Galois cohomology of reductive algebraic groups over the field of real numbers

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Abstract. We describe functorially the first Galois cohomology set $H^1(\mathbb{R}, G)$ of a connected reductive algebraic group $G$ over the field $\mathbb{R}$ of real numbers in terms of a certain action of the Weyl group on the real points of order dividing 2 of the maximal torus containing a maximal compact torus.

This result was announced with a sketch of proof in the author’s 1988 note [3]. Here we give a detailed proof and a few examples.

To the memory of Arkady L’vovich Onishchik

1 Introduction

Let $G$ be a connected reductive algebraic group over the field $\mathbb{R}$ of real numbers. We wish to compute the first Galois cohomology set $H^1(\mathbb{R}, G) = H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), G(\mathbb{C}))$. In terms of Galois cohomology one can state answers to many natural questions; see Serre [19], Section III.1, and Berhuy [1].

The Galois cohomology of classical groups and adjoint groups is well known. The Galois cohomology of compact groups was computed by Borel and Serre [2], Theorem 6.8; see also Serre’s book [19], Section III.4.5. Here we consider the case of a general connected reductive group over $\mathbb{R}$. We describe $H^1(\mathbb{R}, G)$ in terms of a certain action of the Weyl group on the first Galois cohomology of the maximal torus containing a maximal compact torus. Our main result is Theorem 3.1.

Our description of $H^1(\mathbb{R}, G)$ is inspired by Borel and Serre [2]. Our result was announced in [3]; here we give a detailed proof and a few examples.

Since it was announced in [3], our Theorem 3.1 has been used in a few articles, in particular, in [17], [10], and [15]. In [4], Gornitskii, Rosengarten, and the author described, using Theorem 3.1, the Galois cohomology of quasi-connected reductive $\mathbb{R}$-groups (normal subgroups of connected reductive $\mathbb{R}$-groups). Our description in [4] is similar to that of Theorem 3.1. In [5] Evenor and the author used Theorem 3.1 to describe explicitly the
Galois cohomology of simply connected semisimple \( R \)-groups. In [6] and [7], Timashev and the author used Theorem 3.1 to describe explicitly the Galois cohomology of connected reductive \( R \)-groups.

Note that cited articles refer to Theorem 9 of an early preprint version of this note. In this published version, Theorem 9 became Theorem 3.1.

2 Preliminaries

We recall the definition of the first Galois cohomology set \( H^1(R, G) \) of an algebraic group \( G \) defined over \( R \). The set of 1-cocycles is defined by \( Z^1(R, G) = \{ z \in G(\mathbb{C}) \mid z\bar{z} = 1 \} \) where the bar denotes complex conjugation. The group \( G(\mathbb{C}) \) acts on the right on \( Z^1(R, G) \) by

\[
z \cdot x = x^{-1}z\bar{x},
\]

where \( z \in Z^1(R, G) \) and \( x \in G(\mathbb{C}) \). By definition \( H^1(R, G) = Z^1(R, G)/G(\mathbb{C}) \). Let \( G(R)_2 \) denote the subset of elements of \( G(R) \) of order 2 or 1. Then \( G(R)_2 \subset Z^1(R, G) \), and we obtain a canonical map \( G(R)_2 \to H^1(R, G) \).

2.1. Lemma. Let \( S \) be an algebraic \( R \)-torus. Let \( S_0 \) denote the largest compact (that is, anisotropic) \( R \)-subtorus in \( S \), and let \( S_1 \) denote the largest split subtorus in \( S \). Then :

(a) The map \( \lambda: S(R)_2 \to H^1(R, S) \) induces a canonical isomorphism

\[
S(R)_2/S_1(R)_2 \xrightarrow{\sim} H^1(R, S).
\]

(b) The composite map \( \mu: S_0(R)_2 \to H^1(R, S_0) \to H^1(R, S) \) is surjective.

(c) \( (S_0 \cap S_1)(R) = S_0(R)_2 \cap S_1(R)_2 \), and the surjective map \( \mu \) of (b) induces an isomorphism

\[
S_0(R)_2/(S_0 \cap S_1)(R) \xrightarrow{\sim} H^1(R, S).
\]

Proof. Any \( R \)-torus is isomorphic to a direct product of tori of three types, see Casselman [9], Section 2:

(1) \( \mathbb{G}_{m,R} \),

(2) \( R_{C/R} \mathbb{G}_{m,C} \),

(3) \( R^1_{C/R} \mathbb{G}_{m,C} \). Here \( \mathbb{G}_m \) denotes the multiplicative group, \( R_{C/R} \) denotes the Weil restriction of scalars, and

\[
R^1_{C/R} \mathbb{G}_{m,C} = \ker [\text{Nm}_{C/R}: R_{C/R} \mathbb{G}_{m,C} \to \mathbb{G}_{m,R}],
\]

where \( \text{Nm}_{C/R} \) is the norm map.

