On the Generalised Ricci Solitons and Sasakian Manifolds

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Abstract. In this note, we find a necessary condition on odd-dimensional Riemannian manifolds under which both of Sasakian structure and the generalised Ricci soliton equation are satisfied, and we give some examples.

1 Introduction and main results

Let \((M, g)\) be a smooth Riemannian manifold. By \(R\) and \(\text{Ric}\) we denote respectively the Riemannian curvature tensor and the Ricci tensor of \((M, g)\). Thus \(R\) and \(\text{Ric}\) are defined by

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,
\]

\[
\text{Ric}(X,Y) = g(R(X,e_i)e_i,Y),
\]

where \(\nabla\) is the Levi-Civita connection with respect to \(g\), \(\{e_i\}\) is an orthonormal frame, and \(X, Y, Z \in \Gamma(TM)\).

The gradient of a smooth function \(f\) on \(M\) is defined by

\[
g(\text{grad} f, X) = X(f), \quad \text{grad} f = e_i(f)e_i,
\]

where \(X \in \Gamma(TM)\). The Hessian of \(f\) is defined by

\[
(\text{Hess} f)(X,Y) = g(\nabla_X \text{grad} f, Y),
\]

where \(X, Y \in \Gamma(TM)\). For \(X \in \Gamma(TM)\), we define \(X^\flat \in \Gamma(T^*M)\) by

\[
X^\flat(Y) = g(X, Y).
\]

(For more details of previous definitions, see for example [9]).

The generalised Ricci soliton equation in Riemannian manifold \((M, g)\) is defined by (see [8])

\[
\mathcal{L}_X g = -2\epsilon_1 X^\flat \odot X^\flat + 2\epsilon_2 \text{Ric} + 2\lambda g,
\]

where \(X \in \Gamma(TM)\), \(\mathcal{L}_X g\) is the Lie-derivative of \(g\) along \(X\) given by

\[
(\mathcal{L}_X g)(Y,Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y),
\]

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for all $Y, Z \in \Gamma(TM)$, and $c_1, c_2, \lambda \in \mathbb{R}$. Equation (6), is a generalization of Killing’s equation ($c_1 = c_2 = \lambda = 0$), Equation for homotheties ($c_1 = c_2 = 0$), Ricci soliton ($c_1 = 0$, $c_2 = -1$), Cases of Einstein-Weyl ($c_1 = 1$, $c_2 = \frac{1}{n-1}$), Metric projective structures with skew-symmetric Ricci tensor in projective class ($c_1 = 1$, $c_2 = \frac{1}{2}$), Vacuum near-horizon geometry equation ($c_1 = 1$, $c_2 = \frac{1}{2}$), and is also a generalization of Einstein manifolds (For more details, see [1], [4], [5], [6], [8]).

In this paper, we give a new generalization of Ricci soliton equation in Riemannian manifold $(M, g)$, given by the following equation

$$L_{X_1} g = -2c_1 X_2^\lambda \odot X_2^\lambda + 2c_2 \text{Ric} + 2\lambda g,$$

(8)

where $X_1, X_2 \in \Gamma(TM)$.

Note that, if $X_1 = \text{grad} f_1$ and $X_2 = \text{grad} f_2$, where $f_1, f_2 \in C^\infty(M)$, the generalised Ricci soliton equation (8) is given by

$$\text{Hess} f_1 = -c_1 df_2 \odot df_2 + c_2 \text{Ric} + \lambda g.$$

(9)

**Example 1.1.** Let $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ be a 2-dimensional hyperbolic space equipped with the Riemannian metric $g = \frac{dx^2 + dy^2}{y^2}$, the following functions

$$f_1(x, y) = -(\lambda - c_2) \ln y, \quad f_2(x, y) = -\sqrt{\frac{c_1(\lambda - c_2)}{c_1}} \ln y,$$

satisfy the generalised Ricci soliton equation (9) with $c_1(\lambda - c_2) > 0$.

**Example 1.2.** The product Riemannian manifold $M^3 = (0, \infty) \times \mathbb{R}^2$ equipped with the Riemannian metric $g = dx^2 + x^2(dy^2 + dz^2)$ satisfies the generalised Ricci soliton equation (9), with

$$f_1(x, y, z) = \frac{\lambda}{2} x^2 - c_2 \ln x, \quad f_2(x, y, z) = -\sqrt{-c_1 c_2} \frac{c_2}{c_1} \ln x,$$

where $c_1 c_2 < 0$.

