\mathfrak{k}\neq 0$. But $\mathfrak{k}$ is the Lie algebra of the identity component K^0 of K, which is a connected compact algebraic subgroup of H. Thus H contains a nontrivial connected compact algebraic subgroup. □

2.4 Lemma (well-known). *Any two maximal compact tori in a connected reductive real algebraic group H are conjugate under $H(\mathbb{R})$.*

Proof. It suffices to prove that any two maximal compact tori in the derived group $[H,H]$ of H are conjugate. This follows from the following well-known facts from the theory of Lie groups: (1) Any two maximal compact subgroups in a connected semisimple Lie group are conjugate (see, for instance, Gorbatsevich, Onishchik, and Vinberg [11], Section 4.3.4, Theorem 3.5); (2) Any two maximal tori in a connected compact Lie group are conjugate (see, for instance, Onishchik and Vinberg [16], Section 5.2.7, Theorem 15). □

3 Main result

Let G be a connected reductive algebraic group over $\mathbb{R}$. Let T_0 be a maximal compact torus in G. Set $T=\mathcal{Z}(T_0)$, $N_0=\mathcal{N}(T_0)$, $W_0=N_0/T$, where $\mathcal{Z}$ and $\mathcal{N}$ denote the centralizer and the normalizer in G, respectively.

We prove that T is a torus. By Humphreys [13], Theorem 22.3 and Corollary 26.2.A, the centralizer T of T_0 is a connected reductive $\mathbb{R}$-group. The torus T_0 is a maximal compact torus in T, and it is central in T. Since by Lemma 2.4 all the maximal compact tori in T are conjugate under $T(\mathbb{R})$, we see that T_0 is the only maximal compact torus in T. It follows that the derived group $[T,T]$ of T contains no nontrivial compact tori. By Lemma 2.3 every nontrivial semisimple group over $\mathbb{R}$ has a nontrivial compact connected algebraic subgroup, hence a nontrivial compact torus. We conclude that $[T,T]=1$, and hence T is a torus. We see that T is a *fundamental torus in* G, that is, a maximal torus containing a maximal compact torus.

We have a right action of W_0 on T_0 defined by $(t, w) \mapsto t \cdot w := n^{-1}tn$, where $t \in T_0(\mathbb{C})$, $n \in N_0(\mathbb{C})$, n represents $w \in W_0(\mathbb{C})$. This action is defined over $\mathbb{R}$. We prove that $W_0(\mathbb{C})$ acts on T_0 effectively. Indeed, if $w \in W_0(\mathbb{C})$ with representative $n \in N_0(\mathbb{C})$ acts trivially on T_0, then $n^{-1}tn = t$ for any $t \in T_0(\mathbb{C})$, and hence $n \in T(\mathbb{C})$ (because the centralizer of T_0 is T), whence $w = 1$.

We prove that $W_0(\mathbb{C}) = W_0(\mathbb{R})$. We have seen that $W_0(\mathbb{C})$ embeds in $\mathrm{Aut}_{\mathbb{C}}(T_0)$. Since T_0 is a compact torus, all the complex automorphisms of T_0 are defined over $\mathbb{R}$. We see that the complex conjugation acts trivially on $\mathrm{Aut}_{\mathbb{C}}(T_0)$, and hence on $W_0(\mathbb{C})$. Thus $W_0(\mathbb{R}) = W_0(\mathbb{C})$.

Note that N_0 normalizes T; hence W_0 acts on T. We define a right action $*$ of $W_0(\mathbb{R})$ (which is equal to $W_0(\mathbb{C})$) on $H^1(\mathbb{R}, T)$. Let $z \in Z^1(\mathbb{R}, T)$, $n \in N_0(\mathbb{C})$, z represents $\xi \in H^1(\mathbb{R}, T)$, n represents $w \in W_0(\mathbb{R}) = W_0(\mathbb{C})$. We set

$$\xi * w = [n^{-1}z\bar{n}] = [n^{-1}zn \cdot n^{-1}\bar{n}],$$

where brackets [ ] denote the cohomology class.