We prove (a). The composite homomorphism \( S_1(R)_2 \hookrightarrow S(R)_2 \to H^1(R, S) \) factors via \( H^1(R, S_1) = 1 \), and hence it is trivial. We obtain an induced homomorphism

\[
S(R)_2/S_1(R)_2 \to H^1(R, S);
\]
we must prove that it is bijective. It suffices to consider the three cases:

1. $S = \mathbb{G}_{m, \mathbb{R}}$, that is, $S(\mathbb{R}) = \mathbb{R}^\times$. Then $H^1(\mathbb{R}, S) = 1$. We have $S_1 = S$, so $S(\mathbb{R})_2/S_1(\mathbb{R})_2 = 1$. This proves (a) in case (1).

2. $S = R_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{C}}$, that is $S(\mathbb{R}) = \mathbb{C}^\times$. Then $H^1(\mathbb{R}, S) = 1$. We have $S_1 = \mathbb{G}_{m, \mathbb{R}}$, $S_1(\mathbb{R}) = \mathbb{R}^\times$, $S_1(\mathbb{R})_2 = \{1, -1\} = S(\mathbb{R})_2$, so $S(\mathbb{R})_2/S_1(\mathbb{R})_2 = 1$. This proves (a) in case (2).

3. $S = R_{\mathbb{C}/\mathbb{R}}^1 \mathbb{G}_{m, \mathbb{C}}$, that is $S(\mathbb{R}) = \{x \in \mathbb{C}^\times \mid \text{Nm}(x) = 1\}$, where $\text{Nm}(x) = x\bar{x}$. Then by the definition of Galois cohomology $H^1(\mathbb{R}, S) = \mathbb{R}^\times / \text{Nm}(\mathbb{C}^\times) \simeq \{-1, 1\}$. The homomorphism $S(\mathbb{R})_2 = \{-1, 1\} \to H^1(\mathbb{R}, S)$ is an isomorphism. This proves (a) in case (3).

Assertion (b) reduces to the cases (1), (2), (3), where it is obvious (note that only in case (3) we have $H^1(\mathbb{R}, S) \neq 1$).

Concerning (c), we have a commutative diagram

$$
\begin{array}{ccc}
S_0(\mathbb{R})_2 & \xrightarrow{\mu} & S(\mathbb{R})_2 \\
\downarrow & & \downarrow^\lambda \\
H^1(\mathbb{R}, S_0) & \xrightarrow{\sim} & H^1(\mathbb{R}, S).
\end{array}
$$

We see from (a) that $\ker \mu = S_0(\mathbb{R})_2 \cap S_1(\mathbb{R})_2$, and we know from (b) that $\mu$ is surjective. Thus we obtain a canonical isomorphism

$$
S_0(\mathbb{R})_2/(S_0(\mathbb{R})_2 \cap S_1(\mathbb{R})_2) \xrightarrow{\sim} H^1(\mathbb{R}, S).
$$

It remains only to check that $S_0(\mathbb{R})_2 \cap S_1(\mathbb{R})_2 = (S_0 \cap S_1)(\mathbb{R})$. This can be easily checked in each of the cases (1), (2), (3) (note that only in case (2) this group is nontrivial). \qed

2.2. Corollary. Assume that $S$ is an $\mathbb{R}$-torus such that $S = S' \times S''$, where $S'$ is a compact torus and $S'' = R_{\mathbb{C}/\mathbb{R}} T$, where $T$ is a $\mathbb{C}$-torus. Then $H^1(\mathbb{R}, S) = H^1(\mathbb{R}, S') = S'(\mathbb{R})_2$.

Proof. The assertion follows from the proof of Lemma 2.1(a), because $S'$ is a direct product of tori of type (3); hence $H^1(\mathbb{R}, S') = S'(\mathbb{R})_2$, and $S''$ is a direct product of tori of type (2), whence $H^1(\mathbb{R}, S'') = 1$. \qed

We say that a connected real algebraic group $H$ is compact, if the group $H(\mathbb{R})$ is compact, that is, $H$ is reductive and anisotropic. We shall need the following two standard facts.

2.3 Lemma (well-known). Any nontrivial semisimple algebraic group $H$ over $\mathbb{R}$ contains a nontrivial connected compact subgroup.

