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A generalization of certain associated Bessel functions in connection with a group of shifts

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Abstract. Considering the kernel of an integral operator intertwining two realizations of the group of motions of the pseudo-Euclidean space, we derive two formulas for series containing Whittaker's functions or Weber's parabolic cylinder functions. We can consider this kernel as a special function. Some particular values of parameters involved in this special function are found to coincide with certain variants of Bessel functions. Using these connections, we also establish some analogues of orthogonality relations for Macdonald and Hankel functions.

1 Introduction and preliminaries

It is well known that any Lie group depends on a finite set of continuously changing parameters. The cardinality of this set is small for a group of a low dimension. In this case, the matrix elements of representation operators, the matrix elements of bases transformations, kernels of the corresponding integral operators, intertwined different realizations of representations, can be expressed in terms of classical special functions. For more complicated groups, the above matrix elements and kernels of subrepresentations to some subgroups and kernels of intertwining operators are found to yield new special functions, which can be considered either generalizations or analogues of known (classical) special functions. For instance, in [13], the matrix elements of the restriction of the representation of Lorentz group on some diagonal matrices were shown to be able to be expressed in terms of modified hyper Bessel functions of the first kind. The connection between these matrix elements written in two different bases of a representation space leads to new formulas for series containing above-mentioned hyper functions and converging to (ordinary) modified Bessel functions.

Vilenkin [15] showed that many known and new properties of variants of Bessel functions are related to representations of the group $ISO(1,1)$ (denoted by $MH(2)$ there) of motions of the pseudo-Euclidean plane and group $M(n)$ of motions of Euclidean n-dimensional space. In this paper, we consider an one-parameter subgroup in more complicated group $ISO(n,1)$ and show that the kernel of an intertwined integral operator can be considered as a generalization of Macdonald and Hankel functions, in a sense that some simple cases of the kernel coincide with those variants of Bessel functions.

Recall that the pseudo-Euclidean $(n+1)$-dimensional space of signature

$$\{-,\ldots,-,+\}$$

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is the linear space in $\mathbb{R}^{n+1}$ endowed with the bilinear form

$$\mu(x,y) = x_{n+1}y_{n+1} - \sum_{i=1}^{n} x_i y_i \,.$$

Here and throughout, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, positive real numbers, and complex numbers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. Let $\tilde{\mu}(x) = \mu(x,x)$ be the corresponding quadratic form. A motion of the pseudo-Euclidean space is an isometry with respect to the distance $\sqrt{\tilde{\mu}(x)}$ that preserves orientation. These motions form a group denoted by $ISO(n,1)$. For any $g \in ISO(n,1)$ and any $x \in \mathbb{R}^{n+1}$, we have

$$g(x) = g_0 x + g_1 \,,$$

where $g_0 \in SO(n,1)$ and g_1 is a shift vector. Therefore, $g = g(g_0, g_1)$. Let $\mathfrak{D}$ be the linear space consisting of infinitely differentiable functions defined on the upper cone $\tilde{\mu}(x) = 1$, where $x = (x_1, \ldots, x_{n+1})$ with each $x_j \in \mathbb{R}^+$.

For any $\tau \in \mathbb{C}$, we consider a map T_τ defined by

$$T_\tau \colon ISO(n,1) \to GL(\mathfrak{D}), \quad f \mapsto \exp\big(-\tau\mu(g_1,x)\big)\, f(g_0^{-1}x)$$

where $GL(\mathfrak{D})$ is the multiplicative group of linear operators of $\mathfrak{D}$ whose rank is nonzero. Since for any $g, \tilde{g} \in ISO(n,1)$ and any $x \in \mathbb{R}^{n+1}$ we have

$$\tilde{g}g(x) = \tilde{g}(g_0 x + g_1) = \tilde{g}_0 g_0 x + \tilde{g}_0 g_1 + \tilde{g}_1 \,.$$

So we obtain $\tilde{g}g = \hat{g}(\hat{g}_0, \hat{g}_1)$, where $\hat{g}_0 = \tilde{g}_0 g_0$ and $\hat{g}_1 = \tilde{g}_0 g_1 + \tilde{g}_1$. Then

$$\begin{aligned}[T_\tau(\tilde{g})T_\tau(g)](f(x)] &= [T_\tau(\tilde{g})]\big((\exp\big(-\tau\mu(g_1,x)\big)\, f(g_0^{-1}x)\big) \\ &= \exp\big(-\tau, \mu(\hat{g}_1, x)\big)\, f(\hat{g}_0^{-1}x)\,.\end{aligned}$$

We thus find that T_τ is a homomorphism.

In order to simplify the representation of the group $SH(n+1)$ of hyperbolic rotations in $\mathbb{R}^{n+1}$, Vilenkin [15, Chapter 10] employed the so-called horosphere method. Indeed, Vilenkin used the Gelfand-Graev integral transformation, which maps $f \in \mathfrak{D}$ into the space L of $\hat{\sigma}$-homogeneous functions defined on the intersection of the cone $\tilde{\mu}(x) = 0$ and the plane $x_{n+1} = 1$. Let $K = (k_0, \ldots, k_{n-2}) \in \mathbb{Z}^{n-1}$, $k_0 \geqslant k_1 \ldots \geqslant k_{n-3} \geqslant |k_{n-2}|$, C_k^r be the Gegenbauer polynomials, and

$$\begin{aligned}(A_K)^2 &= 2^{n-3}\pi^{-\frac{1}{2}}\Gamma^{-1}\left(\frac{n}{2}\right) \\ &\times \prod_{i=0}^{n-3} \frac{2^{2k_{i+1}-i}(k_i - k_{i+1})!(2k_i + n - 2 - i)\Gamma^2\left(\frac{n-i}{2} + k_{i+1} - 1\right)}{\Gamma(k_i + k_{i+1} + n - i - 2)}\,.\end{aligned}$$

It was shown that for any $f \in \mathfrak{D}$, the 'coordinates' a_K of the image of f (arising after action of Gelfand-Graev transform) with respect to the basis

$$\Xi_K(x) = A_K \prod_{i=0}^{n-3} C_{ki-k_{i+1}+1}^{\frac{n-i-2}{2}}(\cos\varphi_{n-i-1}) \sin^{k_i+1}\varphi_{n-i-1} \exp(\pm\mathbf{i}k_{n-2}\varphi_1)$$

(where A_k is the normalizing factor) can be expressed as the integral transform (see [15, Entry 10.5.4])

$$a_K = \mathsf{I}[f](\sigma) = \int\limits_{\tilde{\mu}(x)=1} \ker(x, K, \sigma) f(x)\, \mathrm{d}x\,.$$

