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Invariant bilinear differential operators

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Abstract. Let M be an n-dimensional manifold, let V be the space of a representation $\rho : GL(n) \longrightarrow GL(V)$. Locally, let $T(V)$ be the space of sections of the tensor bundle with fiber V over a sufficiently small open set $U \subset M$, in other words, $T(V)$ is the space of tensor fields of type V on U. In $T(V)$, the group $\mathrm{Diff}(U)$ of diffeomorphisms of U naturally acts by means of ρ applied to the Jacobi matrix of the diffeomorphism at the point.

Here, I give the details of the classification of the $\mathrm{Diff}(M)$-invariant differential operators $D : T(V_1) \otimes T(V_2) \longrightarrow T(V_3)$ for irreducible fibers with lowest weight. Up to dualization and the permutation of arguments $T(V_1) \otimes T(V_2) \simeq T(V_2) \otimes T(V_1)$ a.k.a. "twist", these operators split into 9 types of operators of order 1, four types of order 2 and 3 types of order 3. The operators of orders 2 and 3 are compositions of 1st order operators with one exception: an indecomposable 3rd order operator which exists only for $n = 1$. There are no operators of order > 3.

Amazingly, almost each 1st order operator determines a Lie superalgebra structure on its domain. Moreover, this Lie superalgebra is almost simple (is a central extension of a simple one or contains a simple ideal of codimension 1).

Contents

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Preface

Several years ago Leites told me that the book [KMS] devoted to natural differential operators contains a complaint that the details of the proof of my classification were never published nor preprinted. Actually, they were deposited to VINITI and contain not only the proof in the general case but also the proof in the divergence-free case. However, it was not easy to retrieve anything from VINITI depositions even during Soviet period, now they are, it seems, totally inaccessible.

Here are the details of the proof in the general case; the divergence-free case adds only several compositions of operators, so I skipped it; for formulation, see [G2]. I also skipped verification of invariance of several operators, the fact being known, more or less, to Niujenhuis, and by now can be considered "well-known".

In view of plans greater than the proof of the claims in my Ph.D. thesis' results (these plans are described in [GLS]), Leites arranged translation and typing of the proof. Here is this version.

I want to warn the reader. There are two detailed expositions of the proof (a draft and the final version of my thesis). In the final version, the proof is absolutely correct, but it is written very succinctly and proof modified (as compared with the draft versions) with the peculiarities of the case under study being taken into account. As a result, the proof is shorter (which was my goal), but is difficult to generalize to other algebras or to infinite-dimensional fibers.

Stockholm, 2006.

Introduction

1. Invariant operators: an overview

By *invariant operators* we will mean operators acting in the spaces of tensor fields (or sections of other types of vector bundles) which have the same form in any (curvilinear) coordinate system on the fixed manifold M.

The importance of such operators became manifest after discovery of the relativity theory. Indeed, according to equivalence principle, the motion of a body in the gravitational field is equivalent to the motion in the absence of the field but in a non-inertial coordinate system, with curvilinear coordinates if the gravitational field is non-homogeneous. Thanks to Einstein equations, the action of the gravitational field on bodies is expressed via the metric of the space. Invariance of the Einstein equations is a mathematical formulation of the equivalence principle.

Similarly, invariant operators should always appear whenever there exists either a relation between tensor fields (or sections of vector bundles depending on higher jets of the diffeomorphism group), or a condition on a tensor field, or an algebraic structure, etc., that do not vary under the changes of coordinates.

Examples: Lie algebra structure on the space of vector fields, the Stokes formula, the equation of a geodesic curve, condition for a local rectifiability of a pair of vector fields, condition for local integrability of a distribution, etc., or, if we confine ourselves to analytic coordinates only, Cauchy–Riemann equations, etc.

By *tensor fields* we will mean sections of the bundle

$$E^p_q(M) = (TM)^{\otimes p} \otimes (TM^*)^{\otimes q}.$$

By λ-*densities* we mean sections of $Vol^\lambda = (\Omega^{\dim M})^{\otimes\lambda}$; the space is well-defined as a module over the group of diffeomorphisms for non-negative (and by dualizing for all) integer values of λ, but infinitesimally, on the level of the "Lie algebra of the group of diffeomorphisms", the action of the Lie algebra can be defined for any λ. As we will show, for such Lie algebra one can take the Lie algebra of vector fields with polynomial or formal coefficients and the action of the vector field in Vol^λ is just the multiplication by divergence with factor λ.

In this paper we consider the unary linear differential operators

$$\Gamma(M, E^p_q \otimes Vol^\lambda(M)) \longrightarrow \Gamma(M, E^r_s \otimes Vol^\mu(M))$$

and bilinear differential operators

$$\Gamma(M, E^{p_1}_{q_1} \otimes Vol^{\lambda_1}(M)) \times \Gamma(M, E^{p_2}_{q_2} \otimes Vol^{\lambda_2}(M)) \longrightarrow \Gamma(M, E^r_s \otimes Vol^\mu(M)).$$

The simplest linear invariant operator is the differential of a function

$$f \mapsto df = \sum_{i=1}^n dx_i \frac{\partial f}{\partial x_i}.$$

The invariance of this operator is one of the fundamental theorems of Calculus.

A generalization of this operator is the exterior differential of differential forms

$$d : \Omega^p \longrightarrow \Omega^{p+1}.$$

It turns out that

$$d \quad \textit{is the only linear invariant differential operator of nonzero order}$$

acting in the spaces of tensor fields with irreducible fibers. This was proven for more and more general tensors: for differential forms ([P], 1959), for covariant tensor fields ([L], 1973) and, finally, for general tensors independently and by different methods by Rudakov [R1] in 1973, Terng (Ph.D. Thesis, 1976, see [T]), and Kirillov [Ki1] in 1977.

Consider bilinear operators. Historically, the first and most known first order differential operator is the *Lie derivative*

$$L : \Gamma(M, TM) \times \Gamma(M, E^p_q \otimes \mathit{Vol}^\lambda(M)) \longrightarrow \Gamma(M, E^p_q \otimes \mathit{Vol}^\lambda(M)).$$

Particular cases of this operator: the bracket of vector fields and operators representable as compositions of d and zero order operators.

In the first half of XX century, after works by Einstein and Hilbert on general relativity, researchers started a systematic search of invariant operators. Veblen explicitly formulated the problem at the 1928 Mathematical Congress in Bolognia [V]. In 1940 and 1954, Schouten found two new invariant operators:

$$\Gamma(M, \Lambda^k TM) \times \Gamma(M, \Lambda^l TM) \longrightarrow \Gamma(M, \Lambda^{k+l-1} TM)$$

and

$$L : \Gamma(M, E^p_q(M)) \times \Gamma(M, E^q_p \otimes \mathit{Vol}^\lambda(M)) \longrightarrow \Gamma(M, T^*M \otimes \mathit{Vol}(M)).$$

called the *anti-symmetric* and *Lagrangian* concomitant, respectively.

Schouten also observed that the Poisson bracket can be interpreted, if one restricts to functions homogenous on fibers, as a first order invariant operator (*symmetric* concomitant)

$$P : \Gamma(M, S^k TM) \times \Gamma(M, S^l TM) \longrightarrow \Gamma(M, S^{k+l-1} TM).$$

In 1955, a student of Schouten, Nijenhuis, found one more invariant operator (the *Nijenhuis bracket*) on the space of vector-valued forms (see [N1])

$$N : \Gamma(M, TM \otimes \Lambda^k T^*M) \times \Gamma(M, TM \otimes \Lambda^l T^*M) \longrightarrow \Gamma(M, TM \otimes \Lambda^{k+l} T^*M).$$

During the next 20 years various applications of these operators were studied ([Bu], [FF1], [FF2], [N2], [Tu]).

In 1977–78 in my BS and MS theses, I have completely classified bilinear invariant differential operators for $\dim M \leq 2$, see [G1]. Three new operators were found, denoted in what follows F, G, and P^*.

A. Kirillov noticed (see [Ki1]) that by means of the invariant pairing (index c indicates that we consider fields with compact support)

$$B : \Gamma_c(M, E^p_q(M)) \times \Gamma(M, E^q_p \otimes Vol(M)) \longrightarrow \mathbb{R}$$

one can define the duals (with respect to the first or second argument) briefly referred in what follows as the *first and second duals* or 1-dual, B^{1*}, and 2-dual, B^{2*}, of B.

Clearly, if B is invariant, so are its duals. It turned out that the lagrangian concomitant is dual to the Lie derivative, whereas the operators dual to the other two Schouten's concomitants and to the Nijenhuis bracket turned out to be new.

In the same paper, Kirillov generalized to dimensions $\dim M > 2$ the above mentioned operator F:

$$F : \Gamma(M, \Lambda^p T^* M \otimes Vol^k) \times \Gamma(M, \Lambda^q T^* M \otimes Vol^l) \longrightarrow \Gamma(M, \Lambda^{p+q+1} T^* M \otimes Vol^{k+l}),$$

where $Vol^k := \begin{cases} (\Lambda^n TM)^{\otimes k} & \text{for } k \geq 0 \\ (\Lambda^n T^* M)^{\otimes(-k)} & \text{for } k \geq 0. \end{cases}$

Observe that when we are not interested in rational representations of the group of linear changes of coordinates but allow ourselves to speak about infinitesimal transformations, we can consider not only integer values of k but any real or complex ones.