Proof. This assertion follows from the classification, (see, for instance, Helgason [12], Section X.6.2, Table V). We prove it without using the classification.
Let $\kappa : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ denote the Killing form on $\mathfrak{h}$. Let $\mathfrak{h} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of the real semisimple Lie algebra $\mathfrak{h} = \text{Lie } H$. This means that the linear transformation

$$\theta : \mathfrak{h} \to \mathfrak{h}, \quad k + p \mapsto k - p \quad \text{for } k \in \mathfrak{k}, p \in \mathfrak{p}$$

is an automorphism of $\mathfrak{h}$, and that the bilinear form

$$b_\theta(x, y) = -\kappa(x, \theta(y))$$

is positive definite on $\mathfrak{h}$. Set

$$K = \{ h \in H \mid \text{Ad } h \in O(\mathfrak{h}, b_\theta) \}.$$ 

Then $K$ is a real algebraic subgroup of $H$. We have $\text{Lie } K = \mathfrak{k}$; see Gorbatsevich, Onishchik, and Vinberg [11], Section 4.3.2. Since $H(\mathbb{R})$ has finitely many connected components and the center of $H(\mathbb{R})^0$ is finite, by [11], Corollary 5 of Theorem 4.3.2, the group $K(\mathbb{R})$ is compact.

Since $\mathfrak{p}$ and $\mathfrak{k}$ are the eigenspaces of $\theta$ with eigenvalues $-1$, and $+1$, respectively, we have $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. If $\mathfrak{k} = 0$, then $\mathfrak{h} = \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] = 0$, whence $\mathfrak{h}$ is commutative, which is clearly impossible. Thus $\mathfrak{k} \neq 0$. But $\mathfrak{k}$ is the Lie algebra of the identity component $K^0$ of $K$, which is a connected compact algebraic subgroup of $H$. Thus $H$ contains a nontrivial connected compact algebraic subgroup.

2.4 Lemma (well-known). Any two maximal compact tori in a connected reductive real algebraic group $H$ are conjugate under $H(\mathbb{R})$.

Proof. It suffices to prove that any two maximal compact tori in the derived group $[H, H]$ of $H$ are conjugate. This follows from the following well-known facts from the theory of Lie groups: (1) Any two maximal compact subgroups in a connected semisimple Lie group are conjugate (see, for instance, Gorbatsevich, Onishchik, and Vinberg [11], Section 4.3.4, Theorem 3.5); (2) Any two maximal tori in a connected compact Lie group are conjugate (see, for instance, Onishchik and Vinberg [16], Section 5.2.7, Theorem 15).

3 Main result

Let $G$ be a connected reductive algebraic group over $\mathbb{R}$. Let $T_0$ be a maximal compact torus in $G$. Set $T = \mathcal{Z}(T_0)$, $N_0 = \mathcal{N}(T_0)$, $W_0 = N_0/T$, where $\mathcal{Z}$ and $\mathcal{N}$ denote the centralizer and the normalizer in $G$, respectively.

We prove that $T$ is a torus. By Humphreys [13], Theorem 22.3 and Corollary 26.2.A, the centralizer $T$ of $T_0$ is a connected reductive $\mathbb{R}$-group. The torus $T_0$ is a maximal compact torus in $T$, and it is central in $T$. Since by Lemma 2.4 all the maximal compact tori in $T$ are conjugate under $T(\mathbb{R})$, we see that $T_0$ is the only maximal compact torus in $T$. It follows that the derived group $[T, T]$ of $T$ contains no nontrivial compact tori. By Lemma 2.3 every nontrivial semisimple group over $\mathbb{R}$ has a nontrivial compact connected algebraic subgroup, hence a nontrivial compact torus. We conclude that $[T, T] = 1$, and
hence $T$ is a torus. We see that $T$ is a fundamental torus in $G$, that is, a maximal torus containing a maximal compact torus.

We have a right action of $W_0$ on $T_0$ defined by $(t, w) \mapsto t \cdot w := n^{-1}tn$, where $t \in T_0(\mathbb{C})$, $n \in N_0(\mathbb{C})$, $n$ represents $w \in W_0(\mathbb{C})$. This action is defined over $\mathbb{R}$. We prove that $W_0(\mathbb{C})$ acts on $T_0$ effectively. Indeed, if $w \in W_0(\mathbb{C})$ with representative $n \in N_0(\mathbb{C})$ acts trivially on $T_0$, then $n^{-1}tn = t$ for any $t \in T_0(\mathbb{C})$, and hence $n \in T(\mathbb{C})$ (because the centralizer of $T_0$ is $T$), whence $w = 1$.