Here the kernel $\ker(x, K, \sigma)$ coincides with the matrix elements of the $SH(n+1)$-representation in the space L situating in the 'zero' column which is exactly described by formula [15, Entry 10.4.9]. The corresponding formulas for the inverse transform are given by (see [15, Entries 10.5.5 and 10.5.6])

$$f(x) = \frac{(-1)^{\frac{n-\eta}{2}}\,\mathbf{i}}{2^n \pi^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)} \sum_K \int\limits_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} (\sigma)_{n-1}\,[\cot(\pi\sigma)]^{1-\eta}\, a_K(\sigma)\, \ker(x, K, 1-n-\sigma)\, \mathrm{d}\sigma\,. \tag{1}$$

Here and elsewhere, $\mathbf{i} = \sqrt{-1}$ and η is the remainder of n divided by 2, that is, either $\eta = 0$ or $\eta = 1$. We also recall some functions and notations, which are used in the following sections. The generalized hypergeometric series ${}_pF_q$ (p, $q \in \mathbb{N}_0$) is defined by (see [12, p. 73]):

$$\begin{aligned} {}_pF_q\begin{bmatrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{bmatrix} z &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)\,, \end{aligned}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol which is defined (for $\lambda, \nu \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

it being understood conventionally that $(0)_0 := 1$. The Gegenbauer function $C_\nu^\lambda(z)$ is defined by (see, e.g., [11, p. 791])

$$C_\nu^\lambda(z) = \frac{\Gamma(2\lambda+\nu)}{\Gamma(2\lambda)\,\Gamma(\nu+1)}\,{}_2F_1\Big(-\nu, 2\lambda+\nu; \lambda+\frac{1}{2}; \frac{1-z}{2}\Big)\,,$$

the cases $\nu = n \in \mathbb{N}_0$ of which are the Gegenbauer polynomials. The associated Legendre function of the first kind $P_\nu^\mu(z)$ is defined by (see, e.g., [11, p. 795])

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\tfrac{z+1}{z-1}\right)^{\frac{\mu}{2}} {}_2F_1\Big(-\nu,\, \nu+1; 1-\mu; \frac{1-z}{2}\Big) \tag{2}$$

$$(|\arg(z \pm 1)| < \pi; \quad \mu \in \mathbb{C} \setminus \mathbb{N})\,.$$

The confluent hypergeometric function Ψ is given by (see, e.g., [8, Entry 13.6.21])

$$\Psi(a, b; z) = z^{-a}\,{}_2F_0\Big(a, a-b+1; -; -\frac{1}{z}\Big)\,.$$

The Whittaker function of the second kind $W_{\kappa,\mu}(z)$ is defined by (see, for instance, [4, p. 264, Eq. (5)] and [8, p. 334, Entry 13.14.3])

$$W_{\kappa,\mu}(z) = e^{-\frac{z}{2}}\, z^\kappa\, {}_2F_0\Big(\frac{1}{2}-\kappa+\mu, \frac{1}{2}-\kappa-\mu; -; -\frac{1}{z}\Big) \tag{3}$$

$$\Big(|\arg(z)| < \pi, |z| > 0; \quad \frac{1}{2}-\kappa+\mu, \frac{1}{2}-\kappa-\mu \in \mathbb{C} \setminus \mathbb{Z}_0^-\Big)\,.$$

Recall the following relation (see, e.g., [8, Entry 13.14.3])

$$W_{\kappa,\mu}(z) = z^{\mu+\frac{1}{2}}\, e^{-\frac{z}{2}}\, \Psi\Big(\frac{1}{2}-\kappa+\mu,\, 2\mu+1; z\Big)\,. \tag{4}$$

The parabolic cylinder function $D_\nu(z)$ is given by (see, for instance, [2, p. 674] and [11, p. 792])

$$D_\nu(z) = 2^{\frac{\nu}{2}} e^{-\frac{z^2}{4}}\, \Psi\Big(-\frac{\nu}{2}, \frac{1}{2}; \frac{z^2}{4}\Big)\,.$$

The modified Bessel function of the 1st kind $I_\nu(z)$ is given by (see, e.g., [11, p. 794])

$$I_\nu(z) = \frac{1}{\Gamma(\nu+1)} \left(\tfrac{z}{2}\right)^\nu {}_0F_1\Big(-; \nu+1; \frac{z^2}{4}\Big) = e^{-\frac{\nu\pi\mathbf{i}}{2}}\, J_\nu(e^{\frac{\pi\mathbf{i}}{2}} z)\,,$$

where $J_\nu(z)$ is the Bessel function of the 1st kind given by (see, e.g., [11, p. 794])

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)} \left(\tfrac{z}{2}\right)^\nu {}_0F_1\Big(-; \nu+1; -\frac{z^2}{4}\Big).$$

The MacDonald function (modified Bessel function of the 3rd kind) $K_\nu(z)$ is given by (see, e.g., [11, p. 794])

$$K_\nu(z) = \frac{\pi[I_{-\nu}(z) - I_\nu(z)]}{2\,\sin(\nu\pi)}, \quad (v \in \mathbb{C} \setminus \mathbb{Z}), \tag{5}$$

whose integral representation of pure imaginary index $\nu = \mathbf{i}t$ and the argument $z = x \in \mathbb{R}$ was used as a definition in [3, p. 873, Eq. (16)]:

$$K_{\mathbf{i}t}(x) = \int_0^\infty e^{-x\cosh u} \cos(tu)\,\mathrm{d}u.$$

The Hankel functions of the first and second kind (the Bessel functions of the third kind) are given, respectively, by (see, e.g., [2, p. 675])

$$H_\nu^{(1)}(z) = J_\nu(z) + \mathbf{i}\,Y_\nu(z) \quad \text{and} \quad H_\nu^{(2)}(z) = J_\nu(z) - \mathbf{i}\,Y_\nu(z),$$

where $Y_\nu(z)$ is the Bessel function of the second kind (the Neumann function) (see, e.g., [2, p. 677])

$$Y_\nu(z) = \frac{\cos(\nu\pi)\,J_\nu(z) - J_{-\nu}(z)}{\cos(\nu\pi)}, \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}).$$

The Kronecker symbol $\delta_{m,n}$ is defined by $\delta_{m,n} = 1$ when $m = n$ and $\delta_{m,n} = 0$ when $m \neq n$.

2 Kernels of integral operators of the subrepresentation to the subgroup of shifts along the axis Ox_{n+1}

Consider the subgroup

$$H = \{h(\lambda) := g(\mathrm{diag}(1,\ldots,1),(0,\ldots,0,\lambda)) \mid \lambda \in \mathbb{R}\}$$

in $M(n,1)$. In this section and elsewhere, we deal with the integral operator B which acts in the 'space of functions $a_K(\sigma)$' and corresponds with the representation operator $T_\tau\big(h(\lambda)\big)$, where $h(\lambda) \in H$.