And the last (as we will see) invariant bilinear differential operator

$$G : \Gamma(M, \Lambda^p T^* M \otimes Vol^k) \times \Gamma(M, \Lambda^q T^* M \otimes Vol^l) \longrightarrow \Gamma(M, \Lambda^{p+q-1} T^* M \otimes Vol^{k+l}),$$

was discovered in 1980, see [G1]. The operator is a generalization of two operators: the anti-symmetric Schouten concomitant and its dual.

The same paper [G1] contains the list of second and third order differential operators. All of them are compositions of the exterior differential d and bilinear operators of orders ≤ 1.

If $\dim M = 1$, there exists one more new operator determined on the weighted densities, not on the usual tensor fields, namely

$$T_2 : f(dx)^{-2/3}, g(dx)^{-2/3} \mapsto (2(f'''g - fg''') - 3(f'g'' - f''g'))\,(dx)^{5/3}.$$

I discovered it in 1977, in my BS thesis.

In 1979, Feigin and Fuchs [FF1] generalized it for the m-linear operators and in 1982 they classified all the multilinear anti-symmetric invariant differential operators acting in the spaces of weighted densities on the line [FF2].

The theorem on complete classification of differential operators acting in the spaces of weighted densities on any manifolds is announced in [G1] and deposited to VINITI; here is a slightly edited translation of the inaccessible deposition.

2. Related results

Let an additional structure on M be fixed, e.g., a volume, or a symplectic structure, or a contact structure. One can consider differential operators invariant with respect to transformations preserving the structure. We can, of course, consider other types of structures, such as metrics or combinations of several structures. But the method we use works well when the Lie algebra of infinitesimal transformations is very asymmetric (has more positive operators than negative ones) and close to simple.

For **linear** (unary) operators, the complete classification was obtained by Rudakov [R1], [R2] for the general, volume preserving and symplectic cases. I. Kostrikin [KoI] described the contact case.

For **bilinear** operators, the complete classification was obtained for the general and volume preserving cases in [G1] and for symplectic case (partly) in [G2], [G3]. I conjecture that these partial results are final as far as indecomposable operators are concerned, i.e., other, new, operators, if any, are compositions of the ones already found. Observe that an explicit description of several operators in symplectic case is to be given though their existence is proved [G3].

For the contact case, only small dimensions on supermanifolds corresponding to some of the "string theories" are considered [LKW].

3. Methods

1) Reduction to canonical forms. For example, one can rectify any vector field or a volume form in a vicinity of any non-singular point; in other words, there are coordinates in which the components of these tensor fields are constants. This means that the rational invariant differential operators in the space $\mathrm{Vect}(M)$ or $Vol(M)$ can only be of order 0. In other words, they are algebraic, point-wise ones.

C.-L. Terng [T] similarly proves that any rational invariant differential operators in the spaces $C^\infty(M)$, or $\Omega^1(M)$, or $\Omega^{n-1}(M)$ can be algebraically expressed via the exterior derivative d.

Epstein [E] similarly proved (making use of Cartan's results) that — and this is an important statement —

any invariant differential operator on the quadratic forms can be algebraically expressed via the curvature tensor and its covariant derivatives.

The tensor fields of more general form can not be reduced to a canonical form by dimension considerations. Nevertheless, any tensor field can be represented as a sum of several tensor fields each of which can be reduced to an *affine* form, i.e., to the form in which the components of the field are vector-valued affine functions

$$f(x) = a + \sum b_i x_i.$$

It is precisely this fact that Palais [P], Leicher [L] and C.-L. Terng [T] used to classify linear operators.

Kirillov in [Ki1] uses another method. He considers any linear invariant operator as a morphism between two pairs of representations of the Lie algebra of vector fields and its subalgebra $\mathfrak{gl}(n)$ of linear vector fields. One further makes use of the sophisticated machinery of representation theory, in particular, Laplace operators on finite dimensional $\mathfrak{gl}(n)$-modules.

Here we come closer to the heart of the matter: the local problem should be solved by local means and Lie algebras should replace global discussions.

Rudakov [R1] *started* with the infinitesimal problem. His method applied to unary operators boils down to simple Linear Algebra only slightly seasoned with some easy facts from representation theory and is applicable to operators of any "arity", not only binary.

The method can be further applied to description of irreducible representations of Lie algebras and superalgebras of vector fields. In some cases the results directly follow from the description of invariant linear differential operators and the Poincaré lemma or its analogs (the general vector fields, see [BL]). Kotchetkov observed that sometimes (when no analog of Poincaré lemma holds) the situation is more subtle, see [Ko], [Ko2].

Bernstein showed [BL] that local Rudakov's problem is equivalent to the global one, initially formulated (somewhat vaguely) by Veblen and in modern and lucid terms by Kirillov.

Let me describe Rudakov's method in more detail: it will be my main tool in this paper.

4. Rudakov's method for solution of Veblen's problem

Let M be a connected n-dimensional manifold over $\mathbb{R}$, and ρ a representation of $GL(n,\mathbb{R})$ in a finite-dimensional space V. Denote by $T(\rho)$ or $T(V)$ the space of tensor fields of type ρ (or, which is the same, of type V), i.e., the collection of the sections of the bundle over M with fiber V (over an open set U). On $T(V)$, the group $\mathrm{Diff}\,M$ of diffeomorphisms of M (the local ones, which send U into itself) naturally acts: let J_A be the Jacobi matrix of A calculated in coordinates of points m and $A^{-1}(m)$, then set:

$$A(t)(m)=\rho(J_A)(t(A^{-1}(m)))\quad \text{for } A\in \mathrm{Diff}\,M,\ m\in M, t\in T(V). \tag{$*$}$$

Any operator $c: T(\rho_1)\longrightarrow T(\rho_2)$ is called *invariant* if it commutes with the $\mathrm{Diff}\,M$-action.

It is instructive to compare Rudakov's and Kirillov's approaches to Veblen's problem. First, they considered different categories, namely Kirillov immediately confined himself to differential operators of finite order and to tensor fields.

Rudakov allowed not only tensor fields but arbitrary jets and did not bind the order of the (differential) operator. His result shows (*a posteriori*) that

1) in spaces of jets higher than tensors (i.e., if the action depends not only on first derivatives of the diffeomorphism, as in ($*$), but on higher derivatives) there are no invariant differential operators (apart from scalar ones);

2) even if we consider arbitrary (irreducible) representations with lowest weight vector, the restrictions on the weight that the invariant operator requires for its existence imply that the representation is finite-dimensional.

Observe that it is only due to the traditional reading of the term “tensor field” that we consider finite-dimensional representations. It is more natural to consider, say, representations with vacuum vector (the lowest for the tensor fields and the highest for the dual spaces), though, strictly speaking, we have to consider indecomposable representations in this infinite-dimensional setting.

Observe also that none of the researchers mention non-local invariant operators: though we all know an example of such an operator — the integral — it is unclear how to study them.

5. The result

This paper contains

1) an enlarged reproduction of my Ph.D. thesis, i.e., I give a detailed proof of the classification of binary differential operators listed in [G1] and [G2].

2) The interpretation of some of the operators in terms of Lie superalgebras seems to be new and might be of interest for theoretical physicists.

Roughly speaking, the list of binary differential operators $D : T(V_1) \otimes T(V_2) \to T(V_3)$ invariant with respect to the group of diffeomorphisms of M runs as follows. Up to dualization and permutation of arguments, the operators split into 9 types of order 1, four types of order 2 and 3 types of order 3. Operators of orders 2 and 3 are compositions of 1st order operators, except one indecomposable operator which only exists for $n = 1$. There are no operators of higher order.

Amazingly, almost all 1-st order operators determine a Lie superalgebra structure on their domain. Moreover, this Lie superalgebra is almost simple: is a central extension of a simple one or contains a simple ideal of codimension 1.

3) In addition to the investigations from my thesis reproduced here, I also considered the infinite-dimensional fibers. The result of this consideration is discouraging: for 2-dimensional manifolds we do not get “really new” operators (the operators we got earlier were realized in functions polynomial fiber-wise; now we consider nonpolynomial functions also but this is all); to consider manifolds of higher dimensions seems to be a wild problem.

6. On open problems

A natural generalization of the above Veblen–Rudakov’s problem: consider operators invariant with respect to other simple Lie algebras (or superalgebras) of vector fields and consider operators of greater *arity*: ternary, etc.

For the review of classification of unary operators (this task is completely performed on manifolds and only partly on supermanifolds), see [L1].

The case of binary operators is, so far, considered on symplectic manifolds, see [G2], [G3], and on general and certain contact supermanifolds [LKW]. Both results are partial.

The exceptional bilinear operator of order 3 was generalized in [FF1], [FF2], where m-ary anti-symmetric operators on the line are classified.

The generalization of the problem in all the directions mentioned is desirable, but we give the priority to the operators invariant with respect to the Lie algebra that preserves the contact form on manifolds and various structures on the supercircle as having more immediate applications.

A mysterious operator

Since in the absence of even coordinates, all operators in superspaces of tensors or jets (with finite dimensional fiber) are *differential* ones (even the integral), Leites hoped that having worked out this finite-dimensional model one could, by analogy, find new non-local invariant operators for manifolds as well. The arity of such operators is, clearly, > 1. So far, no such operator is explicitly written except an example of a *symbol of such a binary* operator acting in the spaces of certain tensors on the line; see Kirillov's review [Ki3].

