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On the Generalised Ricci Solitons and Sasakian Manifolds

A. Mohammed Cherif, K. Zegga and G. Beldjilali

Abstract. In this note, we find a necessary condition on odd-dimensional Riemannian manifolds under which both of Sasakian structure and the generalised Ricci soliton equation are satisfied, and we give some examples.

1 Introduction and main results

Let (M, g) be a smooth Riemannian manifold. By R and Ric we denote respectively the Riemannian curvature tensor and the Ricci tensor of (M, g). Thus R and Ric are defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1}$$

$$\operatorname{Ric}(X,Y) = g(R(X,e_i)e_i, Y), \tag{2}$$

where ∇ is the Levi-Civita connection with respect to g, $\{e_i\}$ is an orthonormal frame, and $X, Y, Z \in \Gamma(TM)$. The gradient of a smooth function f on M is defined by

$$g(\operatorname{grad} f, X) = X(f), \quad \operatorname{grad} f = e_i(f)e_i, \tag{3}$$

where $X \in \Gamma(TM)$. The Hessian of f is defined by

$$(\operatorname{Hess} f)(X,Y) = g(\nabla_X \operatorname{grad} f, Y), \tag{4}$$

where $X, Y \in \Gamma(TM)$. For $X \in \Gamma(TM)$, we define $X^\flat \in \Gamma(T^*M)$ by

$$X^\flat(Y) = g(X,Y). \tag{5}$$

(For more details of previous definitions, see for example [9]).
The generalised Ricci soliton equation in Riemannian manifold (M, g) is defined by (see [8])

$$\mathcal{L}_X g = -2c_1 X^\flat \odot X^\flat + 2c_2 \operatorname{Ric} + 2\lambda g, \tag{6}$$

where $X \in \Gamma(TM)$, $\mathcal{L}_X g$ is the Lie-derivative of g along X given by

$$(\mathcal{L}_X g)(Y,Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y), \tag{7}$$

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Affiliation:
Ahmed Mohammed Cherif – Mascara University, Faculty of Exact Sciences, Department of Mathematics. Laboratory of Geometry, Analysis, Control and Applications. Mascara 29000, Algeria.
E-mail: a.mohammedcherif@univ-mascara.dz
Kaddour Zegga – Mascara University, Faculty of Exact Sciences, Department of Mathematics. Laboratory of Geometry, Analysis, Control and Applications. Mascara 29000, Algeria.
E-mail: zegga.kadour@univ-mascara.dz
Gherici Beldjlali – Mascara University, Faculty of Exact Sciences, Department of Mathematics. Laboratory of Quantum Physics and Mathematical Modeling (LPQ3M). Mascara 29000, Algeria.
E-mail: gherici.beldjilali@univ-mascara.dz

for all $Y, Z \in \Gamma(TM)$, and $c_1, c_2, \lambda \in \mathbb{R}$. Equation (6), is a generalization of Killing's equation ($c_1 = c_2 = \lambda = 0$), Equation for homotheties ($c_1 = c_2 = 0$), Ricci soliton ($c_1 = 0$, $c_2 = -1$), Cases of Einstein-Weyl ($c_1 = 1$, $c_2 = \frac{-1}{n-2}$), Metric projective structures with skew-symmetric Ricci tensor in projective class ($c_1 = 1$, $c_2 = \frac{-1}{n-1}$, $\lambda = 0$), Vacuum near-horzion geometry equation ($c_1 = 1$, $c_2 = \frac{1}{2}$), and is also a generalization of Einstein manifolds (For more details, see [1], [4], [5], [6], [8]).

In this paper, we give a new generalization of Ricci soliton equation in Riemannian manifold (M, g), given by the following equation

$$\mathcal{L}_{X_1} g = -2c_1 X_2^\flat \odot X_2^\flat + 2c_2 \operatorname{Ric} + 2\lambda g, \tag{8}$$

where $X_1, X_2 \in \Gamma(TM)$.