Lemma 2.1. *For any $h(\lambda) \in H$, the integral operator B can be written as a Barnes integral*

$$\mathsf{B}[a_K(\sigma)] = \int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} \widetilde{\ker}(\tau,\lambda;n,K,\sigma,\hat\sigma) a_{\hat K}(\hat\sigma)\,\mathrm{d}\hat\sigma,$$

where the kernel admits the integral representation

$$\widetilde{\ker}(\tau,\lambda;n,K,\sigma,\hat\sigma) = \big[(-1)^{\frac{n}{2}}\cot(\pi\hat\sigma)\delta_{\eta,0} + (-1)^{\frac{n+1}{2}}\delta_{\eta,1}\big]2^{-1}\mathbf{i}(-\sigma)_{k_0} \\ \times (\hat\sigma)_{n-1}(\hat\sigma+n-1)_{k_0} \int_1^\infty P^{1-k_0-\frac{n}{2}}_{\frac{n}{2}+\sigma-1}(t)\,P^{1-k_0-\frac{n}{2}}_{-\frac{n}{2}-\hat\sigma}(t)\,e^{-\tau\lambda t}\,\mathrm{d}t. \tag{6}$$

Proof. The inverse transform formulas (1) (which will be applied in this proof) coincide for even and odd cases up to factor $\pm\cot(\pi\sigma)$. Therefore, it is sufficient to prove the result for an arbitrary odd n. Introduce the following parametrization on $\tilde\mu(x) = 1$, where $x = (x_1,\ldots,x_{n+1})$ with $x_{n+1} \in \mathbb{R}^+$ (see [15, Entry 10.1.1]):

$$x_i = \sum_{j=1}^n \delta_{i,j}\sinh\theta_n \prod_{s=i}^{n-1}\sin\theta_s \sum_{t=2}^n \delta_{t,i}\cos\theta_{i-1} + \delta_{i,n+1}\cosh\theta_n,$$
$$\theta_1 \in [-\pi,\pi], \theta_2,\ldots,\theta_{n-1} \in [0,\pi], \theta_n \in \mathbb{R}_0^+.$$

Given $x = (x_1, \dots, x_{n+1})$ with $x_{n+1} \in \mathbb{R}^+$, the corresponding $SH(n+1)$-invariant measure on $\tilde{\mu}(x) = 1$ is given by (see [15, Entry 10.1.6])

$$\mathrm{d}x = \sinh^{n-1}\theta_n \prod_{i=1}^{n-1} \sin^{i-1}\theta_i \, \mathrm{d}\theta_i \, .$$

Then we find from [15, Entry 10.4.9] that

$$\begin{aligned}
&\ker(x, K, \sigma) \\
&= (-1)^{k_0} 2^{\frac{n}{2}-1} \Gamma(\sigma+1)\Gamma^{-1}(\sigma - k_0 + 1) \sinh^{1-\frac{n}{2}}\theta_n \exp(\pm \mathbf{i} k_{n-2}\theta_1) \\
&\quad \times \Big[\Gamma\Big(\frac{n}{2}\Big) \prod_{i=0}^{n-3} \frac{2^{2k_{i+1}+n-i-4}(n+2k_i-i-2)(k_i-k_{i+1})!\,\Gamma\big(\frac{n-i}{2}+k_{i+1}-1\big)}{\pi^{\frac{n}{2}-1}\Gamma(n+k_i+k_{i+1}-i-2)} \Big]^{\frac{1}{2}} \\
&\quad \times P^{1+k_0-\frac{n}{2}}_{\frac{n}{2}+\sigma-1}(\cosh\theta_n) \prod_{i=0}^{n-3} C^{\frac{n-i}{2}+k_{i+1}-1}_{k_i-k_{i+1}}(\cos\theta_{n-i-1}) \sin^{k_{i+1}}\theta_{n-i-1}
\end{aligned}$$

and $[T_\tau(h)][f(x)] = \exp(-\tau\lambda\cosh\theta_n) f(x)$. Therefore we obtain

$$\begin{aligned}
\mathsf{B}[a_k(\sigma)] = \frac{(-1)^{\frac{n}{2}}\, \mathbf{i} 2^{1-n}}{\pi^{\frac{n}{2}}\Gamma\big(\frac{n}{2}\big)} \sum_{\hat{K}} \int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} (\hat{\sigma})_{n-1}\, a_{\hat{K}}(\hat{\sigma})\, \mathrm{d}\hat{\sigma} \\
\times \ker(x, K, \sigma) \ker(x, \hat{K}, 1-n-\hat{\sigma}) \int_{-\pi}^{\pi} \exp\big[\pm \mathbf{i}(k_{n-2}-\hat{k}_{n-2})\theta_1\big]\, \mathrm{d}\theta_1 \\
\times \prod_{i=0}^{n-3} \int_0^{\pi} C^{\frac{n-i}{2}+k_{i+1}-1}_{k_i-k_{i+1}}(\cos\theta_{n-i-1}) C^{\frac{n-i}{2}+\hat{k}_{i+1}-1}_{\hat{k}_i-\hat{k}_{i+1}}(\cos\theta_{n-i-1}) \\
\times \sin^{k_{i+1}+\hat{k}_{i+1}+n-i-2}\theta_{n-i-1}\, \mathrm{d}\theta_{n-i-1} \\
\times \int_0^{+\infty} P^{1-k_0-\frac{n}{2}}_{\frac{n}{2}+\sigma-1}(\cosh\theta_n) P^{1-\hat{k}_0-\frac{n}{2}}_{-\frac{n}{2}-\hat{\sigma}-1}(\cosh\theta_n)\, \sinh\theta_n\, \mathrm{d}\theta_n.
\end{aligned}$$

Obviously $\mathsf{B}[a_k(\sigma)] = 0$ for $k_{n-2} \neq \hat{k}_{n-2}$. Otherwise, in view of orthogonality property for Gegenbauer polynomials (see, e.g., [1, p. 198])

$$\int_{-1}^{1} (1-x^2)^{\varrho-\frac{1}{2}}\, C^{\varrho}_k(x)\, C^{\varrho}_m(x)\, \mathrm{d}x = \frac{2^{1-2\lambda}\pi\Gamma(k+2\varrho)\delta_{k,m}}{k!(k+\varrho)\Gamma^2(\varrho)} \, ,$$

we see that the integral

$$\int_0^{\pi} C^{\frac{1}{2}+k_{n-2}}_{k_{n-3}-k_{n-2}}(\cos\theta_2) C^{\frac{1}{2}+\hat{k}_{n-2}}_{\hat{k}_{n-3}-\hat{k}_{n-2}}(\cos\theta_2) \sin^{k_{n-2}+\hat{k}_{n-2}+1}\theta_2\, \mathrm{d}\theta_2 = 0$$