We prove that $W_0(\mathbb{C}) = W_0(\mathbb{R})$. We have seen that $W_0(\mathbb{C})$ embeds in $\text{Aut}_C(T_0)$. Since $T_0$ is a compact torus, all the complex automorphisms of $T_0$ are defined over $\mathbb{R}$. We see that the complex conjugation acts trivially on $\text{Aut}_C(T_0)$, and hence on $W_0(\mathbb{C})$. Thus $W_0(\mathbb{R}) = W_0(\mathbb{C})$.

Note that $N_0$ normalizes $T$; hence $W_0$ acts on $T$. We define a right action $*$ of $W_0(\mathbb{R})$ (which is equal to $W_0(\mathbb{C})$) on $H^1(\mathbb{R}, T)$. Let $z \in Z^1(\mathbb{R}, T)$, $n \in N_0(\mathbb{C})$, $z$ represents $\xi \in H^1(\mathbb{R}, T)$, $n$ represents $w \in W_0(\mathbb{R}) = W_0(\mathbb{C})$. We set

$$\xi * w = [n^{-1}zn] = [n^{-1}zn \cdot n^{-1}n],$$

where brackets $[ \ ]$ denote the cohomology class.

We prove that $*$ is a well defined action. First, since $N_0$ normalizes $T$ and $z \in T(\mathbb{C})$, we see that $n^{-1}zn \in T(\mathbb{C})$. Now $w \in W_0(\mathbb{R})$, whence $w^{-1}w = 1$ and $n^{-1}n \in T(\mathbb{C})$. It follows that $n^{-1}zn = n^{-1}zn \cdot n^{-1}n \in T(\mathbb{C})$. We have

$$n^{-1}zn \cdot n^{-1}n = n^{-1}zn \cdot n^{-1}zn^{-1}z = 1$$

because $z \bar{z} = 1$. Thus $n^{-1}zn \in Z^1(\mathbb{R}, T)$. If $z' \in Z^1(\mathbb{R}, T)$ is another representative of $\xi$, then $z' = t^{-1}zt'$ for some $t \in T(\mathbb{C})$, and

$$n^{-1}z'n = n^{-1}t^{-1}zt'n = (n^{-1}tn)^{-1} \cdot n^{-1}zn \cdot n^{-1}tn = (t')^{-1}(n^{-1}zn)\bar{t}$$

where $t' = n^{-1}tn$, $t' \in T(\mathbb{C})$. We see that the cocycle $n^{-1}z'n \in Z^1(\mathbb{R}, T)$ is cohomologous to $n^{-1}zn$. If $n'$ is another representative of $w$ in $N_0(\mathbb{C})$, then $n' = nt$ for some $t \in T(\mathbb{C})$, and $(n')^{-1}z'n' = t^{-1}n^{-1}xnt$. We see that $(n')^{-1}z'n'$ is cohomologous to $n^{-1}zn$. Thus $*$ is indeed a well defined action of the group $W_0(\mathbb{R})$ on the set $H^1(\mathbb{R}, T)$.

Note that in general $[1] * w = [n^{-1}n] \neq [1]$, and therefore, the action $*$ does not respect the group structure in $H^1(\mathbb{R}, T)$.

Let $\xi \in H^1(\mathbb{R}, T)$ and $w \in W_0(\mathbb{R})$. It follows from the definition of the action $*$ that the images of $\xi$ and $\xi * w$ in $H^1(\mathbb{R}, G)$ are equal. We see that the map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$ induces a map $H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \to H^1(\mathbb{R}, G)$.

The following theorem is the main result of this note:

**3.1. Theorem.** Let $G$, $T_0$, $T$, and $W_0$ be as above. The map

$$H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \to H^1(\mathbb{R}, G)$$

induced by the map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$ is a bijection.
Proof. We prove the surjectivity. It suffices to show that the map \( H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G) \) is surjective. This was proved by Kottwitz [14], Lemma 10.2, with a reference to Shelstad [20]. We give a different proof. Let \( \eta \in H^1(\mathbb{R}, G) \), \( \eta = [z] \), \( z \in G(\mathbb{C}) \), \( z \bar{z} = 1 \). Let \( z = su = \bar{s} - 1 \), where \( s \) and \( u \) are the semisimple and the unipotent parts of \( z \), respectively (see Humphreys [13], Theorem 15.3). We have \( us\bar{s} = 1 \), where \( u\bar{s} = \bar{s}u \) (because \( us = su \)). Thus \( us = \bar{u}^{-1} \bar{s}^{-1} \), where \( u \) and \( \bar{u} \) are unipotent, \( s \) and \( \bar{s} \) are semisimple, \( us = su \). From the equality \( u\bar{s} = \bar{s}u \) it follows that \( \bar{u}^{-1} \bar{s}^{-1} = \bar{s}^{-1} \bar{u}^{-1} \). Since the Jordan decomposition in \( G(\mathbb{C}) \) is unique (see Humphreys [13], Theorem 15.3), we conclude that \( s = \bar{s}^{-1} \), \( u = \bar{u}^{-1} \). In other words, \( s\bar{s} = 1 \), \( u\bar{u} = 1 \), that is, \( s \) and \( u \) are cocycles.