Note that, if $X_1 = \operatorname{grad} f_1$ and $X_2 = \operatorname{grad} f_2$, where $f_1, f_2 \in C^\infty(M)$, the generalised Ricci soliton equation (8) is given by

$$\operatorname{Hess} f_1 = -c_1 df_2 \odot df_2 + c_2 \operatorname{Ric} + \lambda g. \tag{9}$$

Example 1.1. Let $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ be a 2-dimensional hyperbolic space equipped with the Riemannian metric $g = \frac{dx^2 + dy^2}{y^2}$, the following functions

$$f_1(x, y) = -(\lambda - c_2) \ln y, \quad f_2(x, y) = -\frac{\sqrt{c_1(\lambda - c_2)}}{c_1} \ln y,$$

satisfy the generalised Ricci soliton equation (9) with $c_1(\lambda - c_2) > 0$.

Example 1.2. The product Riemannian manifold $M^3 = (0, \infty) \times \mathbb{R}^2$ equipped with the Riemannian metric $g = dx^2 + x^2(dy^2 + dz^2)$ satisfies the generalised Ricci soliton equation (9), with

$$f_1(x, y, z) = \frac{\lambda}{2} x^2 - c_2 \ln x, \quad f_2(x, y, z) = -\frac{\sqrt{-c_1 c_2}}{c_1} \ln x,$$

where $c_1 c_2 < 0$.

Remark 1.3. There are Riemannian manifolds that do not admit generalized soliton equation (9) such that $f_1 = f_2$ (for example, the Riemannian manifold given in Example 1.2).

An $(2n+1)$-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$ tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \Gamma(TM)$. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Such a manifold is said to be a contact metric manifold if $d\eta = \phi$, where $\phi(X, Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M. If, in addition, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field X on M. The almost contact metric structure of M is said to be normal if $[\varphi, \varphi](X, Y) = -2d\eta\ (X, Y)\xi$, for any $X, Y \in \Gamma(TM)$, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that a Sasakian manifold is K-contact, and that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{10}$$

for any X,Y. Moreover, for a Sasakian manifold the following equation holds

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

From the formula (10) easily obtains

$$\nabla_X \xi = -\varphi X, \qquad (\nabla_X \eta)Y = -g(\varphi X, Y). \tag{11}$$

(For more details, see [2], [3], [10]).
The main result of this paper is the following:

Theorem 1.4. *Suppose $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold, and satisfies the generalised Ricci soliton equation* (9)*. Then*

$$\zeta \equiv \operatorname{grad} f_1 + c_1\xi(\xi(f_2)) \operatorname{grad} f_2 - c_1\xi(f_2)\nabla_\xi \operatorname{grad} f_2 = \xi(f_1)\xi. \tag{12}$$

Remark 1.5. The condition (12) is necessary for the existence of a Sasakian structure and the generalised Ricci soliton equation (9) on an odd-dimensional Riemannian manifold.

Example 1.6. Consider the Sasakian manifold $(\mathbb{R}^2 \times (0, \pi), \varphi, \xi, \eta, g)$ endowed with the Sasakian structure (φ, ξ, η, g) given by

$$(g_{ij}) = \begin{pmatrix} p^2+q^2 & 0 & -q \\ 0 & p^2 & 0 \\ -q & 0 & 1 \end{pmatrix}, \quad (\varphi_{ij}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -q & 0 \end{pmatrix},$$

$$\xi = \frac{\partial}{\partial z}, \quad \eta = -q dx + dz, \quad p(x,y,z) = \frac{4e^y}{16+e^{2y}}, \quad q(x,y,z) = \frac{-e^{2y}}{16+e^{2y}}.$$

Then, the following smooth functions

$$f_1(x,y,z) = \frac{2c_2+\lambda}{2}\left(\ln(16+e^{2y}) - 2\ln\left(\frac{\sin z}{2c_2+\lambda}\right)\right),$$

$$f_2(x,y,z) = -\frac{1}{2}\sqrt{-\frac{2c_2+\lambda}{c_1}}\left(2\ln(\sin z) - \ln(16+e^{2y})\right),$$

satisfy the generalised Ricci soliton equation (9), where $c_1 < 0$ and $2c_2 + \lambda > 0$. Furthermore,

$$\zeta = \xi(f_1)\xi = -(2c_2+\lambda)\cot(z)\xi.$$

2 Proof of the result

For the proof of Theorem 1.4, we need the following lemmas.