for $k_{n-3} \neq \hat{k}_{n-3}$. Since, for $K = \hat{K}$

$$\int_{-\pi}^{\pi} \exp\big[\pm \mathbf{i}(k_{n-2}-\hat{k}_{n-2})\theta_1\big]\, \mathrm{d}\theta_1 = 2\pi$$

and

$$\prod_{i=0}^{n-3}\int_0^{\pi} C_{k_i-k_{i+1}}^{\frac{n-i}{2}+k_{i+1}-1}(\cos\theta_{n-i-1})C_{\hat{k}_i-\hat{k}_{i+1}}^{\frac{n-i}{2}+\hat{k}_{i+1}-1}(\cos\theta_{n-i-1}) \times \sin^{k_{i+1}+\hat{k}_{i+1}+n-i-2}\theta_{n-i-1}\,\mathrm{d}\theta_{n-i-1} = \pi^{n-2}\Big[\Gamma\Big(\frac{n}{2}\Big)(A_K)^2\Big]^{-1},$$

we have

$$\begin{aligned}\mathsf{B}[a_k(\sigma)] &= (-1)^{\frac{n+1}{2}}\,2\mathbf{i}\Gamma(\sigma+1)\Gamma^{-1}(1+\sigma-k_0)\\ &\times\int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty}(\hat\sigma)_{n-1}(2-n-\hat\sigma)^{k_0}a_K(\hat\sigma)\,\mathrm{d}\hat\sigma\\ &\times\int_1^{+\infty}P_{\frac{n}{2}+\sigma-1}^{1-k_0-\frac{n}{2}}(t)P_{-\frac{n}{2}-\hat\sigma}^{1-k_0-\frac{n}{2}}(t)\exp(-\tau\lambda t)\,\mathrm{d}t\,.\end{aligned}$$

This completes the proof. ■

We find from Lemma 2.1 that the kernel $\widetilde{\ker}(\tau,\lambda;n,K,\sigma,\hat\sigma)$ does not depend on $k_1,\ldots,k_{n-2}$, in fact,

$$\widetilde{\ker}(\tau,\lambda;n,K,\sigma,\hat\sigma)\equiv\widetilde{\ker}(\tau,\lambda;n,k_0,\sigma,\hat\sigma)\,.$$

Lemma 2.2. *The kernel $\widetilde{\ker}(\tau,\lambda;n,k_0,\sigma,\hat\sigma)$ admits the following series representation:*

- *The case $\eta=0$*

$$\begin{aligned}&\widetilde{\ker}(\tau,\lambda;n,k_0,\sigma,\hat\sigma) \qquad (7)\\ &=(-1)^{\frac{n}{2}}(2\tau\lambda)^{-\frac{n+\sigma+\hat\sigma}{2}}\mathbf{i}\cot(\pi\hat\sigma)\\ &\times(-\sigma)_{k_0}(\hat\sigma+n-1)_{k_0}(\hat\sigma)_{n-1}\left[\Gamma\left(\frac{n}{2}+k_0\right)\right]^{-1}\\ &\times\sum_{i=0}^{\infty}\frac{\left(1-\sigma-\frac{n}{2}\right)_i(k_0-\sigma)_i}{i!}\,{}_4F_3\left[\begin{matrix}1-\frac{n}{2}-k_0-i,1+\hat\sigma-\frac{n}{2},k_0-\hat\sigma,-i;\\ \frac{n}{2}+k_0,\frac{n}{2}+\sigma-i,1+\sigma-k_0-i;\end{matrix}\;1\right]\\ &\times W_{\frac{\sigma+\hat\sigma}{2}-k_0-i,\,\frac{n+\sigma+\hat\sigma-1}{2}}(2\tau\lambda). \qquad (8)\end{aligned}$$

- *The other case $\eta=1$*

$$\begin{aligned}\widetilde{\ker}(\tau,\lambda;n,k_0,\sigma,\hat\sigma)&=(-1)^{\frac{n+1}{2}}(2\tau\lambda)^{\frac{\hat\sigma-\sigma-1}{2}}\mathbf{i}(-\sigma)_{k_0}(\hat\sigma+n-1)_{k_0}\\ &\times(\hat\sigma)_{n-1}\left[\Gamma\left(\frac{n}{2}+k_0\right)\right]^{-1}\sum_{i=0}^{\infty}\frac{\left(1-\sigma-\frac{n}{2}\right)_i(k_0-\sigma)_i}{i!}\\ &\times{}_4F_3\left[\begin{matrix}1-\frac{n}{2}-k_0-i,\frac{n}{2}+\hat\sigma,n+k_0+\hat\sigma-1,-i;\\ \frac{n}{2}+k_0,\frac{n}{2}-\sigma-i,1+\sigma-k_0-i;\end{matrix}\;1\right]\\ &\times W_{\frac{1+\sigma-\hat\sigma-n}{2}-k_0-i,\,\frac{\sigma-\hat\sigma}{2}}(2\tau\lambda)\,.\end{aligned}$$

Proof. We prove this theorem only for the case $\eta=1$. Using (2) and a product formula of functions ${}_pF_q$ (see, e.g., [11, p. 441, Entry 7.2.3-44])

$${}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ b_1,\ldots,b_q;\end{matrix}\;cz\right]{}_rF_s\left[\begin{matrix}\hat a_1,\ldots,\hat a_r;\\ \hat b_1,\ldots,\hat b_s;\end{matrix}\;\hat cz\right]=\sum_{i=0}^{\infty}\gamma_i z^i\,,$$

where

$$\gamma_i = \frac{c^i \prod_{j=1}^{p} (a_j)_i}{i! \prod_{j=1}^{q} (b_j)_i} {}_{q+r+1}F_{p+s} \left[\begin{matrix} -i, 1-b_1-i, \ldots, 1-b_q-i, \hat{a}_1, \ldots, \hat{a}_r; \\ 1-a_1-i, \ldots, 1-a_p-i, \hat{b}_1, \ldots, \hat{b}_s; \end{matrix} \; \frac{(-1)^{p+q+1}\hat{c}}{c} \right],$$

we obtain

$$\int_1^\infty P^{1-k_0-\frac{n}{2}}_{\frac{n}{2}+\sigma-1}(t) P^{1-k_0-\frac{n}{2}}_{-\frac{n}{2}-\hat{\sigma}}(t) \exp(-\tau\lambda t)\,\mathrm{d}t \tag{9}$$

$$= \frac{2^{1-\sigma-\hat{\sigma}}}{\Gamma^2\left(\frac{n}{2}+k_0\right)}$$

$$\times \sum_{i=0}^{\infty} \frac{\left(1-\sigma-\frac{n}{2}\right)_i (k_0-\sigma)_i}{i!\,\left(\frac{n}{2}+k_0\right)_i} {}_4F_3 \left[\begin{matrix} -i, \hat{\sigma}+n+k_0-1, \hat{\sigma}+\frac{n}{2}, 1-k_0-i-\frac{n}{2} \\ \frac{n}{2}+k_0, \frac{n}{2}+\sigma-i, 1+\sigma-k_0-i \end{matrix} \; 1 \right]$$

$$\times \exp(-\tau\lambda) \int_0^\infty t^{\frac{n}{2}+k_0+i-1} (t+2)^{\sigma-\hat{\sigma}-k_0-i-\frac{n}{2}} \exp(-\tau\lambda t)\,\mathrm{d}t. \tag{10}$$

