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An existence result for p-Laplace equation with gradient nonlinearity in $\mathbb{R}^N$

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Abstract. We prove the existence of a weak solution to the problem

$$-\Delta_p u + V(x)|u|^{p-2}u = f(u, |\nabla u|^{p-2}\nabla u),$$
$$u(x) > 0 \;\; \forall x \in \mathbb{R}^N,$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator, $1 < p < N$ and the non-linearity $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous and it depends on gradient of the solution. We use an iterative technique based on the Mountain pass theorem to prove our existence result.

1 Introduction

In this article, we prove the existence of a weak solution to the problem:

$$-\Delta_p u + V(x)|u|^{p-2}u = f(u, |\nabla u|^{p-2}\nabla u),$$
$$u(x) > 0 \quad \forall x \in \mathbb{R}^N, \tag{1}$$

where $1 < p < N$ and the non-linearity $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function.

The Problem (1) is non-variational in nature, as the nonlinearity f depends on gradient of the solution. Such type of problems have been studied widely in literature through non-variational techniques, such as method of sub-solution and super-solution [11], [16], [24], degree theory [20], [22] etc. In 2004, Figueiredo *et al.* [7] used an iterative technique based on Mountain Pass Theorem to establish the existence of a positive and a negative solution to the problem:

$$-\Delta u = f(x, u, \nabla u) \quad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega,$$

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where $\Omega \subseteq \mathbb{R}^n$ is a smooth and bounded domain. Motivated by the techniques used by Figueiredo et. al [7], several authors established existence results for second order elliptic equations with gradient nonlinearities, see for instance [6], [12], [13], [15], [18], [21] and references therein.

This work is motivated the existence results of G.M. Figueiredo [12], where the author obtained existence of a positive solution to (1) with $V(x) = 1$. Recently, an existence result for (1) in case of $p = N$ is discussed by Chen et al. [5]. For some existence results for the problems of the type (1) with potential $V(x)$ and without gradient dependence, we refer to [1], [9], [17] and references therein.

The plan of this article is as follows: In section 2, we state our hypotheses and main result. Section 3 deals with the proof of our main result, i.e., Theorem 2.1.

2 Hypotheses and Main Result

In this section, we state hypotheses on the nonlinearity f and the potential V. We assume the following conditions on the nonlinearity f:

(f_1) $f(t, |\xi|^{p-2}\xi) = 0$ for all $t < 0,\ \xi \in \mathbb{R}^N$.

(f_2) $\lim\limits_{|t|\to 0} \dfrac{|f(t, |\xi|^{p-2}\xi)|}{|t|^{p-1}} = 0,\ \forall\ \xi \in \mathbb{R}^N$.

(f_3) There exists $q \in (p, p^*)$ such that $\lim_{|t|\to\infty} \dfrac{|f(t, |\xi|^{p-2}\xi)|}{|t|^{q-1}} = 0,\ \forall\ \xi \in \mathbb{R}^N$, where $p^* = \dfrac{Np}{N-p}$.

(f_4) There exists $\theta > p$ such that

$$0 < \theta F(t, |\xi|^{p-2}\xi) \leq t f(t, |\xi|^{p-2}\xi),$$

for all $t > 0,\ \xi \in \mathbb{R}^N$, where $F(t, |\xi|^{p-2}\xi) = \int_0^t f(s, |\xi|^{p-2}\xi)ds$.

(f_5) There exist positive real numbers a and b such that

$$F(t, |\xi|^{p-2}\xi) \geq a t^\theta - b,$$

for all $t > 0,\ \xi \in \mathbb{R}^N$.

(f_6) There exist positive constants L_1 and L_2 such that

$$|f(t_1, |\xi|^{p-2}\xi) - f(t_2, |\xi|^{p-2}\xi)| \leq L_1 |t_1 - t_2|^{p-1}$$

for all $t_1, t_2 \in [0, \rho_1],\ |\xi| \leq \rho_2$,

$$|f(t, |\xi_1|^{p-2}\xi_1) - f(t, |\xi_2|^{p-2}\xi_2)| \leq L_2 |\xi_1 - \xi_2|^{p-1}$$

for all $t \in [0, \rho_1]$ and $|\xi_1|, |\xi_2| \leq \rho_2$, where ρ_1, ρ_2 depend on q, N and θ. Moreover, L_1 and L_2 satisfy $\left(\dfrac{L_2}{C_p - L_1}\right)^{1/p-1} < 1$, where C_p is the constant in the inequality (3).