**Remark 1.3.** There are Riemannian manifolds that do not admit generalized soliton equation (9) such that $f_1 = f_2$ (for example, the Riemannian manifold given in Example 1.2).

An $(2n + 1)$-dimensional Riemannian manifold $(M, g)$ is said to be an almost contact metric manifold if there exist on $M$ a $(1, 1)$ tensor field $\varphi$, a vector field $\xi$ (called the structure vector field) and a 1-form $\eta$ such that

$$\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \Gamma(TM)$. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Such a manifold is said to be a contact metric manifold if $d\eta = \phi$, where $\phi(X, Y) = g(X, \varphi Y)$ is called the fundamental 2-form of $M$. If, in addition, $\xi$ is a Killing vector field, then $M$ is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla X \xi = -\varphi X$, for any vector field $X$ on $M$. The almost contact metric structure of $M$ is said to be normal if $[\varphi, \varphi](X, Y) = -2d\eta (X, Y)\xi$, for any $X, Y \in \Gamma(TM)$, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of $\varphi$, given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that a Sasakian manifold is K-contact, and that an almost contact metric manifold is Sasakian if and only if

$$(\nabla X \varphi) Y = g(X, Y)\xi - \eta(Y)X,$$

(10)

for any $X, Y$. Moreover, for a Sasakian manifold the following equation holds

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

From the formula (10) easily obtains

$$\nabla X \xi = -\varphi X, \quad (\nabla X \eta) Y = -g(\varphi X, Y).$$

(11)
(For more details, see [2], [3], [10]).

The main result of this paper is the following:

**Theorem 1.4.** Suppose \((M, \varphi, \xi, \eta, g)\) is a Sasakian manifold, and satisfies the generalised Ricci soliton equation (9). Then

\[
\zeta \equiv \text{grad} f_1 + c_1 \xi f_2 \text{grad} f_2 - c_1 \xi f_2 \nabla \xi \text{grad} f_2 = \xi (f_1) \xi.
\]

**Remark 1.5.** The condition (12) is necessary for the existence of a Sasakian structure and the generalised Ricci soliton equation (9) on an odd-dimensional Riemannian manifold.

**Example 1.6.** Consider the Sasakian manifold \((\mathbb{R}^2 \times (0, \pi), \varphi, \xi, \eta, g)\) endowed with the Sasakian structure \((\varphi, \xi, \eta, g)\) given by

\[
(g_{ij}) = \begin{pmatrix} p^2 + q^2 & 0 & -q \\ 0 & p^2 & 0 \\ -q & 0 & 1 \end{pmatrix}, 
(\varphi_{ij}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -q & 0 \end{pmatrix},
\]

\[
\xi = \frac{\partial}{\partial z}, \quad \eta = -qdx + dz, \quad p(x, y, z) = \frac{4e^y}{16 + e^{2y}}, \quad q(x, y, z) = -e^{2y}.
\]

Then, the following smooth functions

\[
f_1(x, y, z) = \frac{2c_2 + \lambda}{2} \left( \ln(16 + e^{2y}) - 2 \ln \left( \frac{\sin z}{2c_2 + \lambda} \right) \right),
\]

\[
f_2(x, y, z) = \frac{1}{2} \sqrt{-2c_2 + \lambda \frac{1}{c_1} \left( 2 \ln(\sin z) - \ln(16 + e^{2y}) \right)},
\]

satisfy the generalised Ricci soliton equation (9), where \(c_1 < 0\) and \(2c_2 + \lambda > 0\). Furthermore,

\[
\zeta = \xi (f_1) \xi = -(2c_2 + \lambda) \cot(z) \xi.
\]

**2 Proof of the result**

For the proof of Theorem 1.4, we need the following lemmas.

**Lemma 2.1.** [7] Let \((M, \varphi, \xi, \eta, g)\) be a Sasakian manifold. Then

\[
(\mathcal{L}_\xi (\mathcal{L}_{X_1} g)) (Y, \xi) = g(X_1, Y) + g(\nabla_\xi \nabla_\xi X_1, Y) + Y g(\nabla_\xi X_1, \xi),
\]

where \(X_1, Y \in \Gamma(TM)\), with \(Y\) is orthogonal to \(\xi\).

**Lemma 2.2.** [7] Let \((M, g)\) be a Riemannian manifold, and let \(f_2 \in C^\infty(M)\). Then

\[
(\mathcal{L}_\xi (df_2 \circ df_2)) (Y, \xi) = Y (\xi (f_2)) \xi (f_2) + Y (f_2) \xi (\xi (f_2)),
\]

where \(\xi, Y \in \Gamma(TM)\).