Since \( u \) is unipotent, the logarithm \( \log(u) \in \text{Lie } G_{\mathbb{C}} \) is defined. We have:

\[
\log(u) + \bar{\log(u)} = 0.
\]

Set \( y = \frac{1}{2} \log(u) \), then \( y + \bar{y} = 0 \). We have \( -y + \log(u) + \bar{y} = 0 \), where \( -y, \bar{y} \) and \( \log(u) \) pairwise commute. Set \( u' = \exp(y) \), then \( (u')^{-1} u \bar{u} = 1 \). Since \( s \) commutes with \( u \), we have \( \text{Ad}(s)y = y \), and hence \( s \) commutes with \( u' \). We obtain \( (u')^{-1} su \bar{u} = s \), and hence the cocycle \( z = su \) is cohomologous to the cocycle \( s \), where \( s \) is semisimple.

We may and shall therefore assume that \( z \) is semisimple. Set \( C = Z_{G_{\mathbb{C}}}(z) \). Since \( \bar{z} = z^{-1} \), we have \( \bar{C} = C \), and hence the algebraic subgroup \( C \) of \( G_{\mathbb{C}} \) is defined over \( \mathbb{R} \). The semisimple element \( z \) is contained in a maximal torus of \( G_{\mathbb{C}} \) (see Humphreys [13], Theorem 22.2); hence \( z \) is contained in the identity component \( C^0 \) of \( C \). The group \( C^0 \) is reductive, see Steinberg [21], Section 2.7(a). Let \( T' \) be a maximal torus of \( C^0 \) defined over \( \mathbb{R} \), then \( z \in T'((\mathbb{C})) \), because \( z \) is contained in the center of \( C^0 \). By Lemma 2.1(b) the class \( \eta \) of \( z \) comes from the maximal compact subtorus \( T'_0 \) of \( T' \). By Lemma 2.4 any compact torus in \( G \) is conjugate under \( G(\mathbb{R}) \) to a subtorus of \( T_0 \). Thus \( \eta \) comes from \( H^1(\mathbb{R}, T_0) \), hence from \( H^1(\mathbb{R}, T) \). This proves the surjectivity in Theorem 3.1.

We prove the injectivity in Theorem 3.1. Let \( z, z' \in T(\mathbb{C}) \), \( zz' = 1 \), \( z'\bar{z} = 1 \), \( z = x^{-1}z'x \), where \( x \in G(\mathbb{C}) \). We shall prove that \( z = n^{-1}z'\bar{n} \) for some \( n \in N_0(\mathbb{C}) \).

For \( g \in G(\mathbb{C}) \) set \( g^\nu = zg^{-1} \). Then \( \nu \) is an involutive antilinear automorphism of \( G_{\mathbb{C}} \), and in this way we obtain a twisted form \( zG \) of \( G \). Since \( z \in T(\mathbb{C}) \), the embeddings of the tori \( T_{\mathbb{C}} \) and \( T_{0,\mathbb{C}} \) into \( zG_{\mathbb{C}} \) are defined over \( \mathbb{R} \). We denote the corresponding \( \mathbb{R} \)-tori of \( zG \) again by \( T \) and \( T_0 \), respectively. The centralizer of \( T_0 \) in \( zG \) is \( T \). The compact torus \( T_0 \) of \( zG \) is contained in some maximal compact torus \( S \) of \( zG \), and clearly \( S \) is contained in the centralizer \( T \) of \( T_0 \) in \( zG \). Since \( T_0 \) is the largest compact subtorus of \( T \), we conclude that \( S = T_0 \). Thus \( T_0 \) is a maximal compact torus in \( zG \).