Recently, this operator was demystified, see [IoMa]. Later on, Bouarroudj and Leites classified bilinear differential operators on 1-dimensional supervariety over algebraically closed fields of characteristic $p > 0$ and found several generalizations of what they called, after Feigin and Fuchs, the "Grozman operator", see [BoLe].

1 The list of operators

Let $\Omega^i = T(\Lambda^i(\mathrm{id}))$ be the space of differential i-forms, $\Omega^{\boldsymbol{\cdot}} = \oplus\, \Omega^i$. Recall that $n = \dim M$; let $\mathit{Vol} = \Omega^n$ and let Vol^λ for $\lambda \in \mathbb{C}$ be the space of λ-densities. This is a rank 1 module over functions $\mathcal{F} = \Omega^0$ with generator vol_x^λ. Observe that the action of $\mathrm{Diff}\, M$ is not defined on Vol^λ unless λ is integer, but the Lie algebra $\mathfrak{vect}(n)$ naturally acts on Vol^λ for any λ by the formula

$$L_D(vol_x^\lambda) = \lambda \mathrm{div}(D) vol_x^\lambda \text{ for any } D \in \mathfrak{vect}(M).$$

We will consider this later wider problem: classification of $\mathfrak{vect}(M)$-invariant differential operators.

1.0. Zero order operators

Obviously any zero order differential operator

$$Z : T(V_1) \otimes T(V_2) \longrightarrow T(W)$$

is just a scalar one and is the uniquely defined extension of a morphism of $\mathfrak{g}_0$-modules in $\mathrm{Hom}(V_1 \otimes V_2, W)$.

1.1. First order operators

$$P_1 : \Omega^r \otimes T(\rho_2) \longrightarrow T(\rho_3) \quad (w, t) \mapsto Z(dw, t); \tag{P_1}$$

where Z is the zero-th order operator — extension of the projection $\rho_1 \otimes \rho_2 \longrightarrow \rho_3$ onto the irreducible component; the operator P_1^{*2} is of the same form, whereas P_1^{*1} is of the form

$$P_1^{*1} : T(\rho_1) \otimes T(\rho_2) \longrightarrow \Omega^r \qquad P_1^{*1}(t_1, t_2) \mapsto d(Z(t_1, t_2)),$$
$$\text{where } Z : T(\rho_1) \otimes T(\rho_2) \longrightarrow \Omega^{r-1} \text{ for } r > 0.$$

The Lie derivative:

$$P_2 : \mathrm{Vect} \otimes T(\rho) \longrightarrow T(\rho); \tag{P_2}$$

the operator P_2^{*2} is also the Lie derivative, whereas P_2^{*1} is Schouten's "lagrangian concomitant".

Schouten's "symmetric concomitant" or the Poisson bracket:

$$P_3 = PB : T(S^p(\mathrm{id}^*)), T(S^q(\mathrm{id}^*)) \longrightarrow T(S^{p+q-1}(\mathrm{id}^*)); \qquad (P_3)$$

for $p = 1$ the operator reduces to the Lie derivative, for $p = 1$ to P_1; the duals of P_3 are also of the same (up to the permutation of arguments T) form:

$$P_3^{*2} : T(S^p(\mathrm{id}^*)), T(S^q(\mathrm{id})) \otimes Vol^* \longrightarrow T(S^{p+q-1}(\mathrm{id})) \otimes Vol^*.$$

The *Nijenhuis bracket*. This bracket is a linear combination of operators P_1, P_1^*1, their composition with the permutation of arguments (a.k.a. twist) operator

$$T : T(V) \otimes T(W) \longrightarrow T(W) \otimes T(V),$$

and an "irreducible" operator sometimes denoted in what follows by N

$$P_4 : \Omega^p \otimes_c \mathrm{Vect}, \Omega^q \otimes_c \mathrm{Vect} \longrightarrow \Omega^{p+q} \otimes_c \mathrm{Vect} \qquad (P_4)$$

defined as follows

$$\begin{array}{l} P_4(\omega_1 \otimes D_1, \omega_2 \otimes D_2) = (\omega_1 \wedge \omega_2) \otimes [D_1, D_2] + \\ \big(\omega_1 \wedge L_{D_1}(\omega_2) + (-1)^{p(\omega_1)} d\omega_1 \wedge \iota_{D_1}(\omega_2)\big) \otimes D_2 + \\ \big(-L_{D_2}(\omega_1) \wedge \omega_2 + (-1)^{p(\omega_2)} \iota_{D_2}(\omega_1) d \wedge \omega_2\big) \otimes D_1. \end{array}$$

The invariance of the Nijenhuis bracket is a corollary of the following observation. It is evident that for a fixed $\omega_1 \otimes D_1 \in \Omega^k \otimes_c \mathrm{Vect}$, the operator $D' : \Omega^k \otimes_c \mathrm{Vect} \times \Omega^{\boldsymbol{\cdot}} \longrightarrow \Omega^{\boldsymbol{\cdot}}$ given by the formula (here $\overline{B}(a, b) := B(b, a)$ is the twisted operator, ι_D is the inner derivation along D)

$$\begin{array}{l} D'(\omega_1 \otimes D_1, \omega_2) = D^{*1}(\omega_1 \otimes D_1, \omega_2) + \overline{D}(\omega_1 \otimes D_1, \omega_2) = \\ d(\omega_1 \wedge L_{D_1}(\omega_2)) + (-1)^{p(\omega_1)} \omega_1 \wedge \iota_{D_1}(d\omega_2) = \\ d\omega_1 \wedge \iota_{D_1} \omega_2 + (-1)^{p(\omega_1)} \omega_1 \wedge L_{D_1}(\omega_2). \end{array}$$

is a superdifferentiation of the supercommutative superalgebra $\Omega^{\boldsymbol{\cdot}}$.

Observe that the $\mathfrak{gl}(n)$-module $\Lambda^k(\mathrm{id}) \otimes \mathrm{id}^*$ is reducible:

$$\Lambda^k(\mathrm{id}) \otimes \mathrm{id}^* = R(1, 0, \dots, 0, \underbrace{-1, \dots, -1}_{k}) \oplus R(0, \dots, 0, \underbrace{-1, \dots, -1}_{k-1}).$$

Therefore, the operator N splits into the direct sum of several operators. One of them, that we did not consider before, will be denoted N or P_4, namely the projection onto the first component:

$$\begin{array}{l} N : T(R(1, 0, \dots, 0, \underbrace{-1, \dots, -1}_{k})) \times T(R(1, 0, \dots, 0, \underbrace{-1, \dots, -1}_{l})) \longrightarrow \\ T(R(1, 0, \dots, 0, \underbrace{-1, \dots, -1}_{k+l})). \end{array}$$

There is also a dual operator:

$$N^{*2}: T(R(1,0,\dots,0,\underbrace{-1,\dots,-1}_{k})) \times T(R(0,\dots,0,\underbrace{-1,\dots,-1}_{l}),-2) \longrightarrow T(R(0,\dots,0,\underbrace{-1,\dots,-1}_{k+l}),-2).$$

The following operator is just a composition of the exterior derivative and a zero order operator:

$$P_5:\Omega^p,\Omega^q \longrightarrow \Omega^{p+q+1};\quad \omega_1,\omega_2 \mapsto (-1)^{p(\omega_1)}a(d\omega_1\omega_2)+b(\omega_1 d\omega_2),\ \text{where } a,b\in\mathbb{C}. \tag{P_5}$$

Let $|\mu|^2+|\nu|^2\neq 0$. Define

$$P_6^{\Omega}:\Omega^p_\mu,\Omega^q_\nu \longrightarrow \Omega^{p+q+1}_{\mu+\nu}$$

by setting

$$\omega_1 vol^{\mu},\omega_2 vol^{\nu} \mapsto \left(\nu(-1)^{p(\omega_1)}d\omega_1\omega_2-\mu\omega_1 d\omega_2\right) vol^{\mu+\nu}. \tag{P_6}$$

Denote the Schouten bracket:

$$P_7: L^p, L^q \longrightarrow L^{p+q-1}. \tag{P_7}$$

Define a generalization $P_8: L^p_\mu, L^q_\nu \longrightarrow L^{p+q-1}_{\mu+\nu}$ of the Schouten bracket (on manifolds, for $p+q\leq n$; on supermanifolds of dimension $n|1$, for $p,q\in\mathbb{C}$) by the formula

$$\begin{gathered}X vol^{\mu}, Y vol^{\nu} \mapsto \big((\nu-1)(\mu+\nu-1)\mathrm{div}X\cdot Y+(-1)^{p(X)}(\mu-1)(\mu+\nu-1)X\mathrm{div}Y-\\ (\mu-1)(\nu-1)\mathrm{div}(XY)\big)\, vol^{\mu+\nu},\end{gathered} \tag{P_8}$$

where the *divergence of a polyvector field* is best described in local coordinates $(x,\check{x})$ on the supermanifold $\check{M}$ associated to any manifold M, cf. [BL].

The operators dual to P_6, P_7, P_8 are, as is not difficult to see, of the same form, respectively.