We prove that $*$ is a well defined action. First, since N_0 normalizes T and $z \in T(\mathbb{C})$, we see that $n^{-1}zn \in T(\mathbb{C})$. Now $w \in W_0(\mathbb{R})$, whence $w^{-1}\bar{w} = 1$ and $n^{-1}\bar{n} \in T(\mathbb{C})$. It follows that $n^{-1}z\bar{n} = n^{-1}zn \cdot n^{-1}\bar{n} \in T(\mathbb{C})$. We have

$$n^{-1}z\bar{n} \cdot \overline{n^{-1}z\bar{n}} = n^{-1}z\bar{n}\bar{n}^{-1}\bar{z}n = 1$$

because $z\bar{z} = 1$. Thus $n^{-1}z\bar{n} \in Z^1(\mathbb{R}, T)$. If $z' \in Z^1(\mathbb{R}, T)$ is another representative of ξ, then $z' = t^{-1}z\bar{t}$ for some $t \in T(\mathbb{C})$, and

$$n^{-1}z'\bar{n} = n^{-1}t^{-1}z\bar{t}\bar{n} = (n^{-1}tn)^{-1} \cdot n^{-1}z\bar{n} \cdot \overline{n^{-1}tn} = (t')^{-1}(n^{-1}z\bar{n})\overline{t'}$$

where $t' = n^{-1}tn$, $t' \in T(\mathbb{C})$. We see that the cocycle $n^{-1}z'\bar{n} \in Z^1(\mathbb{R}, T)$ is cohomologous to $n^{-1}z\bar{n}$. If n' is another representative of w in $N_0(\mathbb{C})$, then $n' = nt$ for some $t \in T(\mathbb{C})$, and $(n')^{-1}z\overline{n'} = t^{-1}n^{-1}x\bar{n}\bar{t}$. We see that $(n')^{-1}z\overline{n'}$ is cohomologous to $n^{-1}z\bar{n}$. Thus $*$ is indeed a well defined action of the group $W_0(\mathbb{R})$ on the set $H^1(\mathbb{R}, T)$.

Note that in general $[1] * w = [n^{-1}\bar{n}] \neq [1]$, and therefore, the action $*$ does not respect the group structure in $H^1(\mathbb{R}, T)$.

Let $\xi \in H^1(\mathbb{R}, T)$ and $w \in W_0(\mathbb{R})$. It follows from the definition of the action $*$ that the images of ξ and $\xi * w$ in $H^1(\mathbb{R}, G)$ are equal. We see that the map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$ induces a map $H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \to H^1(\mathbb{R}, G)$.

The following theorem is the main result of this note:

3.1. Theorem. *Let G, T_0, T, and W_0 be as above. The map*

$$H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \to H^1(\mathbb{R}, G)$$

induced by the map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$ is a bijection.

Proof. We prove the surjectivity. It suffices to show that the map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$ is surjective. This was proved by Kottwitz [14], Lemma 10.2, with a reference to Shelstad [20]. We give a different proof. Let $\eta \in H^1(\mathbb{R}, G)$, $\eta = [z]$, $z \in G(\mathbb{C})$, $z\bar{z} = 1$. Let $z = us = su$, where s and u are the semisimple and the unipotent parts of z, respectively (see Humphreys [13], Theorem 15.3). We have $us\bar{u}\bar{s} = 1$, where $\bar{u}\bar{s} = \bar{s}\bar{u}$ (because $us = su$). Thus $us = \bar{u}^{-1}\bar{s}^{-1}$, where u and $\bar{u}^{-1}$ are unipotent, s and $\bar{s}^{-1}$ are semisimple, $us = su$. From the equality $\bar{u}\bar{s} = \bar{s}\bar{u}$ it follows that $\bar{u}^{-1}\bar{s}^{-1} = \bar{s}^{-1}\bar{u}^{-1}$. Since the Jordan decomposition in $G(\mathbb{C})$ is unique (see Humphreys [13], Theorem 15.3), we conclude that $s = \bar{s}^{-1}$, $u = \bar{u}^{-1}$. In other words, $s\bar{s} = 1$, $u\bar{u} = 1$, that is, s and u are cocycles.

Since u is unipotent, the logarithm $\log(u) \in \mathrm{Lie}\ G_\mathbb{C}$ is defined. We have:

$$\log(u) + \overline{\log(u)} = 0.$$

Set $y = \frac{1}{2}\log(u)$, then $y + \bar{y} = 0$. We have $-y + \log(u) + \bar{y} = 0$, where $-y$, $\bar{y}$ and $\log(u)$ pairwise commute. Set $u' = \exp(y)$, then $(u')^{-1}u\overline{u'} = 1$. Since s commutes with u, we have $\mathrm{Ad}(s)y = y$, and hence s commutes with u'. We obtain $(u')^{-1}su\overline{u'} = s$, and hence the cocycle $z = su$ is cohomologous to the cocycle s, where s is semisimple.