Lemma 2.1. *[7] Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. Then*

$$\big(\mathcal{L}_\xi(\mathcal{L}_{X_1}g)\big)(Y,\xi) = g(X_1, Y) + g(\nabla_\xi\nabla_\xi X_1, Y) + Yg(\nabla_\xi X_1, \xi),$$

where $X_1, Y \in \Gamma(TM)$, with Y is orthogonal to ξ.

Lemma 2.2. *[7] Let (M, g) be a Riemannian manifold, and let $f_2 \in C^\infty(M)$. Then*

$$\big(\mathcal{L}_\xi(df_2 \odot df_2)\big)(Y,\xi) = Y(\xi(f_2))\xi(f_2) + Y(f_2)\xi(\xi(f_2)),$$

where $\xi, Y \in \Gamma(TM)$.

Lemma 2.3. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold of dimension $(2n+1)$, and satisfies the generalised Ricci soliton equation* (9)*. Then*

$$\nabla_\xi \operatorname{grad} f_1 = (\lambda + 2c_2 n)\xi - c_1\xi(f_2) \operatorname{grad} f_2.$$

Proof. Let $Y \in \Gamma(TM)$, we have

$$\begin{aligned} \operatorname{Ric}(\xi, Y) &= g(R(\xi, e_i)e_i, Y) \\ &= g(R(e_i, Y)\xi, e_i) \\ &= \eta(Y)g(e_i, e_i) - \eta(e_i)g(X, e_i) \\ &= (2n+1)\eta(Y) - \eta(Y) \\ &= 2n\eta(Y) \\ &= 2ng(\xi, Y), \end{aligned}$$

where $\{e_i\}$ is an orthonormal frame on M, which implies

$$\begin{aligned} \lambda g(\xi, Y) + c_2 \operatorname{Ric}(\xi, Y) &= \lambda g(\xi, Y) + 2c_2 n g(\xi, Y) \\ &= (\lambda + 2c_2 n) g(\xi, Y). \end{aligned} \tag{13}$$

From equations (9) and (13), we obtain

$$\begin{aligned} (\operatorname{Hess} f_1)(\xi, Y) &= -c_1 \xi(f_2) Y(f_2) + (\lambda + 2c_2 n) g(\xi, Y) \\ &= -c_1 \xi(f_2) g(\operatorname{grad} f_2, Y) + (\lambda + 2c_2 n) g(\xi, Y), \end{aligned} \tag{14}$$

the Lemma follows from equation (14). ∎

Proof of Theorem 1.4. Let $Y \in \Gamma(TM)$, such that $g(\xi, Y) = 0$, from Lemma 2.1, with $X_1 = \operatorname{grad} f_1$, we have

$$\begin{aligned} 2\big(\mathcal{L}_\xi(\operatorname{Hess} f_1)\big)(Y, \xi) &= Y(f_1) + g(\nabla_\xi \nabla_\xi \operatorname{grad} f_1, Y) \\ &\quad + Y g(\nabla_\xi \operatorname{grad} f_1, \xi). \end{aligned} \tag{15}$$

By Lemma 2.3, and equation (15), we get

$$\begin{aligned} 2\big(\mathcal{L}_\xi(\operatorname{Hess} f_1)\big)(Y, \xi) &= Y(f_1) + (\lambda + 2c_2 n) g(\nabla_\xi \xi, Y) \\ &\quad - c_1 g(\nabla_\xi(\xi(f_2) \operatorname{grad} f_2), Y) \\ &\quad + (\lambda + 2c_2 n) Y g(\xi, \xi) - c_1 Y(\xi(f_2)^2). \end{aligned} \tag{16}$$