Consider the embedding \( i_x \colon t \mapsto x^{-1}tx \colon T_{0,\mathbb{C}} \to zG_{\mathbb{C}} \). We have \( i_x(t)^\nu = z\bar{x}^{-1}\bar{t}\bar{z}z^{-1} \). Since \( z\bar{x}^{-1} = x^{-1}z' \), we obtain

\[
\bar{x}^{-1}z'\bar{z}z^{-1} = x^{-1}z'\bar{t}(z')^{-1}x = x^{-1}t\bar{x}x = i_x(\bar{t}) \cdot
\]

We see that \( i_x(t)^\nu = i_x(\bar{t}) \); hence \( i_x \) is defined over \( \mathbb{R} \). Set \( T'_0 = i_x(T_0) \); it is a compact algebraic torus in \( zG \), and \( \dim T'_0 = \dim T_0 \). Therefore, the torus \( T'_0 \) is conjugate to \( T_0 \) under \( zG(\mathbb{R}) \), say, \( T_{0,\mathbb{C}} = h^{-1}T'_{0,\mathbb{C}}h \), where \( h \in zG(\mathbb{R}) \). Set \( n = xh \). Then

\[
n^{-1}T_{0,\mathbb{C}}n = h^{-1}x^{-1}T_{0,\mathbb{C}}xh = h^{-1}T'_{0,\mathbb{C}}h = T_{0,\mathbb{C}}.
\]
whence \( n \in N_0(\mathbb{C}) \). The condition \( h \in \mathcal{Z}(\mathbb{R}) \) means that \( z\bar{h}z^{-1} = h \), or \( h^{-1}z \bar{h} = z \). It follows that

\[
n^{-1}z' \bar{n} = h^{-1}x^{-1}z' \bar{x}h = h^{-1}z \bar{h} = z.
\]

We have proved that there exists \( n \in N_0(\mathbb{C}) \) such that \( z = n^{-1}z' \bar{n} \), and hence the cohomology classes \([z], [z'] \in H^1(\mathbb{R}, T)\) lie in the same orbit of \( W_0(\mathbb{R}) \) in \( H^1(\mathbb{R}, T) \). This proves the injectivity in Theorem 3.1.

3.2. Remark. If \( G \) is a compact group, then Theorem 3.1 asserts that

\[
H^1(\mathbb{R}, G) = T(\mathbb{R})_2/W,
\]

where \( T \) is a maximal torus in \( G \), and \( W \) is the Weyl group with the usual action. This was earlier proved by Borel and Serre [2].

3.3. Remark. The real form \( G_\mathbb{C} \) of \( G_\mathbb{C} \) defines an involutive automorphism \( \tau \) of the based root datum of \( G_\mathbb{C} \), see [6], Proposition 3.7, and hence an involutive automorphism \( \tau_D \) of the Dynkin diagram \( D = D(G_\mathbb{C}) \). This automorphism \( \tau_D \) is trivial if and only if the derived group \( [G, G] \) of \( G \) is an inner form of a compact group, that is, has a compact maximal torus. Let \( \mathcal{D} \) denote the twisted Dynkin diagram corresponding to \( D \) and \( \tau_D \). Then \( W_0 \) is isomorphic to the Weyl group of \( \mathcal{D} \); see [6], Proposition 7.11(iii). This Coxeter group is described in the book of Carter [8], Chapter 13.

4 Examples

In this section, written following a suggestion of the referee, we compute, using Theorem 3.1, the sets \( H^1(\mathbb{R}, G) \) when \( G = \text{GL}_{n,\mathbb{R}}, \text{Sp}_{2m,\mathbb{R}}, \text{SO}_{p,q} \).

4.1. Example. Let \( G = \text{GL}_{2m,\mathbb{R}}, m \in \mathbb{Z}_{>0} \). For \( z = a + bi \in \mathbb{C} \), we write

\[
\mathcal{M}(z) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.
\]

Consider the tori \( T \) and \( T_0 \) in \( G \) such that

\[
T(\mathbb{R}) = \left\{ \text{diag}(\mathcal{M}(z_1), \ldots, \mathcal{M}(z_m)) \mid z_k = a_k + b_k i \in \mathbb{C}^\times, k = 1, \ldots, m \right\},
\]

\[
T_0(\mathbb{R}) = \left\{ \text{diag}(\mathcal{M}(z_1), \ldots, \mathcal{M}(z_m)) \in T(\mathbb{R}) \mid z_k = a_k + b_k i, a_k^2 + b_k^2 = 1 \right\}.
\]

Then \( T_0 \) is a maximal compact torus in \( G \), and \( T \) is a fundamental torus containing \( T_0 \). Since \( T \simeq (\mathbb{R}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}})^m \), by the proof of Lemma 2.1, case (2), we have \( H^1(\mathbb{R}, T) = \{1\} \), and by Theorem 3.1 we conclude that \( H^1(\mathbb{R}, G) = \{1\} \).