We may and shall therefore assume that z is semisimple. Set $C = \mathcal{Z}_{G_\mathbb{C}}(z)$. Since $\bar{z} = z^{-1}$, we have $\overline{C} = C$, and hence the algebraic subgroup C of $G_\mathbb{C}$ is defined over $\mathbb{R}$. The semisimple element z is contained in a maximal torus of $G_\mathbb{C}$ (see Humphreys [13], Theorem 22.2); hence z is contained in the identity component C^0 of C. The group C^0 is reductive, see Steinberg [21], Section 2.7(a). Let T' be a maximal torus of C^0 defined over $\mathbb{R}$, then $z \in T'(\mathbb{C})$, because z is contained in the center of C^0. By Lemma 2.1(b) the class η of z comes from the maximal compact subtorus T'_0 of T'. By Lemma 2.4 any compact torus in G is conjugate under $G(\mathbb{R})$ to a subtorus of T_0. Thus η comes from $H^1(\mathbb{R}, T_0)$, hence from $H^1(\mathbb{R}, T)$. This proves the surjectivity in Theorem 3.1.

We prove the injectivity in Theorem 3.1. Let $z, z' \in T(\mathbb{C})$, $z\bar{z} = 1$, $z'\overline{z'} = 1$, $z = x^{-1}z'\bar{x}$, where $x \in G(\mathbb{C})$. We shall prove that $z = n^{-1}z'\bar{n}$ for some $n \in N_0(\mathbb{C})$.

For $g \in G(\mathbb{C})$ set $g^\nu = z\bar{g}z^{-1}$. Then ν is an involutive antilinear automorphism of $G_\mathbb{C}$, and in this way we obtain a twisted form ${}_zG$ of G. Since $z \in T(\mathbb{C})$, the embeddings of the tori $T_\mathbb{C}$ and $T_{0,\mathbb{C}}$ into ${}_zG_\mathbb{C}$ are defined over $\mathbb{R}$. We denote the corresponding $\mathbb{R}$-tori of ${}_zG$ again by T and T_0, respectively. The centralizer of T_0 in ${}_zG$ is T. The compact torus T_0 of ${}_zG$ is contained in some maximal compact torus S of ${}_zG$, and clearly S is contained in the centralizer T of T_0 in ${}_zG$. Since T_0 is the largest compact subtorus of T, we conclude that the $S = T_0$. Thus T_0 is a maximal compact torus in ${}_zG$.

Consider the embedding $i_x \colon t \mapsto x^{-1}tx \colon T_{0,\mathbb{C}} \to {}_zG_\mathbb{C}$. We have $i_x(t)^\nu = z\bar{x}^{-1}\bar{t}\bar{x}z^{-1}$. Since $z\bar{x}^{-1} = x^{-1}z'$, we obtain

$$z\bar{x}^{-1}\bar{t}\bar{x}z^{-1} = x^{-1}z'\bar{t}(z')^{-1}x = x^{-1}\bar{t}x = i_x(\bar{t})\,.$$

We see that $i_x(t)^\nu = i_x(\bar{t})$; hence i_x is defined over $\mathbb{R}$. Set $T'_0 = i_x(T_0)$; it is a compact algebraic torus in ${}_zG$, and $\dim T'_0 = \dim T_0$. Therefore, the torus T'_0 is conjugate to T_0 under ${}_zG(\mathbb{R})$, say, $T_{0,\mathbb{C}} = h^{-1}T'_{0,\mathbb{C}}h$, where $h \in {}_zG(\mathbb{R})$. Set $n = xh$. Then

$$n^{-1}T_{0,\mathbb{C}}\,n = h^{-1}x^{-1}T_{0,\mathbb{C}}\,xh = h^{-1}T'_{0,\mathbb{C}}h = T_{0,\mathbb{C}},$$

whence $n \in N_0(\mathbb{C})$. The condition $h \in {}_zG(\mathbb{R})$ means that $z\bar{h}z^{-1} = h$, or $h^{-1}z\bar{h} = z$. It follows that

$$n^{-1}z'\bar{n} = h^{-1}x^{-1}z'\bar{x}\bar{h} = h^{-1}z\bar{h} = z.$$

We have proved that there exists $n \in N_0(\mathbb{C})$ such that $z = n^{-1}z'\bar{n}$, and hence the cohomology classes $[z], [z'] \in H^1(\mathbb{R}, T)$ lie in the same orbit of $W_0(\mathbb{R})$ in $H^1(\mathbb{R}, T)$. This proves the injectivity in Theorem 3.1. ☐