**Lemma 2.3.** Let \((M, \varphi, \xi, \eta, g)\) be a Sasakian manifold of dimension \((2n + 1)\), and satisfies the generalised Ricci soliton equation (9). Then

\[
\nabla_\xi \text{grad} f_1 = (\lambda + 2c_2 n) \xi - c_1 \xi (f_2) \text{grad} f_2.
\]

**Proof.** Let \(Y \in \Gamma(TM)\), we have

\[
\text{Ric}(\xi, Y) = g(\text{R}(\xi, e_i) e_i, Y)
\]

\[
= g(\text{R}(e_i, Y) \xi, e_i)
\]

\[
= \eta(Y) g(e_i, e_i) - \eta(e_i) g(X, e_i)
\]

\[
= (2n + 1) \eta(Y) - \eta(Y)
\]

\[
= 2n \eta(Y)
\]

\[
= 2ng(\xi, Y),
\]
where \( \{e_i\} \) is an orthonormal frame on \( M \), which implies
\[
\lambda g(\xi, Y) + c_2 \text{Ric}(\xi, Y) = \lambda g(\xi, Y) + 2c_2 ng(\xi, Y) = (\lambda + 2c_2 n)g(\xi, Y).
\]
From equations (9) and (13), we obtain
\[
(\text{Hess } f_1)(\xi, Y) = -c_1 \xi(f_2)Y(f_2) + (\lambda + 2c_2 n)g(\xi, Y) = -c_1 \xi(f_2)g(\text{grad } f_2, Y) + (\lambda + 2c_2 n)g(\xi, Y),
\]
the Lemma follows from equation (14).

**Proof of Theorem 1.4.** Let \( Y \in \Gamma(TM)\), such that \( g(\xi, Y) = 0 \), from Lemma 2.1, with \( X_1 = \text{grad } f_1 \), we have
\[
2(\mathcal{L}_\xi(\text{Hess } f_1))(Y, \xi) = Y(f_1) + g(\nabla_\xi \nabla_\xi \text{grad } f_1, Y)
+ Yg(\nabla_\xi \text{grad } f_1, \xi).
\]
By Lemma 2.3, and equation (15), we get
\[
2(\mathcal{L}_\xi(\text{Hess } f_1))(Y, \xi) = Y(f_1) + (\lambda + 2c_2 n)g(\nabla_\xi \xi, Y)
- c_1 g(\nabla_\xi \xi(f_2) \text{grad } f_2, Y)
+ (\lambda + 2c_2 n)Yg(\xi(\xi(f_2))^2).
\]
Since \( \nabla_\xi \xi = 0 \) and \( g(\xi, \xi) = 1 \), from equation (16), we obtain
\[
2(\mathcal{L}_\xi(\text{Hess } f_1))(Y, \xi) = Y(f_1) - c_1 \xi(f_2)Y(f_2)
- c_1 \xi(f_2)g(\nabla_\xi \text{grad } f_2, Y)
- 2c_1 \xi(f_2)Y(\xi(f_2)).
\]
Since \( \mathcal{L}_\xi g = 0 \) (i.e. \( \xi \) is a Killing vector field), it implies that \( \mathcal{L}_\xi \text{Ric} = 0 \). Taking the Lie derivative to the generalised Ricci soliton equation (9) yields
\[
2(\mathcal{L}_\xi(\text{Hess } f_1))(Y, \xi) = -2c_1 (\mathcal{L}_\xi(\text{df }_2 \circ df_2))(Y, \xi).
\]
Thus, from equations (17), (18) and Lemma 2.2, we have
\[
Y(f_1) - c_1 \xi(f_2)Y(f_2)
- c_1 \xi(f_2)g(\nabla_\xi \text{grad } f_2, Y)
- 2c_1 \xi(f_2)Y(\xi(f_2)) = -2c_1 Y(\xi(f_2))^2(\xi(f_2),
\]
which is equivalent to
\[
Y(f_1) + c_1 \xi(f_2)Y(f_2) - c_1 \xi(f_2)g(\nabla_\xi \text{grad } f_2, Y) = 0,
\]
that is, the vector field
\[
\zeta = \text{grad } f_1 + c_1 \xi(f_2) \text{grad } f_2 - c_1 \xi(f_2) \nabla_\xi \text{grad } f_2,
\]
is parallel to \( \xi \). The proof is completed.

**References**


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