In the following, we state conditions on the potential V :

(V_1) $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$;

(V_2) $V(x)$ is a continuous 1-periodic function, i.e., $V(x+y) = V(x)$, $\forall y \in \mathbb{Z}^N$ and $\forall x \in \mathbb{R}^N$.

For further details about the periodic potential V, we refer to [1] and references therein. Let

$$W = \{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx < \infty\}.$$

W is a reflexive Banach space with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx\right)^{1/p}.$$

Moreover, we have the continuous inclusions $W \hookrightarrow W^{1,p}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for all $s \in [p, p^*]$. For the details, we refer to [9, Lemma 2.1].

Next, in the spirit of Figueiredo et al. [7], we associate with (1), a family of problems with no dependence on the gradient of solution. To be precise, for every, $w \in W \cap C^{1,\beta}_{loc}(\mathbb{R}^N)$ with $0 < \beta < 1$, we consider the problem

$$\begin{gathered} -\Delta_p u + V(x)|u|^{p-2}u = f(u, |\nabla w|^{p-2}\nabla w), \\ u(x) > 0 \quad \forall x \in \mathbb{R}^N. \end{gathered} \tag{2}$$

Problem (2) is variational in nature and the critical points of the functional

$$I_w(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} F(u, |\nabla w|^{p-2}\nabla w)dx$$

are the weak solutions to (2).

To prove our main result, we will use the following inequality [10] :

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq C_p|x-y|^p, \tag{3}$$

for all $x, y \in \mathbb{R}^N$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^N$. Now, we state our main result:

Theorem 2.1. *Suppose that the conditions* $(f_1) - (f_6)$ *and* (V_1), (V_2) *are satisfied. Then, there exists a positive solution to (1).*

3 Proof of Theorem 2.1

This section deals with the proof of Theorem 2.1. The proof is divided in a series of lemmas.

Lemma 3.1. *Let $w \in W \cap C^{1,\beta}_{loc}(\mathbb{R}^N)$ with $0 < \beta < 1$. Then there exist positive real numbers α and ρ independent of w such that*

$$I_w(u) \geq \alpha > 0, \quad \forall u \in W \text{ such that } \|u\| = \rho.$$

Proof. From (f_2) we have, for any $\epsilon > 0$, there exists $\delta_1 > 0$ such that

$$|f(s, |\xi|^{p-2}\xi)| \leq \epsilon |s|^{p-1}, \quad \forall |s| < \delta_1, \ \xi \in \mathbb{R}^N. \tag{4}$$

From (f_3) we have, for any $\epsilon > 0$, there exists $\delta_2 > 0$ such that

$$|f(s, |\xi|^{p-2}\xi)| \leq \epsilon |s|^{q-1}, \quad \forall |s| > \delta_2, \ \xi \in \mathbb{R}^N. \tag{5}$$

By (4) and (5) we have,

$$|F(u, |\nabla w|^{p-2}\nabla w)| \leq \frac{1}{p}\epsilon|u|^p + \frac{1}{q}\epsilon|u|^q, \quad \forall u \in W. \tag{6}$$

Thus,

$$I_w(u) = \frac{1}{p}||u||^p - \int_{\mathbb{R}^N} F(u, |\nabla w|^{p-2}\nabla w)dx.$$

It follows from (6) and embedding result that

$$I_w(u) \geq (\frac{1}{p} - c_1\epsilon)||u||^p - c_2\epsilon||u||^q.$$

Now choose ϵ such that $\frac{1}{p} - c_1\epsilon > 0$ and $\rho \leq \left(\frac{\frac{1}{p} - c_1\epsilon}{c_2\epsilon}\right)^{\frac{1}{q-p}}$. This completes the proof. ∎

Lemma 3.2. *Let $w \in W \cap C^{1,\beta}_{loc}(\mathbb{R}^N)$ with $0 < \beta < 1$. Fix $v_0 \in C_0^\infty(\mathbb{R}^N)$ with $||v_0|| = 1$. Then $\exists\ t_0 > 0$, independent of w, such that*

$$I_w(tv_0) \leq 0, \quad \forall t \geq t_0.$$

Proof. By (f_5) we get,

$$\begin{aligned} I_w(tv_0) &\leq \frac{t^p}{p} - \int_{\text{Supp}(v_0)} (at^\theta v_0^\theta - b)dx \\ &= \frac{t^p}{p} - at^\theta \int_{\text{Supp}(v_0)} v_0^\theta dx + b|\text{Supp}(v_0)|, \end{aligned}$$

Since $\theta > p$, the result follows. ∎

Lemma 3.3. *Let conditions* $(f_1)-(f_5)$ *and* (V_1), (V_2) *hold. Then, the Problem* (2) *admits a positive solution* $u_w \in W$.