1.2. Operators of order >1

All of them are reduced to compositions of operators of orders ≤ 1:

$$S_1:\Omega^p\times\Omega^q \longrightarrow T(R(0,\dots,0,\underbrace{-1,\dots,-1}_{k},\underbrace{-2,\dots,-2}_{l})),\ \text{where } k+2l=p+q+2, \tag{S_1}$$

defined to be

$$S_1(\omega_1,\omega_2)=Z(d\omega_1,d\omega_2);$$

$$S_1^*:\Omega^p\times T(R(1,\dots,1,0,\dots,0,-1,\dots,-1)\longrightarrow\Omega^q, \tag{S_1^*}$$

$$S_1^*(\omega,t)=dZ(\omega,t);$$

$$S_2 : \Omega^{n-1} \times \Omega^p \otimes Vol^k \longrightarrow \Omega^{p+1} \otimes Vol^{k+1}, \tag{S_2}$$

defined to be

$$S_2(\omega, t) = F(d\omega, t);$$

$$S_2^* : \Omega^p \otimes Vol^k \times \Omega^{n-1-p} \otimes Vol^{-k-1} \longrightarrow \Omega^1, \tag{S_2^*}$$

defined to be

$$S_2^*(a, b) = dF(a, b);$$

$$T_1 \Omega^{n-1} \times \Omega^{n-1} \longrightarrow \Omega^1 \otimes Vol^2, \tag{T_1}$$

defined to be

$$T_1(\omega_1, \omega_2) = F(d\omega_1, d\omega_2);$$

$$T_1^* : \Omega^{n-1} \times \Omega^{n-1} \otimes Vol^{-2} \longrightarrow \Omega^1, \tag{T_1^*}$$

defined to be

$$T_1^*(\omega, t) = dF(\omega, t);$$

If we abandon requirement of rationality of densities in the definition of operators F, G and S_2, then for $n = 1$ we obtain one more (irreducible, i.e., not factorizable in a composition) invariant operator

$$T_2 : Vol^{-2/3} \times Vol^{-2/3} \longrightarrow Vol^{5/3}, \tag{T_1}$$

defined to be

$$T_2 : (f vol^{-2/3}, g vol^{-2/3}) \longmapsto (2f'''g - 2fg''' + 3f''g' - 3f'g'')vol^{5/3}.$$

1.2.1. Theorem. *Every bilinear invariant differential operator acting in tensor fields on a connected smooth manifold is a linear combination of the above operators and the ones obtains from them by a transposition of the arguments.*

2 The beginning of the proof

Consider the group $G = \mathrm{Diff}\, M$ of local diffeomorphisms of M. In some sense, its Lie algebra is $\mathfrak{g} = \mathfrak{vect}(M)$, the Lie algebra of vector fields on M. Let $G(x_0)$ be the stabilizer of $x_0 \in M$; its Lie algebra is $\mathfrak{g}(x_0) = \{\xi \in g \mid \xi(x_0) = 0\}$. There exists a neighborhood U of x_0 over each point of which the fibers can be identified with a "standard" fiber V, then $\mathfrak{g}$ acts on the tensor fields, the elements from $T(V)$, via the formula

$$L_\xi(\varphi \otimes v) = \sum_{i=1}^{n} \xi_i \frac{\partial \varphi}{\partial x_i} \otimes v + \sum_{i,j=1}^{n} \frac{\partial \xi_i}{\partial x_j} \otimes \rho(E_i^j)v, \tag{1}$$

where $\varphi \in C^\infty(U)$, $v \in V$, $\xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} \in \mathfrak{g}$ and $\{E_i^j\}_{i,j=1}^n$ is the standard basis of $\mathfrak{gl}(n)$ consisting of matrix units. The space $I(V^*) = \mathbb{K}(\partial_1, \dots, \partial_m) \otimes V^*$ of differential operators whose coefficients are linear functionals on V is a $\mathfrak{g}$-invariant subspace of $(T(V))^*$. The pairing of $I(V^*)$ with $T(V)$ is determined by the formula

$$\langle \partial \otimes v', f \otimes v \rangle = \partial f \mid_{x=0} \langle v', v \rangle,$$

where $\partial \in \mathbb{K}[\partial_1, \ldots, \partial_m]$, $f \in C^\infty(M)$, $v \in V$, $v' \in V^*$ and $f \otimes v$ is a representation of the section $s \in T(V)$ in coordinates. The action of $\mathfrak{g}$ is found from the formula

$$\begin{aligned} &\langle I_\xi(\partial \otimes v'), f \otimes v \rangle = -\langle \partial \otimes v', L_\xi(f \otimes v) \rangle = \\ &-(\partial \circ \xi f) \mid_{x=0} \langle v', v \rangle - \textstyle\sum_{i,j} \partial(\frac{\partial \xi_i}{\partial x_j} f) \mid_{x=0} \langle v', \rho(\varepsilon_i^j) v \rangle = \\ &\langle -\partial \circ \xi \mid_{x=0} \otimes v' + \textstyle\sum_{i,j} (\partial \circ \frac{\partial \xi_i}{\partial x_j}) \mid_{x=0} \otimes \rho^*(\varepsilon_i^j) v', f \otimes v \rangle. \end{aligned}$$

Thus,

$$I_\xi(\partial \otimes v') = (\partial \circ \xi) \mid_{x=0} \otimes \sum_{i,j} (\partial \circ \frac{\partial \xi_i}{\partial x_j}) \mid_{x=0} \otimes \rho^*(E_i^j) v'. \tag{2}$$

The space $I(V^*)$ is graded $I(V^*) = \otimes_{m \geq 0} I^m(V^*)$, where $I^m(V^*)$ consists of homogeneous polynomials of degree m in $\partial_1, \ldots, \partial_n$. Let

$$\mathcal{I}_m(V^*) = \oplus_{k=0}^m I^k(V^*)$$

denote the space of polynomials of degree $\leq m$. Observe, that each $\mathcal{I}_m(V^*)$ is $\mathfrak{g}(x_0)$-invariant, in particular, $\mathcal{I}_0(V^*) = V^*$.

Together with $\mathfrak{g}(x_0)$, consider the Lie algebra $\mathcal{L}_0 = \mathfrak{vect}(n)$ of polynomial vector fields on $\mathbb{K}^n$ that vanish at the origin, x_0. Determine the $\mathcal{L}_0$-action on $I(V^*)$ by the same formula (2). Clearly, with $\mathcal{L}_0$ a grading is associated

$$\mathcal{L}_0 = L_0 \oplus L_1 \oplus \ldots,$$

where L_m consists of vector fields whose coefficients (i.e., the coefficients of the ∂_i) are homogeneous polynomials of degree $m+1$. The Lie subalgebra L_0 is isomorphic to $\mathfrak{gl}(n)$ under the correspondence

$$x_i \partial_j \longleftrightarrow E_j^i.$$

Observe that if $\xi \in L_0$, then $I_\xi(1 \otimes v) = 1 \otimes \rho^*(\xi) v$, whereas $\xi \in \mathcal{L}_1 = L_1 \oplus L_2 \oplus \ldots$ annihilates $\mathcal{I}_0(V^*)$.

2.1. Lemma. *$\mathcal{I}_m(V^*)$ is the annihilator of $Z_{x_0}^{m+1}(V) = I_{x_0}^{m+1} \cdot T(V)$.*

Proof. If $\deg \partial \leq m$ and $\varphi \in I_{x_0}^{m+1}$, then $\partial(\varphi s)(0) = 0$. Hence, $\langle \mathcal{I}_m(V^*), Z_{x_0}^{m+1}(V) \rangle = 0$, but $\dim \mathcal{I}_m(V^*) = \mathrm{codim} Z_{x_0}^{m+1}(V)$; hence, $\mathcal{I}_m(V^*)$ is the annihilator of $Z_{x_0}^{m+1}(V)$.

Let $D : T(V_1) \to T(V_2)$ be a G-invariant differential operator of order m. Then,

$$D(Z_{x_0}^{m+1}(V_1)) \subset Z_{x_0}^1(V_2),$$

and therefore $D^*(\mathcal{I}_0(V_2^*)) \subset \mathcal{I}_m(V_1^*)$. □

2.2. Remark. By means of the standard theorems of Linear Algebra one can prove that for any $\mathfrak{g}(x_0)$-invariant operator $D^0 : V_2^* \to \mathcal{I}_m(V_1^*)$, there exists a unique $\mathfrak{g}$-invariant operator $D : T(V_1) \to T(V_2)$ such that $D^* \mid_{\mathcal{I}_0(V_2^*)} = D^0$. The $\mathfrak{g}(x_0)$-action on $\mathcal{I}_m$ depends only on the first $m+1$ derivatives (from the 0-th to $(m+1)$-st inclusively) at the origin x_0 of the vector fields from $\mathfrak{g}(x_0)$. Hence, the sets of operators

$$\{I_\xi : \xi \in g(x_0\} \text{ and } \{I_\xi : \xi \in \mathcal{L}\}$$

coincide, and therefore the $\mathfrak{g}(x_0)$-invariance of the operator $D^* : V_2 \to \mathcal{I}_m(V_1^*)$ is equivalent to its $\mathcal{L}$-invariance.

The fact that $D^*(V_2^*) = W \subset \mathcal{I}_m(V_1^*)$ is an $\mathcal{L}$-submodule isomorphic to V_2^* (which is clear, since V_2^* is irreducible) implies that

a) W is an L_0-submodule isomorphic to V_2^*;

b) $\mathcal{L}_1$ annihilates W.

The problem of description of $\mathfrak{g}$-invariant differential operators is, therefore, equivalent to the following problem:

in $I(V_1^*)$, find all L_0-submodules isomorphic to V_2^* and such that $\mathcal{L}_1$ annihilates them.