Using a known integral formula (see, e.g., [10, Entry 2.3.6.-9)])

$$\int_0^\infty t^{\alpha-1}(t+u)^{-\beta} e^{-pt}\,\mathrm{d}t = \Gamma(\alpha) u^{\alpha-\beta} U(\alpha; 1+\alpha-\beta; up)$$

$$(|\arg(u)| < \pi, \min\{\Re(\alpha), \Re(p)\} > 0)$$

and the relation (4), we complete the proof for the case $\eta = 1$. Similarly the case $\eta = 0$ can be shown. ■

We find from Lemma 2.2 that the kernel $\widetilde{\ker}(\tau, \lambda; n, k_0, \sigma, \hat{\sigma})$ depends on the product $u := \tau\lambda$, in fact,

$$\widetilde{\ker}(\tau, \lambda; n, k_0, \sigma, \hat{\sigma}) \equiv \widetilde{\ker}(u; n, k_0, \sigma, \hat{\sigma}) \,.$$

It is noted that some particular values of parameters of $\widetilde{\ker}(u; n, k_0, \sigma, \hat{\sigma})$ can yield certain other series representations. For example, replacing $\hat{\sigma}$ by $\sigma + \frac{1}{2}$ in (10), we can use a known formula (see, e.g., [10, Entry 2.3.6.-12])

$$\int_0^\infty t^{\alpha-1}(x+u)^{-\alpha-\frac{1}{2}} e^{-pt}\,\mathrm{d}t = 2^\alpha u^{-\frac{1}{2}} \Gamma(\alpha) e^{\frac{pu}{2}} D_{-2\alpha}\left(\sqrt{2pu}\right) \tag{11}$$

$$(|\arg(u)| < \pi, \min\{\Re(\alpha), \Re(p)\} > 0) \,.$$

Then we may obtain the series involving the product of ${}_4F_3$ and the parabolic cylinder function.

3 Series representations of Macdonald functions

Using the result in Lemma 2.2, the Macdonald function in (5) can be expressed as a series involving the Whittaker function of the second kind $W_{\kappa,\mu}(z)$ (3), asserted in the following theorem.

Theorem 3.1. *Let n be even and $\Re(u) > 0$. Then*

$$K_{\sigma+\frac{n-1}{2}}(u) = (2u)^{-\frac{\sigma+1}{2}} \pi^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(-\sigma)_i \left(1-\sigma-\frac{n}{2}\right)_i}{i!} W_{\frac{\sigma}{2}-i, \frac{\sigma+n-1}{2}}(2u) \,. \tag{12}$$

Proof. Considering that, for any permutation s acting on the set $\{a_1, \ldots, a_p\}$ containing zero,

$$ {}_pF_q\left[\begin{array}{c} s(a_1), \ldots, s(a_p); \\ b_1, \ldots, b_q; \end{array} z\right] = 1\,, $$

we obtain from (8) that

$$ \widetilde{\ker}(u; n, 0, \sigma, 0) = \frac{(-1)^{\frac{n}{2}}\mathbf{i}\Gamma(n-1)}{\pi(2u)^{\frac{n+\sigma}{2}}\Gamma\left(\frac{n}{2}\right)} \sum_{i=0}^{\infty} \frac{(-\sigma)_i\left(1-\sigma-\frac{n}{2}\right)_i}{i!} W_{\frac{\sigma}{2}-i, \frac{\sigma+n-1}{2}}(2u)\,. \tag{13} $$

Also, in view of the identity

$$ \frac{\pi}{\sin(\pi z)} = \Gamma(z)\Gamma(1-z)\,, \tag{14} $$

we have

$$ \widetilde{\ker}(u; n, 0, \sigma, 0) = \frac{(-1)^{\frac{n}{2}}\mathbf{i}(n-2)!}{2\pi} \int_1^{\infty} P^{1-\frac{n}{2}}_{\frac{n}{2}+\sigma-1}(t) P^{1-\frac{n}{2}}_{-\frac{n}{2}}(t) \exp(-ut)\, \mathrm{d}t\,. $$

From (2), we get

$$ \begin{aligned} P^{1-\frac{n}{2}}_{-\frac{n}{2}}(t) &= \left\{\Gamma\left(\frac{n}{2}\right)\right\}^{-1} \left(\tfrac{t+1}{t-1}\right)^{\frac{1}{2}-\frac{n}{4}} {}_1F_0\left(1-\frac{n}{2}; -; \frac{1-t}{2}\right) \\ &= 2^{1-\frac{n}{2}} \left\{\Gamma\left(\frac{n}{2}\right)\right\}^{-1} (t^2-1)^{\frac{n}{4}-\frac{1}{2}}\,. \end{aligned} $$

Therefore,

$$ \widetilde{\ker}(u; n, 0, \sigma, 0) = \frac{(-1)^{\frac{n}{2}}\mathbf{i}(n-2)!}{2^{\frac{n}{2}}\pi\Gamma\left(\frac{n}{2}\right)} \int_1^{\infty} P^{1-\frac{n}{2}}_{\frac{n}{2}+\sigma-1}(t)(t^2-1)^{\frac{n}{4}-\frac{1}{2}} \exp(-ut)\, \mathrm{d}t. $$

Using a known integral formula (see, e.g., [11, Entry 2.17.7.-5])

$$ \int_b^{\infty} (t^2-b^2)^{-\frac{\mu}{2}} \exp(-pt) P^{\mu}_{\nu}\left(\frac{t}{b}\right) \mathrm{d}t = \left(\tfrac{2b}{\pi}\right)^{\frac{1}{2}} p^{\mu-\frac{1}{2}} K_{\nu+\frac{1}{2}}(bp) \tag{15} $$

$$ (b \in \mathbb{R}^+, \Re(p) > 0, \Re(\mu) < 1)\,, $$

we get

$$ \widetilde{\ker}(u; n, 0, \sigma, 0) = \frac{(-1)^{\frac{n}{2}}\mathbf{i}(2u)^{\frac{1-n}{2}}(n-2)!}{\pi^{\frac{3}{2}}\Gamma\left(\frac{n}{2}\right)} K_{\frac{n-1}{2}+\sigma}(u)\,. \tag{16} $$

Finally, equating (13) and (16) leads to the desired identity (12). ■

The particular case $n = 2$ of (12) gives

$$ K_{\sigma+\frac{1}{2}}(u) = (2u)^{-\frac{\sigma+1}{2}} \pi^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{[(-\sigma)_j]^2}{j!} W_{\frac{\sigma}{2}-j, \frac{\sigma+1}{2}}(2u)\,. $$

4 A series involving parabolic cylinder functions

A series associated with parabolic cylinder functions can be evaluated as in the following theorem.