3.2. Remark. If G is a compact group, then Theorem 3.1 asserts that

$$H^1(\mathbb{R}, G) = T(\mathbb{R})_2/W,$$

where T is a maximal torus in G, and W is the Weyl group with the usual action. This was earlier proved by Borel and Serre [2].

3.3. Remark. The real form G of $G_{\mathbb{C}}$ defines an involutive automorphism τ of the *based root datum of* $G_{\mathbb{C}}$, see [6], Proposition 3.7, and hence an involutive automorphism τ_D of the Dynkin diagram $D = D(G_{\mathbb{C}})$. This automorphism τ_D is trivial if and only if the derived group $[G, G]$ of G is an *inner form* of a compact group, that is, has a compact maximal torus. Let $\overline{D}$ denote the *twisted Dynkin diagram* corresponding to D and τ_D. Then W_0 is isomorphic to the Weyl group of $\overline{D}$; see [6], Proposition 7.11(iii). This Coxeter group is described in the book of Carter [8], Chapter 13.

4 Examples

In this section, written following a suggestion of the referee, we compute, *using Theorem 3.1*, the sets $H^1(\mathbb{R}, G)$ when $G = \mathrm{GL}_{n,\mathbb{R}}$, $\mathrm{Sp}_{2m,\mathbb{R}}$, $\mathrm{SO}_{p,q}$.

4.1. Example. Let $G = \mathrm{GL}_{2m,\mathbb{R}},\ m \in \mathbb{Z}_{>0}$. For $z = a + b\boldsymbol{i} \in \mathbb{C}$, we write

$$\mathcal{M}(z) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Consider the tori T and T_0 in G such that

$$T(\mathbb{R}) = \big\{ \operatorname{diag}(\mathcal{M}(z_1), \dots, \mathcal{M}(z_m)) \ \big|\ z_k = a_k + b_k\boldsymbol{i} \in \mathbb{C}^\times,\ k = 1, \dots, m \big\},$$
$$T_0(\mathbb{R}) = \big\{ \operatorname{diag}(\mathcal{M}(z_1), \dots, \mathcal{M}(z_m)) \in T(\mathbb{R}) \ \big|\ z_k = a_k + b_k\boldsymbol{i},\ a_k^2 + b_k^2 = 1 \big\}.$$

Then T_0 is a maximal compact torus in G, and T is a fundamental torus containing T_0. Since $T \simeq (\mathrm{R}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{\mathrm{m},\mathbb{C}})^m$, by the proof of Lemma 2.1, case (2), we have $H^1(\mathbb{R}, T) = \{1\}$, and by Theorem 3.1 we conclude that $H^1(\mathbb{R}, G) = \{1\}$.

Similarly, if $G = \mathrm{GL}_{2m+1,\mathbb{R}}$, then G has a fundamental torus

$$T \simeq \mathbb{G}_{\mathrm{m},\mathbb{R}} \times (\mathrm{R}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{\mathrm{m},\mathbb{C}})^m.$$

Again we have $H^1(\mathbb{R}, T) = \{1\}$ and $H^1(\mathbb{R}, G) = \{1\}$. Note that it is well known that $H^1(K, \mathrm{GL}_n) = \{1\}$ for any n and any field K; see Serre [18], Section X.1, Proposition 3.