The vectors that $\mathcal{L}_1$ annihilates will be called *singular* ones. Thus, our problem is to describe highest weight singular vectors.

In case of the bilinear operators $B : T(V_1) \otimes T(V_2) \longrightarrow T(V_3)$ the above procedure should be modified as follows. Observe that $T(V_1) \otimes T(V_2)$ is a $C^\infty(M) \otimes C^\infty(M)$-module, whereas $\mathcal{I}_{\rho_0} = I_{\rho_0} \otimes 1 + \otimes I_{\rho_0}$ is a maximal ideal of $C^\infty(M) \otimes C^\infty(M)$. Let

$$Z^m_{x_0}(V_1, V_2) = \mathcal{I}^m_{x_0}(T(V_1) \otimes T(V_2)).$$

The space

$$I(V_1^*, V_2^*) = I(V_1^*) \otimes I(V_2^*) = \mathbb{K}[\partial_1', \dots, \partial_n'] \otimes V_1^* \otimes \mathbb{K}[\partial_1'', \dots, \partial_n''] \otimes V_2^* =$$
$$\mathbb{K}[\partial_1', \dots, \partial_n''] \otimes (V_1^* \otimes V_2^*)$$

is graded by the total degree of the polynomials in ∂' and ∂''. Clearly,

$$I(V_1^*, V_2^*) = \oplus_{m=0}^\infty I^m(V_1^*, V_2^*);$$

and

$$\mathcal{I}_m(V_1^*, V_2^*) = \oplus_{i=0}^m I^i(V_1^*, V_2^*)$$

is the annihilator of $Z^{m+1}_{x_0}(V_1, V_2)$. If B is a $\mathfrak{g}$-invariant bilinear operator of order m, then

$$B(Z^{m+1}_{x_0}(V_1, V_2)) \subset Z^1_{x_0}(V_3),$$

hence,

$$B^*(\mathcal{I}_0(V_3^*)) \subset \mathcal{I}_m(V_1^*, V_2^*).$$

Therefore, to find all such B's it remains to find in $I(V_1^*, V_2^*)$ all L_0-submodules annihilated by $\mathcal{L}_1$.

3 Solution ($n = 1$)

All irreducible finite-dimensional and diagonalizable modules of $\mathfrak{gl}(1)$ are 1-dimensional; let V_1^* and V_2^* be such modules. Let $v \in V_1^*$ and $w \in V_2^*$ be nonzero vectors of weight l and m, respectively, i.e.,

$$(x\partial)v = lv, \quad (x\partial)w = mw.$$

Since for $n = 1$, there is no notion of highest weight $\mathfrak{gl}(1)$-vector, it suffices to describe the singular vectors in $I(V_1^*, V_2^*)$. Since $\mathcal{L}_1$ is generated by $\varepsilon_1 = x^2\partial$ and $\varepsilon_2 = x^3\partial$, it suffices to find all weight solutions of the system

$$\varepsilon_1 f = 0, \quad \varepsilon_2 f = 0$$

for a homogeneous vector $f \in I(V_1^*, V_2^*)$ of degree d:

$$f = \sum_{i+j=d} \frac{1}{i!j!} c_i \partial^i v \otimes \partial^j w,$$

where the factor $\frac{1}{i!j!}$ is inserted for further convenience.

Observe that

$$(x^2\partial)(\partial^i \otimes v) = 2i\partial^{i-1} \otimes (x\partial)v - i(i-1)\partial^{i-1} \otimes v = i(2l - i + 1)\partial^{i-1}v;$$

$$\begin{aligned}&(x^3\partial)(\partial^i v) = 3i(i-1)\partial^{i-2} \otimes (x\partial)v - i(i-1)(i-2)\partial^{i-2} \otimes v = \\ &i(i-1)(3l - i + 2)\partial^{i-2}v.\end{aligned}$$

Hence,

$$\begin{aligned}&0 = (x^2\partial)f = \textstyle\sum_{i+j=d} \frac{c_i}{i!j!}(i(2l - i + 1)\partial^{i-1}v \otimes \partial^j w + j(2m - j + 1)\partial^i v \otimes \partial^{j-1}w) = \\ &\textstyle\sum_{i+j=d} \left(\frac{c_{i+1}}{i!j!}(2l - i)\partial^i v \otimes \partial^{j-1}w + \frac{c_i}{i!j!}(2m - j + 1)\partial^i v \otimes \partial^{i-1}w\right)\end{aligned}$$

implying

$$(2l - i)c_{i+1} + (2m - j + 1)c_i = 0$$

and

$$\begin{aligned}&0 = (x^3\partial)f = \\ &\textstyle\sum_{i+j=d} \frac{c_i}{i!j!}(i(i-1)(3l - i + 2)\partial^{i-2}v \otimes \partial^j w + j(j-1)(3m - j + 2)\partial^i v \otimes \partial^{j-2}w = \\ &\textstyle\sum_{i+j=d} \frac{1}{i!j!}\left(c_{i+2}(3l - i)\partial^i v \otimes \partial^{j-2}w + c_i(3m - j + 2)\partial^i v \otimes \partial^{j-2}w\right)\end{aligned}$$

implying

$$(3l - i)c_{i+2} + (2m - j + 2)c_i = 0.$$

Let $i + j = d$, $1 \leq i$, $j \leq d - 1$. The above formulas impose the constraints on c_{i-1}, c_i and c_{i+1}:

$$\begin{cases} (2l - i)c_{i+1} + (2m - j + 1)c_i = 0 \\ (2l - i + 1)c_i + (2m - j)c_{i-1} = 0 \\ (3l - i + 1)c_{i+1} + (3m - j + 1)c_{i-1} = 0. \end{cases}$$

The determinant of this system is

$$\triangle_i = \begin{vmatrix} 2l-i & 2m-j+1 & 0 \\ 0 & 2l-i+1 & 2m-j \\ 3l-i+1 & 0 & 3m-j+1 \end{vmatrix} =$$
$$(2l-i)(2l-i+1)(3m-j+1)+(2m-j)(2m-j+1)(3l-i+1).$$

Observe that if $c_i = 0$ and $c_{i+1} = 0$, then

$$(2m-j)c_{i-1} = (3m-j+1)c_{i-1} = 0$$

but since $2m-j$ and $3m-j+1$ cannot vanish simultaneously for $j \geq 0$, it follows that $c_{i-1} = 0$. Hence, if two neighboring coefficients in a row vanish, then the coefficients neighboring them also vanish.

So, for a nonzero solution to exist, it is necessary (but not sufficient) that $\triangle_i = 0$ for any i such that $1 \leq i \leq d-1$.

Set $x = 2l-i+1$ and $y = 2m-j+1$. Then, in terms of $\mathcal{D} := x+y = 2l+2m-d+2$ we have

$$\triangle_i = \begin{vmatrix} x-1 & y & 0 \\ 0 & x & y-1 \\ x+l & 0 & y+m \end{vmatrix} = (x+l)y(y-1)+(x+1)x(y+m) =$$
$$x^2y+xy^2-2xy+ly(y-1)+mx(x-1) =$$
$$x^2(\mathcal{D}-x)+x(\mathcal{D}-x)^2-2x(\mathcal{D}-x)+l(\mathcal{D}-x)(\mathcal{D}-x-1)+mx(x-1) =$$
$$x^2(l+m+2-\mathcal{D})+x(\mathcal{D}-2\lambda\mathcal{D}+l-m)+l\mathcal{D}^2-l\mathcal{D} = 0.$$

If $d \geq 4$, then the quadratic equation has ≥ 3 solutions which is only possible if all the coefficients vanish:

$$\lambda+\mu+2-\mathcal{D} = 0 \text{ or } l+m+2 = 2l+2m-d+2$$

implying $d = l+m$ and

$$\mathcal{D}^2+2\lambda\mathcal{D}-2\mathcal{D}+l-m = -l^2+m^2-l+m = (m-l)(l+m+1) = 0.$$

But $m+l = d-1 \neq 0$ yields $l = m$; so

$$l\mathcal{D}^2-l\mathcal{D} = l\mathcal{D}(\mathcal{D}-1),$$

but $\mathcal{D} = l+m+2 = d+2 \geq 6$; hence, $l = 0$, but then $m = 0$ as well, implying $d = l+m = 0$. This is a contradiction.

There are no nonzero solutions, hence, on the 1-dimensional manifold, there are no bilinear operators of order > 3.