Theorem 4.1. *Let* $\Re(u) > 0$. *Then*

$$ \sum_{j=0}^{\infty} 2^{2j}\,\Gamma\left(j+\frac{1}{2}\right) D_{-2j-1}(2\sqrt{u}) = \frac{\sqrt{\pi}\,e^{-2u}}{2\sqrt{u}}\,. \tag{17} $$

Proof. Taking the definition

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\,\mathrm{d}t \quad (\Re(z) > 0)$$

of the Gamma function, from (6) we obtain

$$\begin{aligned}\widetilde{\ker}\Big(u;1,0,-\frac{1}{2},0\Big) &= \frac{1}{2\mathbf{i}}\int_1^\infty P^{\frac{1}{2}}_{-1}(t)\,P^{\frac{1}{2}}_{-\frac{1}{2}}(t)\,\exp(-ut)\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}\mathbf{i}}\int_1^\infty (t-1)^{-\frac{1}{2}}\,\exp(-ut)\,\mathrm{d}t = \frac{1}{\mathbf{i}\sqrt{2\pi u}\exp(u)}\,.\end{aligned} \tag{18}$$

From the proof of Lemma 2.2 and (11) we get

$$\begin{aligned}\widetilde{\ker}\Big(u;1,0,-\frac{1}{2},0\Big) &= \frac{\sqrt{2}}{\pi\mathbf{i}}\sum_{j=0}^\infty 2^{\mathbf{i}}\int_0^\infty \frac{t^{j-\frac{1}{2}}\,\mathrm{d}t}{(t+2)^{j+1}\exp(ut)} \\ &= \frac{\sqrt{2}\,e^u}{\pi\mathbf{i}}\sum_{j=0}^\infty 2^{2j}\Gamma\Big(j+\frac{1}{2}\Big)D_{-2j-1}\big(\sqrt{4u}\big)\,.\end{aligned} \tag{19}$$

Now, equating (18) and (19) yields the desired identity (17). ■

5 Orthogonality relations for kernels $\widetilde{\ker}(u;n,k_0,\sigma,\hat{\sigma})$

It is noted that some particular cases of the kernel $\widetilde{\ker}(u;n,k_0,\sigma,\hat{\sigma})$ coincide with known variants of Bessel functions. For example, applying (14) and two known integral formulas (see [10, Entries 2.3.5.-4 and 2.3.5.-5])

$$\int_a^\infty \big(t^2-a^2\big)^{\beta-1}e^{-pt}\,\mathrm{d}t = \frac{\Gamma(\beta)}{\sqrt{\pi}}\left(\frac{2a}{p}\right)^{\beta-\frac{1}{2}}K_{\beta-\frac{1}{2}}(ap)$$

$$(\min\{\Re(\beta),\Re(p)\} > 0)$$

and

$$\int_a^\infty e^{\varepsilon\mathbf{i}\lambda t}\big(t^2-a^2\big)^{\beta-1}\,\mathrm{d}t = \varepsilon\mathbf{i}\sqrt{\pi}\Gamma(\beta)\,2^{\beta-\frac{3}{2}}\left(\frac{\lambda}{a}\right)^{\frac{1}{2}-\beta}H^{\left(\frac{3}{2}-\frac{\varepsilon}{2}\right)}_{\frac{1}{2}-\beta}(a\lambda)$$

$$[\varepsilon=\pm1;\Re(\beta)>0,\pm\Re(\mathbf{i}\lambda)<0 \quad (0<\Re(\beta)<1,\Re(\mathbf{i}\lambda)=0)]$$

to (6), respectively, we obtain, for even n,

$$\widetilde{\ker}(u;n,0,0,0) = (-1)^{\frac{n}{2}}(2u)^{\frac{1-n}{2}}\pi^{-\frac{3}{2}}\mathbf{i}(n-2)!\left[\Gamma\left(\frac{n}{2}\right)\right]^{-1}K_{\frac{n-1}{2}}(u) \quad (\Re(u)>0)$$

and

$$\widetilde{\ker}(\varepsilon\mathbf{i}\omega;n,0,0,0) = (-1)^{\frac{n}{2}}2^{-\frac{n+1}{2}}\omega^{\frac{1-n}{2}}\pi^{-\frac{1}{2}}\mathbf{i}^{\frac{n+1}{2}}(n-1)!\left[\Gamma\left(\frac{n}{2}\right)\right]^{-1}H^{(1+\eta)}_{\frac{1-n}{2}}(\omega)$$

$$\Big((-1)^\eta\Re(\mathbf{i}\omega)>0\Big)\,.$$

Therefore, the kernel $\widetilde{\ker}(u;n,k_0,\sigma,\hat{\sigma})$ as a function can be considered as a generalization of some functions associated with the Bessel functions. Certain properties of $\widetilde{\ker}(u;n,k_0,\sigma,\hat{\sigma})$ become generalizations and analogues of those associated Bessel functions. Here, in order to prove a family of orthogonality relations for

$\widetilde{\ker}(u;n,0,\sigma,0)$ and $\widetilde{\ker}(u;n,0,0,\hat{\sigma})$, we use a known integral formula belonging to the Kontorovich-Lebedev integral transform (see, e.g., [6, Eq. (1.1)] and two references therein include the origin of this transformation)

$$\int_0^\infty u^{-1}K_{\mathbf{i}\rho}(u)K_{\mathbf{i}\hat{\rho}}(u)\,\mathrm{d}u = \frac{\pi^2\,\delta(\rho-\hat{\rho})}{2\rho\sinh(\pi\rho)} \quad (\min\{\Re(\rho),\Re(\hat{\rho})\}>0), \tag{20}$$

where δ is the Dirac delta function. In [9], several approaches of proof of (20) were given. In [14], (20) was proved in a simpler way than those in [9] by appealing to a technique occasionally used in mathematical physics.