Proof. Lemmas 3.1 and 3.2 tell us that the functional I_w satisfies the geometric conditions of the Mountain Pass Theorem. Hence, by the version of Mountain Pass Theorem without (PS) conditions [23], there exist a sequence $\{u_n\} \subset W$ such that

$$I_w(u_n) \to c_w \quad \text{and} \quad I_w^{'}(u_n) \to 0, \quad \text{as } n \to \infty$$

where

$$c_w = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_w(\gamma(t)) > 0,$$

with

$$\Gamma = \{\gamma \in C([0,1], W) : \gamma(0) = 0,\ \gamma(1) = t_0 v_0\}$$

where t_0 and v_0 are as in Lemma 3.2.

By (f_4), we have $c\|u_n\|^p \leq c_w + \|u_n\|$. This implies that $\{u_n\}$ is bounded in W, hence there exists its subsequence still denoted by $\{u_n\}$, as

$$u_n \rightharpoonup u_w \quad \text{in} \quad W, \tag{7}$$

$$u_n \to u_w \quad \text{in} \quad L^s_{loc} \quad \text{for} \quad p \leq s < p^*. \tag{8}$$

On following the arguments from [8, Proposition 4.4], we obtain

$$\frac{\partial u_n}{\partial x_i}(x) \to \frac{\partial u_w}{\partial x_i}(x) \quad \text{a.e. in } \mathbb{R}^N. \tag{9}$$

This implies,

$$\nabla u_n(x) \to \nabla u_w(x) \quad \text{a.e. in } \mathbb{R}^N. \tag{10}$$

Using (10), we get

$$|\nabla u_n|^{p-2} \nabla u_n \to |\nabla u_w|^{p-2} \nabla u_w \quad \text{a.e. in } \mathbb{R}^N.$$

Since $\{|\nabla u_n|^{p-2} \nabla u_n\}$ is bounded in $L^{p/(p-1)}$, we get,

$$|\nabla u_n|^{p-2} \nabla u_n \rightharpoonup |\nabla u_w|^{p-2} \nabla u_w \quad \text{in } L^{p/(p-1)}(\mathbb{R}^N).$$

By the definition of weak convergence, we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \, dx \to \int_{\mathbb{R}^N} |\nabla u_w|^{p-2} \nabla u_w \nabla \varphi \, dx \quad \text{for all } \varphi \in W.$$

In view of Brezis-Lieb lemma [4], we have

$$\int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \varphi \, dx \to \int_{\mathbb{R}^N} V(x) |u_w|^{p-2} u_w \varphi \, dx \quad \text{for all } \varphi \in W.$$

By the help of [4] and Lebesgue Generalized Theorem [3], we get

$$\int_{\mathbb{R}^N} f(u_n, |\nabla w|^{p-2}\nabla w)\varphi \, dx \to \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)\varphi \, dx \quad \text{for all } \varphi \in W.$$

Therefore, we have

$$\begin{aligned} I'_w(u_w)\varphi = \int_{\mathbb{R}^N} |\nabla u_w|^{p-2}\nabla u_w \nabla\varphi \, dx + \int_{\mathbb{R}^N} V(x)|u_w|^{p-2}u_w\varphi \, dx \\ - \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)\varphi \, dx = 0, \text{ for all } \varphi \in W. \end{aligned} \tag{11}$$

This implies, u is the weak solution of (2).