For $d = 3$, the equation $\triangle_i = 0$ has two roots, 1 and 2:

$$\begin{cases} (2l-1)\cdot 2l\cdot(3m-1) + (2m-2)(2m-1)\cdot 3l = 0 \\ (2l-2)(2l-1)\cdot 3m + (2m-1)\cdot 2m(3l-1) = 0. \end{cases}$$

a) $l = 0$. Then, $6m = 4m^2 + 2m$ implying either $m = 0$ or $m = 2$. Thus, there are solutions $(0,0)$, $(0,2)$ and a symmetric solution $(2,0)$.

b) $l \neq 0$, $m \neq 0$. Then,

$$\begin{gathered} (2l-1)(3m-1) + 3(m-1)(2m-1) = 0 \Longrightarrow \\ 3(2l-1)(3m-1)(l-1) + 9(m-1)(2m-1)(l-1) = 0; \\ 3(l-1)(2l-1) + (2m-1)(3l-1) = 0 \Longrightarrow \\ 3(l-1)(2l-1)(3m-1) + (2m-1)(3l-1)(3m-1) = 0; \end{gathered}$$

implying

$$9(m-1)(l-1)(2m-1) = (3m-1)(2m-1)(3l-1).$$

Hence, either $m = \frac{1}{2}$ and then $(2l-1)(3m-1) = 0 \Longrightarrow l = \frac{1}{2}$ or

$9(m-1)(l-1) = (3m-1)(3l-1)$, i.e., $9lm - 9l - 9m + 9 = 9lm - 3l - 3m + 1 \Longrightarrow l + m = \frac{4}{3}$;

$3(l-1)(2l-1) = (\frac{8}{3} - 2l - 1)(3l-1)$, $2l = \frac{4}{3}$ so, finally, $l = m = \frac{2}{3}$.

But the condition was not a sufficient one; the complete condition is

$$\begin{cases} (2m-2)c_0 + 2lc_1 = 0, \\ (2m-1)c_1 + (2l-1)c_2 = 0, \\ 2mc_2 + (2l-2)c_3 = 0, \\ (3m-1)c_0 + 3lc_2 = 0, \\ 3mc_1 + (3l-1)c_3 = 0. \end{cases}$$

It is routine to verify that in all the cases except for $l = m = \frac{1}{2}$ there is a solution and this solution is unique up to multiplication by a constant; whereas for $l = m = \frac{1}{2}$ there are no solutions. (This is in agreement with our list of operators.)

Let $d = 2$, then the equation $\triangle_1 = 0$ is equivalent to

$$(2l-1)\cdot 2l\cdot 2m + (2m-1)\cdot 2m\cdot 3l = 0.$$

a) $l = 0$, m is arbitrary. The matrix $\begin{pmatrix} -1 & 2m & 0 \\ 0 & 0 & 2m-1 \\ 0 & 0 & 3m \end{pmatrix}$ is of rank 2, hence, there is one solution in this case (namely, the operator S_2, which turns into S_1 for $m = 0$).

The case $m = 0$ is similar.

b) $l \neq 0$, $m \neq 0$. Then, $(2l-1) + (2m-1) = 0$, $l + m = 1$, $m = 1 - l$ and the rank of $\begin{pmatrix} 2l-1 & 2-2l & 0 \\ 0 & 2l & 1-2l \\ 3l & 0 & 3-3l \end{pmatrix}$ is always equal to 2; hence, we have only one operator, S_2^{*1}.

In case $d = 1$, there remains one condition $2l \cdot c_0 + 2m \cdot c_1 = 0$. If l and m do not vanish simultaneously, we have just one operator, P_4, and if $l = m = 0$, then we have two operators (both of type P_1):

$$B(\varphi, \psi) = a\varphi d\psi + bd\varphi \cdot \psi.$$

4 Solution ($n = 2$)

We denote the operator $\xi \in \mathcal{L}$ acting on $I(V_1^*) \otimes I(V_2^*)$ by the same symbol ξ as the element itself, but ξ' indicates that ξ acts on the first factor of $I(V_1^*) \otimes I(V_2^*)$ whereas ξ'' acts only on the second factor.

We identify the linear vector fields with 2×2 matrices and use the following shorthand notations:

$$X_+ = x_1\partial_2, \quad X_- = x_2\partial_1, \quad h_1 = x_1\partial_1, \quad h_2 = x_2\partial_2.$$

The weights of representations ρ_1^*, ρ_2^* and ρ_3^* with respect to h_1 and h_2 will be denoted by

$$\bar{\lambda} = (l_1, l_2), \quad \bar{\mu} = (m_1, m_2), \quad \bar{\nu} = (n_1, n_2);$$

the weights with respect to $h_1 - h_2$ (in other words, with respect to $\mathfrak{sl}(2)$) are:

$$\lambda = l_1 - l_2, \quad \mu = m_1 - m_2, \quad \nu = n_1 - n_2.$$

Sometimes the subscript 2 will be omitted.

In V_1^* and V_2^*, fix weight bases $v_0, v_1, \dots, v_\lambda, \dots$ and $w_0, w_1, \dots, w_\mu, \dots$ such that

$$X_+v_i = (\lambda - i + 1)v_{i-1} \text{ for } i < 0, \quad X_+v_0 = 0$$

$$X_-v_i = (i+1)v_{i+1} \begin{cases} \text{for all } i & \text{if } \lambda \notin \mathbb{Z}_+, \\ \text{for } i < \lambda & \text{if } \lambda \in \mathbb{Z}_+. \end{cases}$$

If $\lambda \in \mathbb{Z}_+$, we set $X_-v_\lambda = 0$; moreover, we only consider $v_0, \dots, v_\lambda$. Similar formulas apply to the $w_0, \dots \in V_2^*$.

In $I(V_1^*, V_2^*)$, the weight basis consists of vectors

$$(\partial_1')^\alpha(\partial_2')^\beta(\partial_1'')^\gamma(\partial_2'')^\delta v_i \otimes w$$

for the above v_i, w_j. Observe that the weights of ∂_1' and ∂_1'' are $(-1, 0)$, the weights of ∂_2' and ∂_2'' are $(0, -1)$. The weight vector f is a highest one if $X_+f = 0$.

The irreducible finite-dimensional $\mathfrak{gl}(2)$-module is determined by its highest weight vector up to an isomorphism. In particular, in order to find all finite-dimensional irreducible L_0 submodules of $I(V_1^*, V_2^*)$ it suffices to find all the highest weight vectors in $I(V_1^*, V_2^*)$.

Further, among these vectors, we have to find the singular vectors, the ones that $\mathcal{L}_1$ annihilates. To this end, it suffices to solve the system of equations $(x_2^2\partial_1)f = 0$ and $(x_2^2\partial_2)f = 0$ because every element from $\mathcal{L}_1$ can be expressed in terms of $x_2^2\partial_1$, $x_2^2\partial_2$ and X_+.

Thus, our problem is to find the homogeneous (with respect to weight) solutions of the system
$$X_+f = (x_2^2\partial_1)f = (x_2^2\partial_2)f = 0. \tag{3}$$
The action of $\mathcal{L}$ on $I(V_1^*, V_2^*)$ is compatible with the grading, and therefore any solution of system is the sum of homogeneous solutions. We will seek only homogeneous solutions. Observe that the homogeneity degree of the singular vector coincides with the order of the corresponding differential operator.

We will look for solutions in the form
$$f = \sum P_i(\partial_1', \partial_2', \partial_1'', \partial_2'') \otimes u_i,$$
where P_i are monomials of degree d and $u_i \in V_1^* \otimes V_2^*$ (in what follows we will often omit the sign of the tensor product).

4.1. Lemma. *Weight solutions of the equation $(X_+)^{d+1}u = 0$, where $u \in V_1^* \otimes V_2^*$, are of the form*
$$u = \sum_{i=0}^{s}(-1)^i(\lambda - i)!(\mu - s + i)!P(i)v_i \otimes w_{s-i}, \tag{4}$$
where $P(i)$ is a polynomial of degree not greater than d.

I will denote the elements u of the form (4) by $(s; P(i))$. The weight of such an element is equal to $(l_1 + m_1 - s, l_2 + m_2 + s)$.

Proof of the lemma follows from the formula
$$X_+(s, P(i)) = (s - 1, P(i) - P(i + 1))$$
(that is the $\deg(P(i) - P(i+1)) - \deg P(i) = 1$). Observe also that the action of X_- is as follows:
$$x_-'(s, P(i)) = (s + 1, i(i - \lambda - 1)P(i - 1)) \qquad (i = 0, 1, \dots, s + 1). \tag{5}$$
$$x_-''(s, P(i)) = (s + 1, -(i + \mu - s)(i - s - 1)P(i)) \qquad (i = 0, 1, \dots, s + 1). \tag{6}$$
In what follows we will see that the elements u_j which correspond to monomials of the form $P(\partial_2', \partial_2'')$ of degree d in the decomposition (4) of f satisfy
$$(X_+)^{d+1}u = 0.$$
The weight of such u is equal to $(l_1 + m_1 - s, l_2 + m_2 + s)$, and therefore the weight of f is equal to $(l_1 + m_1 - s, l_2 + m_2 + s - d)$.

If $B : (T(V_1), T(V_2)) \to T(V_3)$ is a homogeneous differential operator of order d, then its first and second duals are of the same order. This can be deduced by integrating by parts
$$\int_M z(B(s_1, s_2), s_3),$$

where z is the pairing $z : (T(V_3),\ T_c(V_3^+)) \to Vol$, and $V^+ := V^* \otimes \mathrm{tr}$. The displayed formula makes sense if the supports of s_1, s_2 and s_3 belong to one neighborhood $\mathcal{U}$; due to the locality of the operators this suffices.

Let the weight of the representation ρ_3^* be

$$(n_1, n_2) = (l_1 + m_1 - s, l_2 + m_2 + s - d); \quad \nu = n_1 - n_2 = \lambda + \mu - 2s + d.$$

Since the dimensions of the spaces V and V^+ are equal, it suffices to look for operators such that $\lambda \leq \mu \leq \nu$ (the other operators will be 1-dual or 2-dual to such operators).