Similarly, if \( G = \text{GL}_{2m+1,\mathbb{R}} \), then \( G \) has a fundamental torus

\[
T \simeq \mathbb{G}_{m,\mathbb{R}} \times (\mathbb{R}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}})^m.
\]

Again we have \( H^1(\mathbb{R}, T) = \{1\} \) and \( H^1(\mathbb{R}, G) = \{1\} \). Note that it is well known that \( H^1(K, \text{GL}_n) = \{1\} \) for any \( n \) and any field \( K \); see Serre [18], Section X.1, Proposition 3.
4.2. Example. Let \( G = \text{SL}_2, \mathbb{R} \). It has a maximal torus \( T = T_0 \) with group of \( \mathbb{R} \)-points

\[
T(\mathbb{R}) = \{ \mathcal{M}(z) \mid z = a + bi, \ a^2 + b^2 = 1 \}.
\]

Set \( n = \text{diag}(i, -i) \in G(\mathbb{C}) \); then \( n \in N(\mathbb{C}) \), where \( N = N_0 = N_G(T) \). We have \( \#H^1(\mathbb{R}, T) = 2 \) with representatives \( \mathcal{M}(1), \mathcal{M}(-1) \). An easy calculation shows that

\[
n^{-1}\mathcal{M}(-1)\bar{n} = n^{-1}\mathcal{M}(-1)n \cdot n^{-1}\bar{n} = \mathcal{M}(1) = 1.
\]

Thus \( H^1(\mathbb{R}, T)/W_0 = \{1\} \), and by Theorem 3.1 we have \( H^1(\mathbb{R}, G) = 1 \).

Note that in this case the group \( W = W_0 = N_0/T \) has two elements, and that the element \( w := [n^{-1}] \in W(\mathbb{R}) \) has no representative in \( N_0(\mathbb{R}) \), because otherwise we would have \([1] * w = [1]\). Thus in this case \( W_0(\mathbb{R}) \neq N_0(\mathbb{R})/T(\mathbb{R}) \).

4.3. Example. Let \( G = \text{Sp}_{2m} = \text{Sp}(\mathbb{R}^{2m}, \psi) \), where \( \psi \) is the skew-symmetric bilinear form with matrix

\[
M_\psi = \text{diag}(J, \ldots, J) \quad \text{where} \quad J = \mathcal{M}(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The group \( G \) has a compact maximal torus \( T = T_0 \) with

\[
T(\mathbb{R}) = \{ \text{diag}(\mathcal{M}(z_1), \ldots, \mathcal{M}(z_m)) \mid z_k = a_k + b_ki, \ a_k^2 + b_k^2 = 1 \}.
\]

We have

\[
T(\mathbb{R})_2 = \{ \text{diag}(\mathcal{M}(z_1), \ldots, \mathcal{M}(z_m)) \in T(\mathbb{R}) \mid z_k = \pm 1 \}.
\]

Let \( t = \text{diag}(\mathcal{M}(z_1), \ldots, \mathcal{M}(z_m)) \in T(\mathbb{R})_2 \). Write

\[
n = \text{diag}(n_1, \ldots, n_m), \quad \text{where} \quad n_k = \begin{cases} \text{diag}(i, -i) & \text{if} \ z_k = -1, \\ \text{diag}(1, 1) & \text{if} \ z_k = 1. \end{cases}
\]

Then \( n \in N_0(\mathbb{C}) \) and

\[
n^{-1} \cdot t \cdot \bar{n} = 1.
\]

We see that for any \([t] \in H^1(\mathbb{R}, T)\) there exists \( w = [n] \in W_0(\mathbb{R}) \) with

\[
[t] * w = [1].
\]

Thus \( H^1(\mathbb{R}, T)/W_0 = \{1\} \), and by Theorem 3.1 we have \( H^1(\mathbb{R}, G) = \{1\} \). Note that it is well known that \( H^1(K, \text{Sp}_{2m}) = \{1\} \) for any \( m \) and any field \( K \); see Serre [19, Section III.1.2, Proposition 3].

4.4. Example. Let \( G = \text{SO}(p', p'') = \text{SO}(\mathbb{R}^{p'+p''}, f) \), where \( f \) is the diagonal quadratic form with matrix

\[
M_f = \text{diag}(+1, \ldots, +1, -1, \ldots, -1).
\]
We consider the case when both \( p' \) and \( p'' \) are even: \( p' = 2r' \), \( p'' = 2r'' \). Our group \( G \) has a compact maximal torus \( T = T_0 \) with group of \( \mathbb{R} \)-points

\[
T(\mathbb{R}) = \{ \text{diag } (\mathcal{M}(z_1), \ldots, \mathcal{M}(z_{r'+r''})) \mid z_k = a_k + ib_k, \; a_k^2 + b_k^2 = 1 \}.
\]

We have

\[
T(\mathbb{R})_2 = \{ \text{diag } (\mathcal{M}(z_1), \ldots, \mathcal{M}(z_{r'+r''})) \mid z_k = \pm 1 \}.
\]

The Weyl group \( W = W(G_{\mathbb{C}}, T_{\mathbb{C}}) \) is isomorphic to \((\pm 1)^{r'+r''-1} \times S_{r'+r''}, \) where \( S_{r'+r''} \) is the symmetric group on the \( r' + r'' \) symbols \( 1, \ldots, r' + r'' \). The subgroup \((\pm 1)^{r'+r''-1} \) acts on \( T(\mathbb{R})_2 \) trivially. We compute \( T(\mathbb{R})_2/S_{r'+r''} \).