Let $u_w \not\equiv 0$. Next, we show that $u_w > 0$. By taking $\varphi = u_w^-$ in (11), we get

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_w|^{p-2}\nabla(u_w^+ - u_w^-)\nabla u_w^- \, dx + \int_{\mathbb{R}^N} V(x)|u_w|^{p-2}(u_w^+ - u_w^-)u_w^- \, dx \\ = \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)u_w^- \, dx, \end{aligned}$$

which gives

$$-\int_{\mathbb{R}^N} |\nabla u_w|^{p-2}|\nabla u_w^-|^2 \, dx = \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)u_w^- \, dx + \int_{\mathbb{R}^N} V(x)|u_w|^{p-2}(u_w^-)^2 \, dx.$$

Thus,

$$\int_{\mathbb{R}^N} V(x)|u_w|^{p-2}(u_w^-)^2 = 0.$$

This implies, $|u_w|^{p-2}(u_w^-)^2 = |u_w^+ - u_w^-|^{p-2}(u_w^-)^2 = 0$ as $V(x) > 0$. Therefore, we have

$$0 = |u_w^+(x) - u_w^-(x)|^{p-2}(u_w^-(x))^2 = \begin{cases} 0 \, ; & u_w(x) \geq 0 \\ |u_w^-|^p \, ; & u_w(x) < 0. \end{cases}$$

Hence, $u_w = u_w^+ - u_w^- = u_w^+ \geq 0$. Moreover, Harnack inequality implies that $u_w(x) > 0$ for all $x \in \mathbb{R}^N$.

If $u_w \equiv 0$, then there exist a sequence $\{z_n\} \subset \mathbb{R}^N$ and $\delta, R > 0$ such that

$$\int_{B_R(z_n)} |u_n|^p dx \geq \delta. \tag{12}$$

For, if on the contrary

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{B_R(x)} |u_n|^p dx = 0,$$

then by using [14, Lemma 1.1], $u_n \to 0$ in $L^s(\mathbb{R}^N)$ with $p < s < p^*$, which implies that $I_w(u_n) \to 0$ as $n \to \infty$. It contradicts the fact that $c_w > 0$.

Let us define $v_n(x) = u_n(x + z_n)$. Since $V(x)$ is a 1-periodic function, we can use the invariance of $\mathbb{R}^N$ under translations to conclude that $I_w(v_n) \to c_w$ and $I'_w(v_n) \to 0$. Moreover, up to a subsequence, $v_n \rightharpoonup v_w$ in W and $v_n \to v_w$ in $L^p(B_R(0))$, where v_w is a critical point of I_w. By (12), we conclude that v_w is non zero. Arguing as above, we get that v_w is a positive solution to (2). This completes the proof. ■

Lemma 3.4. *Let $w \in W \cap C^{1,\beta}_{loc}(\mathbb{R}^N)$ with $0 < \beta < 1$. Then there exists positive real number η independent of w, such that*

$$||u_w|| \leq \eta,$$

where u_w is the solution of (2) obtained in Lemma 3.3.

Proof. Using the characterization of c_w, we have

$$c_w \leq \max_{t \geq 0}(tu).$$

Fix $v \in W$ such that $||v|| = 1$. By (f_5), we have

$$c_w \leq \max_{t \geq 0} I_w(tv) \leq \max_{t \geq 0} \left(\frac{t^p}{p} - c_6 t^\theta - c_7 \right) = \eta_0.$$

By (f_4), we have

$$I_w(u_w) \geq \frac{1}{p}||u_w||^p - \frac{1}{\theta} \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2} \nabla w) u_w dx. \tag{13}$$

Also, we have

$$I'_w(u_w)(u_w) = ||u_w||^p - \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2} \nabla w) u_w dx. \tag{14}$$

By (13) and (14), we obtain

$$\left(\frac{1}{p} - \frac{1}{\theta} \right) ||u_w||^p \leq I_w(u_w) - \frac{1}{\theta} I'_w(u_w)(u_w).$$

Next, by using the fact that $I'_w(u_w)(u_w) = 0$ and $I_w(u_w) = c_w$ we get

$$\left(\frac{1}{p} - \frac{1}{\theta} \right) ||u_w||^p \leq c_w \leq \eta_0,$$

and the proof is complete. ■

Lemma 3.5. *If u_w is a positive solution of the equation (2) obtained in Lemma 3.3, then $u_w \in C^{1,\beta}_{loc} \cap L^\infty_{loc}(\mathbb{R}^N)$ with $0 < \beta < 1$. Moreover, there exist positive numbers ρ_1 and ρ_2, independent of w, such that $\|u_w\|_{C^{0,\beta}_{loc}(\mathbb{R}^N)} \leq \rho_1$ and $\|\nabla u_w\|_{C^{0,\beta}_{loc}(\mathbb{R}^N)} \leq \rho_2$.*