Thus, let us solve our system under the condition that $\lambda \leq \mu \leq \lambda + \mu - 2s + d$, i.e.,

$$\mu \geq \lambda \geq 2s - d.$$

4.2. Lemma. *The system*

$$\begin{array}{l} a_j x'_- u_j + b_j x''_- u_{j+1} = 0, \ \textit{where} \\ j = 1, 2, \dots, d; a_j \neq 0, b_j \neq 0, u_j \in V_1^* \otimes V_2^* \end{array}$$

has no solutions of the form (4) *if* $s > d$ *and* $\lambda, \mu \geq 2s - d$.

Proof. Having multiplied each u_j by the corresponding coefficient we may assume that $a_j = b_j = 1$. Let $u_j = (s, P_j(i))$, where $\deg P_j \leq d$. By formulas (5), (6), we have

$$i(i-\lambda-1)P_j(i-1) = (i+\mu-s)(i-s-1)P_{j+1}(i) \quad \text{for all } i \in \{0, 1, \dots, s+1\},\ j \in \{1, 2, \dots, d\}.$$

Observe that $(i + \mu - s)(i - s - 1) \neq 0$ for $0 \leq i \leq d$.

For $i = 0$, we get

$$0 = P_{j+1}(0)(\mu - s)(-s - 1)$$

implying that $P_j = 0$ for $j = 2, 3, \dots, d$.

For $i = 1$, we get

$$P_j(1) = \frac{1 \cdot (1 - \lambda - 1)}{(1 + \mu - s)(1 - s - 1)} P_{j-1}(0) = 0$$

for $j = 3, 4, \dots, d$. Substituting $i = 2, 3, \dots, d - 1$ we get

$$P_d(i) = 0 \text{ for } i = 0, 1, \dots, d - 2; \quad P_{d+1}(i) = 0 \text{ for } i = 0, 1, \dots, d - 1.$$

Moreover, substituting $i = s + 1$ in the last equation we get $P_d(s) = 0$. Since $P_d(i)$ and $P_{d+1}(i)$ are of degree d, it follows that

$$P_d(i) = ai(i - 1) \dots (i - d + 2)(i - s); \quad P_{d+1}(i) = bi(i - 1) \dots (i - d + 1).$$

Having substituted

$$\begin{array}{l} i(i - \lambda - 1)a(i - 1)(i - 2) \dots (i - d + 1)(i - s - 1) = \\ (i + \mu - s)(i - s - 1)bi(i - 1) \dots (i - d + 1) \end{array}$$

in the last equation we deduce for $i = d, d + 1$ that

$$a(i - \lambda - 1) = b(i + \mu - s).$$

Therefore, either $a = b = 0$ or $s = \lambda + \mu + 1$; the latter contradicts the conditions $s > d$ and $\lambda, \mu \geq 2s - d$. $\square$

5 The solutions of degree $d = 0$

All the vectors of degree 0 are annihilated by $\mathcal{L}_1$ and the L_0-action on them coincides with $\rho_1^* \otimes \rho_2^*$. Therefore, the solutions of system (3) are all the highest weight vectors from $\mathcal{I}_0(V_1^*, V_2^*) = 1 \otimes (V_1^* \otimes V_2^*)$. To find them, we have to decompose the representation $\rho_1^* \otimes \rho_2^*$ into the sum of irreducible representations. This is a classical problem (for its solution in some cases and an algorithm, see Table 5 in [OV]). The embedding $V_3^* \to V_1^* \otimes V_2^*$ generates the map

$$Z^* : I(V_3^*) \to I(V_1^*) \otimes I(V_2^*)$$

and the dual projection $V_1 \otimes V_2 \to V_3$ gives rise to the operator

$$Z : T(V_1) \otimes T(V_2) \to T(V_3).$$

The above arguments hold for any $n = \dim M$, even for supermanifolds and any arity of the operator, not only binary.

6 Solutions of degree $d = 1$

The generic degree 1 element is of the form

$$f = \partial_1' u_1 + \partial_2' u_2 + \partial_1'' u_3 + \partial_2'' u_4.$$

We have

$$X_+ f = -\partial_2' u_1 + \partial_1'(X_+ u_1) + \partial_2'(X_+ u_2) - \partial_2'' u_3 + \partial_1''(X_+ u_3) + \partial_2''(X_+ u_4)$$

wherefrom

$$u_1 = X_+ u_2, \quad X_+ u_1 = 0, \quad u_3 = X_+ u_4, \quad X_+ u_3 = 0$$

or

$$(X_+)^2 u_2 = (X_+)^2 u_4 = 0.$$

Hence, u_2 and u_4 are of the form (4). The remaining two equations yield

$$(x_2^2 \partial_2) f = 2h_2' u_2 + 2h_2'' u_4 = 0, \quad (x_2^2 \partial_1) f = 2x_-' u_2 + 2x_-'' u_4 = 0$$

wherefrom Lemma 4.2 implies that for $s \geq 2$ there are no solutions such that $\lambda \leq \mu \leq \nu$. It remains to consider the cases $s = 0, 1$.

Let $s = 1$. Then, $\mu \geq \lambda \geq 2s - d = 1$. The generic form of the elements u_2 and u_4 is

$$u_2 = a v_0 \otimes w_1 + b v_1 \otimes w_0 \text{ shortly } a(01) + b(10)$$

and

$$u_4 = \alpha(01) + \beta(10).$$

We have to find all the u_2 and u_4 satisfying

$$x_-' u_2 + x_-'' u_4 = 0, \quad h_2' u_2 + h_2'' u_4 = 0.$$

Consider the following 3 cases:

1) $\lambda \geq 2$, $\mu \geq 2$. Then,

$$x'_{-}u_2 + x''_{-}u_4 = a(11) + 2b(20) + 2\alpha(02) + \beta(11) = 0$$

implying $\alpha = b = 0$, $\beta = -a$. We have

$$h'_2 u_2 + h''_2 u_4 = la(01) - ma(10)$$

implying $l = m = 0$. Thus,

$$\bar{\lambda} = (\lambda, 0), \quad \bar{\mu} = (\mu, 0), \quad \bar{\nu} = (\lambda + \mu - s, 0 + 0 + s - 1) = (\lambda + \mu - 1, 0).$$

The corresponding operator is the Schouten concomitant (the operator P_3 on our list).

2) $\lambda = 1$, $\mu \geq 2$. We have

$$x'_{-}u_2 + x''_{-}u_4 = a(11) + 2\alpha(02) + \beta(11) \Longrightarrow \alpha = 0, \beta = -a,$$

then

$$h'_2 u_2 + h''_2 u_4 = la(01) + (l+1)b(10) - ma(10) = 0$$

implying $la = 0, (l+1)b - ma = 0$. There are two cases:

a) $a = 0$ and then $b \neq 0$ (since otherwise $f = 0$) and $l = -1$. We have

$$\bar{\lambda} = (0, -1), \bar{\mu}(m_1, m_2), \bar{\nu} = (m_1 - 1, m_2 - 2).$$

The corresponding operator is $B(w, s) = dw \circ s$ (the operator P_1 on our list).

b) $a \neq 0 \Longrightarrow l = 0, b = ma$. Hence,

$$\bar{\lambda} = (1, 0), \bar{\mu} = (m_1, m_2), \bar{\nu} = (m_1, m_2).$$

The corresponding operator is $B(\xi, s) = L_\xi s$, the Lie derivative (the operator P_2 on our list).

3) $\lambda = \mu = 1$. We have

$$x'_{-}u_2 + x''_{-}u_4 = a(11) + \beta(11) = 0 \Longrightarrow \beta = -a,$$

and we have

$$h'_2 u_2 + h''_2 u_4 = la(01) + (l+1)b(10) + (m+1)\alpha(01) - ma(1,0) = 0$$

or, equivalently,

$$\begin{cases} la + (m+1)\alpha = 0 \\ (l+1)b - ma = 0 \end{cases}$$

implying $a = (l+1)(m+1)x$, $b = m(m+1)x$, $\alpha = -l(l+1)x$.

Therefore,

$$\bar{\lambda} = (l+1, l), \quad \bar{\mu} = (m+1, m) \quad \bar{\nu} = (l+m+1, l+m)$$

that is the corresponding operator is of type P_4 in our notations.

The case $m=0$, $l=-1$ as well as $l=0$, $m=-1$ and $l=m=-1$ are particular cases which correspond to *two* operators each:

$$B(\xi, w) = a\xi dw + bL_\xi w$$

in the first two cases and

$$B(w_1, w_2) = aw_1 dw_2 + bw_2 dw_1$$

in the third case. (All these operators correspond to operators P_1 and P_2 on our list).

Let now $s=0$, $u_2 = a(00)$, $u_4 = b(00)$. We have:

1) $\lambda \geq 1$, $\mu \geq 1$, then

$$x'_- u_2 + x''_- u_4 = a(10) + b(01) = 0 \Longrightarrow a = b = 0.$$

2) $\lambda = 0$, $\mu \geq 1$, then

$$x'_- u_2 + x''_- u_4 = b(01) = 0 \Longrightarrow b = 0$$

and

$$h'_2 u_2 + h''_2 u_4 = la(00) \Longrightarrow l = 0.$$

Since

$$\bar{\lambda} = (00), \quad \bar{\mu}(m_1, m_2), \quad \bar{\nu}(m_1, m_2 - 1),$$

the corresponding operator is P_1.