4.2. Example. Let $G = \mathrm{SL}_{2,\mathbb{R}}$. It has a maximal torus $T = T_0$ with group of $\mathbb{R}$-points

$$T(\mathbb{R}) = \left\{\mathcal{M}(z) \;\middle|\; z = a + b\boldsymbol{i},\ a^2 + b^2 = 1\right\}.$$

Set $n = \mathrm{diag}(\boldsymbol{i}, -\boldsymbol{i}) \in G(\mathbb{C})$; then $n \in N(\mathbb{C})$, where $N = N_0 = \mathcal{N}_G(T)$. We have $\# H^1(\mathbb{R}, T) = 2$ with representatives $\mathcal{M}(1), \mathcal{M}(-1)$. An easy calculation shows that

$$n^{-1}\mathcal{M}(-1)\bar{n} = n^{-1}\mathcal{M}(-1)n \cdot n^{-1}\bar{n} = \mathcal{M}(1) = 1.$$

Thus $H^1(\mathbb{R}, T)/W_0 = \{1\}$, and by Theorem 3.1 we have $H^1(\mathbb{R}, G) = 1$.

Note that in this case the group $W = W_0 = N_0/T$ has two elements, and that the element $w := [n^{-1}] \in W(\mathbb{R})$ has no representative in $N_0(\mathbb{R})$, because otherwise we would have $[1] * w = [1]$. Thus in this case $W_0(\mathbb{R}) \neq N_0(\mathbb{R})/T(\mathbb{R})$.

4.3. Example. Let $G = \mathrm{Sp}_{2m} = \mathrm{Sp}(\mathbb{R}^{2m}, \psi)$, where ψ is the skew-symmetric bilinear form with matrix

$$M_\psi = \mathrm{diag}(J, \dots, J) \quad \text{where } J = \mathcal{M}(\boldsymbol{i}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The group G has a compact maximal torus $T = T_0$ with

$$T(\mathbb{R}) = \left\{ \mathrm{diag}(\mathcal{M}(z_1), \dots, \mathcal{M}(z_m)) \;\middle|\; z_k = a_k + b_k\boldsymbol{i},\ a_k^2 + b_k^2 = 1\right\}.$$

We have

$$T(\mathbb{R})_2 = \left\{ \mathrm{diag}(\mathcal{M}(z_1), \dots, \mathcal{M}(z_m)) \in T(\mathbb{R}) \;\middle|\; z_k = \pm 1\right\}.$$

Let $t = \mathrm{diag}(\mathcal{M}(z_1), \dots, \mathcal{M}(z_m)) \in T(\mathbb{R})_2$. Write

$$n = \mathrm{diag}(n_1, \dots, n_m), \quad \text{where } n_k = \begin{cases} \mathrm{diag}(\boldsymbol{i}, -\boldsymbol{i}) & \text{if } z_k = -1, \\ \mathrm{diag}(1, 1) & \text{if } z_k = 1. \end{cases}$$

Then $n \in N_0(\mathbb{C})$ and

$$n^{-1} \cdot t \cdot \bar{n} = 1.$$

We see that for any $[t] \in H^1(\mathbb{R}, T)$ there exists $w = [n] \in W_0(\mathbb{R})$ with

$$[t] * w = [1].$$

Thus $H^1(\mathbb{R}, T)/W_0 = \{1\}$, and by Theorem 3.1 we have $H^1(\mathbb{R}, G) = \{1\}$. Note that it is well known that $H^1(K, \mathrm{Sp}_{2m}) = \{1\}$ for any m and any field K; see Serre [19, Section III.1.2, Proposition 3] .

4.4. Example. Let $G = \mathrm{SO}(p', p'') = \mathrm{SO}(\mathbb{R}^{p'+p''}, f)$, where f is the diagonal quadratic form with matrix

$$M_f = \mathrm{diag}\big(\underbrace{+1, \dots, +1}_{p' \text{ times}}, \underbrace{-1, \dots, -1}_{p'' \text{ times}}\big).$$

We consider the case when both p' and p'' are even: $p' = 2r'$, $p'' = 2r''$. Our group G has a compact maximal torus $T = T_0$ with group of $\mathbb{R}$-points

$$T(\mathbb{R}) = \left\{ \operatorname{diag}\left(\mathcal{M}(z_1), \dots \mathcal{M}(z_{r'+r''})\right) \;\middle|\; z_k = a_k + \boldsymbol{i} b_k,\ a_k^2 + b_k^2 = 1 \right\}.$$

We have

$$T(\mathbb{R})_2 = \left\{ \operatorname{diag}\left(\mathcal{M}(z_1), \dots \mathcal{M}(z_{r'+r''})\right) \;\middle|\; z_k = \pm 1 \right\}.$$

The Weyl group $W = W(G_\mathbb{C}, T_\mathbb{C})$ is isomorphic to $(\pm 1)^{r'+r''-1} \rtimes S_{r'+r''}$, where $S_{r'+r''}$ is the symmetric group on the $r' + r''$ symbols $1, \dots, r' + r''$. The subgroup $(\pm 1)^{r'+r''-1}$ acts on $T(\mathbb{R})_2$ trivially. We compute $T(\mathbb{R})_2 / S_{r'+r''}$.