Since $\nabla_\xi \xi = 0$ and $g(\xi, \xi) = 1$, from equation (16), we obtain

$$\begin{aligned} 2\big(\mathcal{L}_\xi(\operatorname{Hess} f_1)\big)(Y, \xi) &= Y(f_1) - c_1 \xi(\xi(f_2)) Y(f_2) \\ &\quad - c_1 \xi(f_2) g(\nabla_\xi \operatorname{grad} f_2, Y) \\ &\quad - 2c_1 \xi(f_2) Y(\xi(f_2)). \end{aligned} \tag{17}$$

Since $\mathcal{L}_\xi g = 0$ (i.e. ξ is a Killing vector field), it implies that $\mathcal{L}_\xi \operatorname{Ric} = 0$. Taking the Lie derivative to the generalised Ricci soliton equation (9) yields

$$2\big(\mathcal{L}_\xi(\operatorname{Hess} f_1)\big)(Y, \xi) = -2c_1 \big(\mathcal{L}_\xi(df_2 \odot df_2)\big)(Y, \xi). \tag{18}$$

Thus, from equations (17), (18) and Lemma 2.2, we have

$$\begin{aligned} &Y(f_1) - c_1 \xi(\xi(f_2)) Y(f_2) \\ &- c_1 \xi(f_2) g(\nabla_\xi \operatorname{grad} f_2, Y) - 2c_1 \xi(f_2) Y(\xi(f_2)) \\ &= -2c_1 Y(\xi(f_2)) \xi(f_2) - 2c_1 Y(f_2) \xi(\xi(f_2)), \end{aligned} \tag{19}$$

which is equivalent to

$$Y(f_1) + c_1 \xi(\xi(f_2)) Y(f_2) - c_1 \xi(f_2) g(\nabla_\xi \operatorname{grad} f_2, Y) = 0, \tag{20}$$

that is, the vector field

$$\zeta = \operatorname{grad} f_1 + c_1 \xi(\xi(f_2)) \operatorname{grad} f_2 - c_1 \xi(f_2) \nabla_\xi \operatorname{grad} f_2, \tag{21}$$

is parallel to ξ. The proof is completed. ∎

References

[1] Besse, Arthur L.: *Einstein manifolds. Classics in Mathematics*. Springer-Verlag, Berlin (2008).

[2] Blair, David E.: *Contact manifolds in Riemannian geometry. Lecture Notes in Mathematics, Vol. 509*. Springer-Verlag, Berlin-New York (1976).

[3] Boyer, Charles P. and Galicki, Krzysztof: *Sasakian geometry. Oxford Mathematical Monographs*. Oxford University Press, Oxford (2008).

[4] Chruściel, Piotr T. and Reall, Harvey S. and Tod, Paul: On non-existence of static vacuum black holes with degenerate components of the event horizon. *Classical Quantum Gravity* 23 (2) (2006) 549–554.

[5] Jezierski, Jacek: On the existence of Kundt's metrics and degenerate (or extremal) Killing horizons. *Classical Quantum Gravity* 26 (3) (2009) 035011, 11.

[6] H. K. Kunduri and J. Lucietti: Classification of near-horizon geometries of extremal black holes. *Living Rev. Relativity* 16 (8) (2013) .

[7] Mekki, Mohammed El Amine and Cherif, Ahmed Mohammed: Generalised Ricci solitons on Sasakian manifolds. *Kyungpook Math. J.* 57 (4) (2017) 677–682.

[8] Nurowski, Pawełand Randall, Matthew: Generalized Ricci solitons. *J. Geom. Anal.* 26 (2) (2016) 1280–1345.

[9] O'Neill, Barrett: *Semi-Riemannian geometry. Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York (1983).

[10] Yano, Kentaro and Kon, Masahiro: *Structures on manifolds. Series in Pure Mathematics*. World Scientific Publishing Co., Singapore (1984).

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