Applying the the relation between Hankel and MacDonald functions (see [7, Eq. 5.33])

$$H_\nu^{(1+\eta)}(u) = (-1)^\eta 2(\pi\mathbf{i})^{-1}\,\exp\left(\frac{(-1)^{\eta+1}\nu\pi\mathbf{i}}{2}\right)K_\nu\left[\exp\left(\frac{(-1)^{\eta+1}\pi\mathbf{i}}{2}\right)u\right]$$

to (20), we get an orthogonality formula for Hankel functions

$$\int_0^\infty u^{-1}H_{\mathbf{i}\rho}^{(1+\eta)}(u)H_{\mathbf{i}\hat{\rho}}^{(1+\eta)}(u)\,\mathrm{d}u = -\frac{2\exp\left((-1)^\eta\pi\rho\right)\delta(\rho-\hat{\rho})}{\rho\sinh(\pi\rho)}$$

$$(\min\{\Re(\rho),\Re(\hat{\rho})\}>0).$$

In [6], the Kontorovich-Lebedev transform with Hankel function as a kernel was discussed in a detailed manner.

Assume here that $u\in\mathbb{R}$. The orthogonality formulae for the kernel functions are given in the following theorems.

Theorem 5.1. *The following integral formula holds.*

$$\int_0^\infty \widetilde{\ker}\left(u;2,0,-\frac{1}{2}+\mathbf{i}\rho,0\right)\widetilde{\ker}\left(u;2,0,-\frac{1}{2}+\mathbf{i}\hat{\rho},0\right)\mathrm{d}u = -\frac{\delta(\rho-\hat{\rho})}{4\pi\rho\,\sinh(\pi\rho)} \quad (\min\{\Re(\rho),\,\Re(\hat{\rho})\}>0)\,. \tag{21}$$

Proof. In view of (16), for any even n we have

$$\widetilde{\ker}\left(u;n,0,\frac{1-n}{2}+\mathbf{i}\rho,0\right)\widetilde{\ker}\left(u;n,0,\frac{1-n}{2}+\mathbf{i}\hat{\rho},0\right) = -\frac{[(n-2)!]^2}{(2u)^{n-1}\pi\left(\frac{n}{2}-1\right)!}K_{\mathbf{i}\rho}(u)K_{\mathbf{i}\hat{\rho}}(u)\,. \tag{22}$$

Then, setting here $n=2$ and integrating both sides of (22) with respect to u from 0 to ∞ with the aid of (20) gives (21). ■

Theorem 5.2. *The following integral formula holds.*

$$\int_0^\infty \widetilde{\ker}\left(u;2,0,0,-\frac{1}{2}+\mathbf{i}\rho\right)\widetilde{\ker}\left(u;2,0,0,-\frac{1}{2}+\mathbf{i}\hat{\rho}\right)\mathrm{d}u = -\frac{\left(-\frac{1}{2}+\mathbf{i}\rho\right)^2\pi\sinh(\pi\rho)}{4\rho\cosh^2(\pi\rho)}\delta(\rho-\hat{\rho}) \quad (\min\{\Re(\rho),\Re(\hat{\rho})\}>0).$$

Proof. We derive from (2) that $P_\nu^{-\nu}(z) = 2^{-\nu}\{\Gamma(\nu+1)\}^{-1}(z^2-1)^{\frac{\nu}{2}}$. Applying it to (6), we obtain

$$\widetilde{\ker}(u;n,0,0,\hat{\sigma}) = \frac{(-1)^{\frac{n}{2}}\mathbf{i}(\hat{\sigma})_{n-1}}{2^{\frac{n}{2}}\tan(\pi\hat{\sigma})\Gamma\left(\frac{n}{2}\right)}\int_0^\infty P^{1-\frac{n}{2}}_{-\frac{n}{2}-\hat{\sigma}}(t)(t^2-1)^{\frac{n}{4}-\frac{1}{2}}e^{-ut}\,\mathrm{d}t\,.$$

Using (15), we get

$$\widetilde{\ker}(u;n,0,0,\hat{\sigma}) = \frac{(-1)^{\frac{n}{2}}\mathbf{i}\cot(\pi\hat{\sigma})(\hat{\sigma})_{n-1}}{(2u)^{\frac{n-1}{2}}\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}K_{\hat{\sigma}+\frac{n-1}{2}}(u)\,,$$

and, therefore,

$$\int_0^\infty \widetilde{\ker}\left(u;2,0,0,-\frac{1}{2}+\mathbf{i}\rho\right)\widetilde{\ker}\left(u;2,0,0,-\frac{1}{2}+\mathbf{i}\hat{\rho}\right)\mathrm{d}u$$

$$= -\frac{\left(-\frac{1}{2}+\mathbf{i}\rho\right)\left(-\frac{1}{2}+\mathbf{i}\hat{\rho}\right)}{2\pi\coth(\pi\rho)\coth(\pi\hat{\rho})}\int_0^\infty u^{-1}K_{\mathbf{i}\rho}(u)K_{\mathbf{i}\hat{\rho}}(u)\,\mathrm{d}u.$$

Applying (20), we complete the proof. ■

Theorem 5.3. *The following integral formula holds.*

$$\int_0^\infty \widetilde{\ker}\left(u;2,0,-\frac{1}{2}+\mathbf{i}\rho,0\right)\widetilde{\ker}\left(u;2,0,0,-\frac{1}{2}+\mathbf{i}\hat{\rho}\right)\mathrm{d}u$$

$$= -\frac{\mathbf{i}\cosh(\pi\rho)\delta(\rho-\hat{\rho})}{4\rho\sinh^2(\pi\rho)}\quad (\min\{\Re(\rho),\Re(\hat{\rho})\}>0). \tag{23}$$

Proof. Integrating both sides of the equality

$$\widetilde{\ker}\left(u;2,0,-\frac{1}{2}+\mathbf{i}\rho,0\right)\widetilde{\ker}\left(u;2,0,0,-\frac{1}{2}+\mathbf{i}\hat{\rho}\right) = \frac{\mathbf{i}\coth(\pi\rho)}{2\pi^2}K_{\mathbf{i}\rho}(u)K_{\mathbf{i}\hat{\rho}}(u)$$

with respect to u from 0 to ∞ with the aid of (20), we derive (23). ■

Theorem 5.4. *The following integral formula holds.*

$$\int_0^\infty \widetilde{\ker}(u;1,0,0,\mathbf{i}\rho)\widetilde{\ker}(u;3,0,0,-1+\mathbf{i}\hat{\rho})\,\mathrm{d}u$$

$$= -\frac{\rho+\mathbf{i}}{2\sinh(\pi\rho)}\delta(\rho-\hat{\rho})\quad (\min\{\Re(\rho),\Re(\hat{\rho})\}>0).$$