For a subset $\Xi \subseteq \{1, \dots, r' + r''\}$, we set

$$c_\Xi = \operatorname{diag}\left(\mathcal{M}(z_1), \dots, \mathcal{M}(z_{r'+r''})\right) \text{ with } z_k = \begin{cases} -1 & \text{if } k \in \Xi, \\ +1 & \text{otherwise.} \end{cases}$$

Then

$$T(\mathbb{R})_2 = \left\{ c_\Xi \;\middle|\; \Xi \subseteq \{1, \dots, r' + r''\} \right\}.$$

Consider the subgroup $S_{r'} \times S_{r''} \subseteq S_{r'+r''}$, where $S_{r''}$ is the symmetric group on the r'' symbols $r' + 1, \dots, r' + r''$. Its elements are represented by elements of $N(\mathbb{R})$, and hence they act on $T(\mathbb{R})_2$ by the usual conjugation:

$$c_\Xi * \sigma = c_{\sigma^{-1}\Xi} \quad \text{for } \sigma \in S_{r'} \times S_{r''}.$$

Write $\Xi = \Xi' \cup \Xi''$ where

$$\Xi' = \Xi \cap \{1, \dots, r'\}, \quad \Xi'' = \Xi \cap \{r' + 1, \dots, r' + r''\}.$$

We see that the W-orbit of c_Ξ depends only on the cardinalities of Ξ' and Ξ''.

The group $S_{r'+r''}$ is generated by its subgroup $S_{r'} \times S_{r''}$ and $\sigma_{1,r'+1} = (1, r' + 1)$. In order to compute the action of $\sigma_{1,r'+1}$ on $T(\mathbb{R})_2$, we consider the case $r' = 1$, $r'' = 1$, $G = \mathrm{SO}(2, 2)$. Consider the block matrix

$$n = \boldsymbol{i} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \quad \text{where } I_2 = \operatorname{diag}(1, 1).$$

One can check that $n \in N(\mathbb{C}) \subset G(\mathbb{C})$ and n represents $\sigma_{1,2}$. Let

$$c = c_{\{1,1\}} = \operatorname{diag}(-1, -1, -1, -1).$$

We have

$$n^{-1} c \bar{n} = c n^{-1} \bar{n} = 1.$$

Returning to the case of arbitrary r' and r'', we see that when both Ξ' and Ξ'' are non-empty, an element of Ξ' can be cancelled with an element of Ξ''. Thus the W-orbit of a cocycle c_Ξ depends only on the difference $\#\Xi' - \#\Xi''$. For s' and s'' such that $1 \le s' \le r'$, $1 \le s'' \le r''$, we write

$$c'_{s'} = c_{\Xi'} \quad \text{with} \ \ \Xi' = \{1, \dots, s'\},$$
$$c''_{s''} = c_{\Xi''} \quad \text{with} \ \ \Xi'' = \{r'+1, \dots, r'+s''\}.$$

By Theorem 3.1 we conclude that

$$\#H^1(\mathbb{R}, G) = r' + r'' + 1$$

with representatives

$$1 \cup \left\{c'_{s'} \mid 1 \le s' \le r'\right\} \cup \left\{c''_{s''} \mid 1 \le s'' \le r''\right\}.$$

The cases $G = \mathrm{SO}(2r', 2r''+1)$ and $G = \mathrm{SO}(2r'+1, 2r''+1)$ are similar to the case $\mathrm{SO}(2r', 2r'')$; in both cases we have $\#H^1(\mathbb{R}, G) = r' + r'' + 1$.

Alternatively, one can use the fact that $H^1(\mathbb{R}, G)$ for $G = \mathrm{SO}(\mathbb{R}^n, f)$ classifies isomorphism classes of real quadratic forms f' on $\mathbb{R}^n$ with $\det M_{f'} = \det M_f$ (see Serre [18], Section X.2, Proposition 4), and one can classify the isomorphism classes of such f' using Sylvester's law of inertia.

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