Proof. By using the fact that $V(x) \geq V_0$ and $u_w > 0$, we have

$$f(u_w, |\nabla w|^{p-2}\nabla w) - V(x)|u_w|^{p-2}u_w \leq f(u_w, |\nabla w|^{p-2}\nabla w) - V_0|u_w|^{p-1}.$$

By the help of (f_2) and (f_3), one gets

$$|f(u_w, |\nabla w|^{p-2}\nabla w)| \leq \epsilon|u_w|^{p-1} + \epsilon|u_w|^{q-1} + V_0|u_w|^{p-1} \leq (\epsilon + V_0)(|u_w|^{p-1} + |u_w|^{q-1}).$$

By [19, Theorem 2.2], for any compact set $K \subseteq \mathbb{R}^N$, we have $\|u_w\|_{L^\infty(K)} \leq C$, where the constant C depends on p, q, N and $\|u_w\|_{L^{p^*}(K)}$. By Sobolev embedding theorem and Lemma 3.4, there exist C_0 independent of w such that $\|u_w\|_{L^\infty(K)} \leq C_0$. By [2, Theorem 1], $\|\nabla u_w\|_{L^\infty(K)} \leq C_1$, for some constant C_1 dependent on p, q, N and $\|u_w\|_{L^\infty(K)}$. Hence there exists a constant C_2 independent of w such that $\|\nabla u_w\|_{L^\infty(K)} \leq C_2$.

By [2, Theorem 2], we obtain $\|u_w\|_{C^{1,\beta}_{loc}(\mathbb{R}^N)} \leq C_3$, where C_3 is dependent on p, q, N and $\|\nabla u_w\|_{L^\infty(K)}$. Thus there exists a positive number ρ independent of w such that, $\|u_w\|_{C^{1,\beta}_{loc}(\mathbb{R}^N)} \leq \rho$. Subsequently, there exist positive real numbers ρ_1 and ρ_2, independent of w, such that $\|u_w\|_{C^{0,\beta}_{loc}(\mathbb{R}^N)} \leq \rho_1$ and $\|\nabla u_w\|_{C^{0,\beta}_{loc}(\mathbb{R}^N)} \leq \rho_2$. This completes the proof. ■

Lemma 3.6. *Let $w \in W \cap C^{1,\beta}_{loc}(\mathbb{R}^N)$ with $0 < \beta < 1$. Then there exists positive real number λ independent of w, such that*

$$||u_w|| \geq \lambda,$$

where u_w is the solution of (2) *obtained in Lemma 3.3.*

Proof. Since u_w is the weak solution of (2) obtained in Lemma 3.3, for all $v \in W$, we have $I'_w(u_w)(v) = 0$. In particular, by putting $v = u_w$ we get

$$\int_{\mathbb{R}^N} |\nabla u_w|^p dx + \int_{\mathbb{R}^N} V(x)|u_w|^p dx = \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)u_w dx$$

$$||u_w||^p = \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)u_w dx.$$

By using (4) and (5) we have,

$$||u_w||^p \leq c_4\epsilon||u_w||^p + c_5\epsilon||u_w||^q.$$

Since $q > p$, we get $||u_w|| \geq \left(\dfrac{1 - c_4\epsilon}{c_5\epsilon}\right)^{q-p}$. This completes the proof. ■

Now, we are in position to prove Theorem 2.1.

Proof of Theorem 2.1: Starting with an arbitrary $u_0 \in W \cap C^{1,\beta}_{loc}(\mathbb{R}^N)$ with $0 < \beta < 1$, we construct a sequence $\{u_n\} \subseteq W$ as solution of

$$-\Delta_p u_n + V(x)|u_n|^{p-2}u = f(u_n, |\nabla u_{n-1}|^{p-2}\nabla u_{n-1}), \text{ in } \mathbb{R}^N \tag{P_n}$$

obtained in Lemma 3.3. By Lemma 3.5, $\{u_n\} \subseteq W \cap C^{1,\beta}_{loc}(\mathbb{R}^N)$ with $0 < \beta < 1$, $\|u_n\|_{C^{0,\beta}_{loc}(\mathbb{R}^N)} \leq \rho_1$ and $\|\nabla u_n\|_{C^{0,\beta}_{loc}(\mathbb{R}^N)} \leq \rho_2$. Since u_{n+1} is the weak solution of (P_{n+1}), we have

$$\int_{\mathbb{R}^N} |\nabla u_{n+1}|^{p-2}\nabla u_{n+1}\nabla\varphi \, dx + \int_{\mathbb{R}^N} V(x)|u_{n+1}|^{p-2}u_{n+1}\varphi \, dx \tag{15}$$

$$= \int_{\mathbb{R}^N} f(u_{n+1}, |\nabla u_n|^{p-2}\nabla u_n)\varphi \, dx, \qquad \forall\varphi \in W. \tag{16}$$