Proof. We obtain from (2) that for any odd n

$$\begin{aligned}\widetilde{\ker}(u;n,0,0,\hat{\sigma}) &= \frac{(-1)^{\frac{n-1}{2}}(\hat{\sigma})_{n-1}}{2\mathbf{i}}\int_1^\infty P^{1-\frac{n}{2}}_{\frac{n}{2}-1}(t)P^{1-\frac{n}{2}}_{-\frac{n}{2}-\hat{\sigma}}(t)e^{-ut}\,\mathrm{d}t\\ &= \frac{(-1)^{\frac{n+1}{2}}\mathbf{i}(\hat{\sigma})_{n-1}}{(2u)^{\frac{n-1}{2}}\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}K_{\hat{\sigma}+\frac{n-1}{2}}(u)\,.\end{aligned} \tag{24}$$

Integrating both sides of

$$\widetilde{\ker}(u;1,0,0,\mathbf{i}\rho)\widetilde{\ker}(u;3,0,0,-1+\mathbf{i}\hat{\rho}) = -\frac{\hat{\rho}^2+\mathbf{i}\hat{\rho}}{\pi^2 u}K_{\mathbf{i}\rho}(u)K_{\mathbf{i}\hat{\rho}}(u)$$

and using (20), we complete the proof. ■

Theorem 5.5. *The following integral formula holds.*

$$\int_0^\infty \widetilde{\ker}\,(u;1,0,\mathbf{i}\rho,0)\;\widetilde{\ker}\,(u;3,0,0,-1+\mathbf{i}\hat{\rho})\,\mathrm{d}u$$

$$= -\frac{(\rho+\mathbf{i})\delta(\rho-\hat{\rho})}{2\sinh(\pi\rho)} \quad (\min\{\Re(\rho),\Re(\hat{\rho})\}>0). \quad (25)$$

Proof. For any odd n, we have

$$\begin{aligned}\widetilde{\ker}(u;n,0,\sigma,0) &= \frac{(-1)^{\frac{n-1}{2}}(0)_{n-1}}{2\mathbf{i}}\int_1^\infty P^{1-\frac{n}{2}}_{\frac{n}{2}+\sigma-1}(t)P^{1-\frac{n}{2}}_{-\frac{n}{2}}(t)e^{-ut}\,\mathrm{d}t \\ &= \frac{(-1)^{\frac{n+1}{2}}\mathbf{i}(0)_{n-1}}{(2u)^{\frac{n-1}{2}}\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}K_{\sigma+\frac{n-1}{2}}(u)\,.\end{aligned} \quad (26)$$

The equality (25) follows from the identity

$$\widetilde{\ker}(u;1,0,\mathbf{i}\rho,0)\widetilde{\ker}(u;3,0,0,-1+\mathbf{i}\hat{\rho}) = \frac{\mathbf{i}\hat{\rho}(-1+\mathbf{i}\hat{\rho})}{\pi u}K_{\mathbf{i}\rho}(u)K_{\mathbf{i}\hat{\rho}}(u). \qquad \blacksquare$$

6 Concluding Remarks

In this paper we have shown that the kernel $\widetilde{\ker}(u;n,k_0,\sigma,\hat{\sigma})$ plays the same role as the function P^l_{mn} and the kernel $K(w,z,g)$ in [11] in the sense that some particular cases of P^l_{mn} reduce to Jacobi and Legendre polynomials, Legendre and Bessel functions, the kernel $K(w,z,g)$ can be expressed in terms of gamma function. Here, properties of the kernel function $\widetilde{\ker}(u;n,k_0,\sigma,\hat{\sigma})$ yield those identities corresponding to variants of Bessel functions and their related functions. For example, choosing $\sigma=0$ in Theorem 3.1, we obtain the well-known relation [5, Entry 9.235.2]

$$K_{\frac{n-1}{2}}(u) = \sqrt{\frac{\pi}{2u}}\,W_{0,\frac{n-1}{2}}(2u)\,.$$

For other examples, the particular cases of of (24) and (26) when $\hat{\rho}=\rho$ give the following integral formula:

$$\int_0^\infty \widetilde{\ker}^2(u;1,0,\mathbf{i}\rho,0)\,\mathrm{d}u = \int_0^\infty \widetilde{\ker}^2(u;1,0,0,\mathbf{i}\rho)\,\mathrm{d}u = -\frac{\operatorname{sech}(\pi\rho)}{16\pi^2}\,,$$

which may be considered as an analogue of the following known integral formula [3, Lemma 2.3]

$$\int_0^\infty [K_{\mathbf{i}\rho}(2\pi u)]^2\,\mathrm{d}u = \frac{\pi}{8\cosh(\pi\rho)}.$$

References

[1] Bell, W.W.: *Special Functions for Scientists and Engineers*. D. van Nostrand Company (1968).

[2] Brychkov, Yu.A.: *Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas*. CRC Press, Taylor & Fancis Group (2008).

[3] Chamizo, F.; Raboso, D.: On the Kuznetsov formula. *J. Func. Anal.* 268 (4) (2015) 869–886.

[4] Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G.: *Higher Transcendental Functions*. McGraw-Hill Book Company (1953).

[5] Gradshteyn, I.S.; Ryzhik, I.M.: *Table of Integrals, Series, and Products*. Academic press (2007).

[6] Gutiérrez-Tovar, Y.E.; Méndez-Pérez, J.M.R.: The Kontorovich–Lebedev integral transformation with a Hankel function kernel in a space of generalized functions of doubly exponential descent. *J. Math. Anal. Appl.* 328 (1) (2007) 359–369.

[7] Miller, W.: *Symmetry and Separation of Variables*. Addison-Wesley Publishing Company (1977).

[8] Olver, F.W.J.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W.: *NIST Handbook of Mathematical Functions and CD-ROM*. Cambridge university press (2010).

[9] Passian, A.; Simpson, H.; Kouchekian, S.; Yakubovich, S.B.: On the orthogonality of the MacDonald's functions. *J. Math. Anal. Appl.* 360 (2) (2009) 380–390.

[10] Prudnikov, A.P.; Brychkov, Yu.A.; Marichev, O.I.; Romer, R.H.: *Integrals and Series*. Overseas Publishers Association (1986).

[11] Prudnikov, A.P.; Brychkov,Yu.A.; Marichev, O.I.: *Integrals and Series: More Special Functions*. Overseas Publishers Association (1986).

[12] Rainville, E.D.: *Special functions*. Chelsea Publishing Company (1971).

[13] Shilin, I.A.; Choi, J.: Some formulas for Bessel and hyper-Bessel functions related to the proper Lorentz group. *Nonlinear Func. Anal. Appl.* 23 (1) (2018) 167–179.

[14] Szmytkowski, R.; Bielski, S.: Comment on the orthogonality of the Macdonald functions of imaginary order. *J. Math. Anal. Appl.* 365 (1) (2010) 195–197.

[15] Vilenkin, N.J.: *Special Functions and the Theory of Group Representations*. American Mathematical Soc. (1968).

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