For a subset \( \Xi \subseteq \{ 1, \ldots, r' + r'' \} \), we set

\[
c_\Xi = \text{diag } (\mathcal{M}(z_1), \ldots, \mathcal{M}(z_{r'+r''})) \text{ with } z_k = \begin{cases} -1 & \text{if } k \in \Xi, \\ +1 & \text{otherwise}. \end{cases}
\]

Then

\[
T(\mathbb{R})_2 = \{ c_\Xi \mid \Xi \subseteq \{ 1, \ldots, r' + r'' \} \}.
\]

Consider the subgroup \( S_{r'} \times S_{r''} \subseteq S_{r'+r''} \), where \( S_{r''} \) is the symmetric group on the \( r'' \) symbols \( r' + 1, \ldots, r' + r'' \). Its elements are represented by elements of \( N(\mathbb{R}) \), and hence they act on \( T(\mathbb{R})_2 \) by the usual conjugation:

\[
c_\Xi \ast \sigma = c_{\sigma^{-1}\Xi} \text{ for } \sigma \in S_{r'} \times S_{r''}.
\]

Write \( \Xi = \Xi' \cup \Xi'' \) where

\[
\Xi' = \Xi \cap \{ 1, \ldots, r' \}, \quad \Xi'' = \Xi \cap \{ r' + 1, \ldots, r' + r'' \}.
\]

We see that the \( W \)-orbit of \( c_\Xi \) depends only on the cardinalities of \( \Xi' \) and \( \Xi'' \).

The group \( S_{r'+r''} \) is generated by its subgroup \( S_{r'} \times S_{r''} \) and \( \sigma_{1,r'+1} = (1, r' + 1) \). In order to compute the action of \( \sigma_{1,r'+1} \) on \( T(\mathbb{R})_2 \), we consider the case \( r' = 1, \; r'' = 1, \; G = \text{SO}(2,2) \). Consider the block matrix

\[
n = i \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \text{ where } I_2 = \text{diag}(1,1).
\]

One can check that \( n \in N(\mathbb{C}) \subset G(\mathbb{C}) \) and \( n \) represents \( \sigma_{1,2} \). Let

\[
c = c_{(1,1)} = \text{diag}(-1, -1, -1, -1).
\]

We have

\[
n^{-1}cn = cn^{-1}n = 1.
\]

Returning to the case of arbitrary \( r' \) and \( r'' \), we see that when both \( \Xi' \) and \( \Xi'' \) are non-empty, an element of \( \Xi' \) can be cancelled with an element of \( \Xi'' \). Thus the \( W \)-orbit of
a cocycle $c_{\Xi}$ depends only on the difference $\#\Xi' - \#\Xi''$. For $s'$ and $s''$ such that $1 \leq s' \leq r'$, $1 \leq s'' \leq r''$, we write

$$c_{s'} = c_{\Xi'} \quad \text{with} \quad \Xi' = \{1, \ldots, s'\},$$
$$c_{s''} = c_{\Xi''} \quad \text{with} \quad \Xi'' = \{r' + 1, \ldots, r' + s''\}.$$ 

By Theorem 3.1 we conclude that

$$\#H^1(\mathbb{R}, G) = r' + r'' + 1$$

with representatives

$$1 \cup \{c_{s'} \mid 1 \leq s' \leq r'\} \cup \{c_{s''} \mid 1 \leq s'' \leq r''\}.$$ 

The cases $G = \text{SO}(2r', 2r'' + 1)$ and $G = \text{SO}(2r' + 1, 2r'' + 1)$ are similar to the case $\text{SO}(2r', 2r'')$; in both cases we have $\#H^1(\mathbb{R}, G) = r' + r'' + 1$.

Alternatively, one can use the fact that $H^1(\mathbb{R}, G)$ for $G = \text{SO}(\mathbb{R}^n, f)$ classifies isomorphism classes of real quadratic forms $f'$ on $\mathbb{R}^n$ with $\det M_{f'} = \det M_f$ (see Serre [18], Section X.2, Proposition 4), and one can classify the isomorphism classes of such $f'$ using Sylvester’s law of inertia.

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**References**


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