Similarly, u_n is the weak solution of (P_n), we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi \, dx + \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n\varphi \, dx \tag{17}$$

$$= \int_{\mathbb{R}^N} f(u_n, |\nabla u_{n-1}|^{p-2}\nabla u_{n-1})\varphi \, dx, \qquad \forall\varphi \in W. \tag{18}$$

Set $\varphi = u_{n+1} - u_n$. On subtracting (17) from (15) and by using the inequality (3), we get

$$\begin{aligned}\|u_{n+1} - u_n\|^p \leq \frac{1}{C_p}\int_{\mathbb{R}^N} &[f(u_{n+1}, |\nabla u_n|^{p-2}\nabla u_n) - f(u_n, |\nabla u_n|^{p-2}\nabla u_n)](u_{n+1} - u_n)dx \\ &+ \frac{1}{C_p}\int_{\mathbb{R}^N} [f(u_n, |\nabla u_n|^{p-2}\nabla u_n) - f(u_n, |\nabla u_{n-1}|^{p-2}\nabla u_{n-1})](u_{n+1} - u_n)dx.\end{aligned}$$

By using (f_6), we obtain

$$\|u_{n+1} - u_n\|^p \leq \frac{L_1}{C_p}\int_{\mathbb{R}^N} |u_{n+1} - u_n|^{p-1}(u_{n+1} - u_n)dx + \frac{L_2}{C_p}\int_{\mathbb{R}^N} |u_n - u_{n-1}|^{p-1}(u_{n+1} - u_n)dx.$$

On simplification, we have

$$\frac{C_p - L_1}{C_p}\|u_{n+1} - u_n\|^p \leq \frac{L_2}{C_p}\int_{\mathbb{R}^N} |u_n - u_{n-1}|^{p-1}(u_{n+1} - u_n)dx.$$

Thanks to Hölder inequality, we get

$$\|u_{n+1} - u_n\| \leq \left(\frac{L_2}{C_p - L_1}\right)^{1/p-1} \|u_n - u_{n-1}\| =: d\|u_n - u_{n-1}\|,$$

where $d = \left(\frac{L_2}{C_p - L_1}\right)^{1/p-1}$. Since $d < 1$, $\{u_n\}$ is a Cauchy sequence in W, there exists $u \in W$ such that $\{u_n\}$ converges to u in W. Next, we will prove that u is a solution of the Problem (1). Since $\|\nabla u_n\|_{C^{0,\beta}_{loc}(\mathbb{R}^N)} \leq \rho_2$, we have $|\, |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi| \leq \rho_2^{p-1}|\nabla\varphi|$. Then, by the help of Lebesgue's Dominated Convergence Theorem, we get

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi \, dx \to \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla\varphi \, dx \quad \text{for all } \varphi \in W.$$

In view of Brezis-Lieb lemma [4], we have

$$\int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n\varphi\,dx \to \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\varphi\,dx \quad \text{for all } \varphi \in W.$$

By the help of Lebesgue Generalized Theorem [3], we get

$$\int_{\mathbb{R}^N} f(u_n, |\nabla u_{n-1}|^{p-2}\nabla u_{n-1})\varphi\,dx \to \int_{\mathbb{R}^N} f(u, |\nabla u|^{p-2}\nabla u)\varphi\,dx \quad \text{for all } \varphi \in W.$$

Therefore, as $n \to \infty$, (17) implies

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\varphi\,dx - \int_{\mathbb{R}^N} f(u, |\nabla u|^{p-2}\nabla u)\varphi\,dx = 0,$$

for all $\varphi \in W$. This implies that u is the weak solution of Problem (1). By Lemma 3.6, $u > 0$ in $\mathbb{R}^N$. ■

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