3) $\lambda = \mu = 0$. We see that the condition $x'_- u_2 + x''_- u_4 = 0$ holds always. The other condition takes the form

$$h'_2 u_2 + h''_2 u_4 = la(00) + mb(00).$$

If l and m do not vanish simultaneously, then $a = mx$ and $b = -lx$. We have

$$\bar{\lambda} = (l, l), \quad \bar{\mu} = (m, m), \quad \bar{\nu} = (l+m, l+m-1),$$

hence, the corresponding operator is of type P_4.

If $l = m = 0$, then we have two operators of type P_1:

$$B(f, g) = afdg + bgdf.$$

Thus, we have verified that for $n = 2$ all the first order operators are listed.

7 Solutions of degree 2

The generic form of a degree 2 vector is

$$f = \partial'^2_1 u_1 + \partial'_1\partial'_2 u_2 + \partial'^2_2 u_3 + \partial'_1\partial''_1 u_4 + \partial'_1\partial''_2 u_5 +$$
$$\partial'_2\partial''_1 u_6 + \partial'_2\partial''_2 u_7 + \partial''_2\partial'_1 u_8 + \partial''_1\partial''_2 u_9 + \partial''^2_2 u_{10}.$$

We have

$$X_+ f = -2\partial'_1\partial'_2 u_1 + \partial'^2_1(X_+u_1) - \partial'^2_2 u_2 + \partial'_1\partial'_2(X_+u_2) + \partial'^2_2(X_+u_3) - \partial'_2\partial''_1 u_4 -$$
$$\partial'_1\partial''_2 u_4 + \partial'_1\partial''_1(X_+u_4) - \partial'_2\partial''_2 u_5 + \partial'_1\partial''_2(X_+u_5) - \partial'_2\partial''_2 u_6 + \partial'_2\partial''_1(X_+u_6) +$$
$$\partial'_2\partial''_2(X_+u_7) - 2\partial''_1\partial''_2 u_8 + \partial''^2_1(X_+u_8) - \partial''^2_2 u_9 + \partial''_1\partial''_2(X_+u_9) + \partial''^2_2(X_+u_{10}) = 0$$

which implies

$$\begin{array}{ccc} X_+u_1 = 0, & X_+u_4 = 0, & X_+u_8 = 0, \\ X_+u_2 - 2u_1 = 0, & X_+u_5 - u_4 = 0, & X_+u_9 - 2u_8 = 0, \\ X_+u_3 - u_2 = 0, & X_+u_6 - u_4 = 0, & X_+u_{10} - u_9 = 0, \\ & X_+u_7 - u_5 - u_6 = 0. & \end{array}$$

All these vectors can be expressed in terms of u_3, u_7, u_{10} and u_0:

$$\begin{array}{ccc} u_1 = \frac{1}{2}(X_+)^2 u_3, & u_4 = \frac{1}{2}(X_+)^2 u_7, & u_8 = \frac{1}{2}(X_+)^2 u_{10}, \\ u_2 = X_+u_3, & u_5 = \frac{1}{2}X_+u_7 - u_0, & u_9 = X_+u_{10}, \\ & u_6 = \frac{1}{2}X_+u_7 + u_0. & \end{array}$$

Moreover,

$$(X_+)^3 u_3 = (X_+)^3 u_7 = (X_+)^3 u_{10} = 0, \qquad X_+u_0 = 0.$$

The condition $(x_2^2\partial_1)f = 0$ implies

$$2\partial'_1(x'_- u_2) - 2\partial'_1 u_3 + 4\partial'_2(x'_- u_3) + 2\partial'_1(x''_- u_5) + 2\partial''_2(x'_- u_6) + 2\partial''_2(x''_- u_7) +$$
$$2\partial''_1(x''_1 u_9) - 2\partial''_1 u_{10} + 4\partial''_2(x''_- u_{10}) = 0$$

wherefrom

$$2x'_- u_3 + x''_- u_7 = 0 \tag{7}$$

$$x'_- u_7 + 2x''_- u_{10} = 0 \tag{8}$$

$$x'_- u_2 + x''_- u_5 - u_3 = 0 \tag{9}$$

$$x'_- u_6 + x''_- u_9 - u_{10} = 0 \tag{10}$$

Equations (7), (8) imply thanks to Lemma 4.2 that for $s \geq 3$ and $\mu \geq \lambda \geq 2s-2$

$$u_3 = u_7 = u_{10} = 0.$$

But then formula (9) implies that $x''_- u_0 = 0$ and (10) implies $x'_- u_0 = 0$. Hence,

$$u_0 = cv_\lambda \otimes w_\mu$$

but $X_+(v_\lambda \otimes w_\mu) \neq 0$ for λ and μ indicated; hence, $u_0 = 0$, i.e., $f = 0$.

There remain the cases $s = 0$, 1, 2.

Let $s = 2$. Then, $\mu \geq \lambda \geq 2s - d = 2$. The equation $x_2^2 \partial_2 f = 0$ implies

$$h'_2 u_2 + h''_2 \cdot u_5 = 0 \tag{11}$$

$$(2h'_2 - 1)u_3 + h''_2 u_7 = 0 \tag{12}$$

$$h'_2 u_6 + h''_2 u_9 = 0 \tag{13}$$

$$h'_2 u_7 + (2h''_2 - 1)u_{10} = 0 \tag{14}$$

The generic form of the elements u_3, u_7, u_{10}, u_0 is as follows:

$$\begin{array}{rr} u_3 = a \cdot (02) + b(11) + c(20), & u_{10} = x(02) + y(11) + z(20), \\ u_7 = 2(\alpha(02) + \beta(11) + \gamma(20), & u_0 = p(01) + q(10). \end{array}$$

Therefore,

$$\begin{array}{ll} u_2 & = ((\mu - 1)a + \lambda b)(01) + (\mu b + (\lambda - 1)c)(10), \\ u_5 & = ((\mu - 1)\alpha + \lambda\beta - p)(01) + (\mu\beta + (\lambda - 1)\gamma - q)(10), \\ u_6 & = ((\mu - 1)\alpha + \lambda\beta + p)(01) + (\mu\beta + (\lambda - 1)\gamma + q)(10), \\ u_9 & = ((\mu - 1)x + \lambda y)(01) + (\mu y + (\lambda - 1)z)(10). \end{array}$$

Let us substitute u_j in the system (7)–(8). We get

$$-a + 2(\mu - 1)\alpha + 2\lambda\beta - 2p = 0 \quad \text{for } \mu \geq 2 \tag{15}$$

$$(\mu - 1)a + (\lambda - 1)b + \mu\beta + (\lambda - 1)\gamma - q = 0 \quad \text{for } \lambda \geq 1,\ \mu \geq 1 \tag{16}$$

$$2\mu b + (2\lambda - 3)c = 0 \quad \text{for } \lambda \geq 2 \tag{17}$$

$$(2\mu - 3)x + 2\lambda y = 0 \quad \text{for } \mu \geq 2 \tag{18}$$

$$(\mu - 1)\alpha + \lambda\beta + p + (\mu - 1)y + (\lambda - 1)z = 0 \quad \text{for } \lambda \geq 1,\ \mu \geq 1 \tag{19}$$

$$2\mu\beta + 2(\lambda - 1)\gamma + 2q - z = 0 \quad \text{for } \lambda \geq 2 \tag{20}$$

One more equation is obtained from the condition $X_+ u_0 = 0$, namely,

$$\mu p + \lambda q = 0. \tag{21}$$

Substituting u_j into the system (9)–(10) we get

$$\begin{array}{ll} \alpha = 0, \quad x = 0 & \text{for } \mu \geq 3 \\ a + 2\beta = 0, \quad \alpha + 2y = 0 & \text{for } \lambda \geq 1,\ \mu \geq 2 \\ 2b + \gamma = 0, \quad 2\beta + z = 0 & \text{for } \lambda \geq 2,\ \mu \geq 1 \\ c = 0, \quad \gamma = 0 & \text{for } \lambda \geq 3 \end{array} \tag{22}$$

Consider the 3 cases:

1) $\lambda \geq 3$, $\mu \geq 3$. Then, (22) implies that $b = c = \alpha = \gamma = x = y = 0$ and $a = z = -2\beta$ and we have

$$\begin{array}{lcc} (15) & \Longrightarrow & \left.\begin{array}{c} 2\beta + 2\lambda\beta - 2p = 0 \\ \lambda\beta - 2(\lambda-1)\beta + p = 0 \end{array}\right\} \Longrightarrow \beta = p = 0, \\ (19) & \Longrightarrow & \\ (21) & \Longrightarrow & \mu p + \lambda q = 0 \quad \Longrightarrow q = 0. \end{array}$$

No solutions.

2) $\lambda = 2$, $\mu \geq 3$. Then, (22) implies $\alpha = x = y = 0$, $a = z = -2\beta$, $\gamma = -2b$ and we have

$$\begin{array}{lccl} (19) & \Longrightarrow & +2\beta + p - 2\beta = 0 & \Longrightarrow p = 0, \\ (21) & \Longrightarrow & \mu p + 2q = 0 & \Longrightarrow q = 0, \\ (15) & \Longrightarrow & 2\beta + 4\beta - 2p = 0 & \Longrightarrow \beta = 0, \\ (16) & \Longrightarrow & -2(\mu-1)\beta + b + \mu\beta - 2b - q = 0 & \Longrightarrow b = 0, \\ (17) & \Longrightarrow & 2\mu b + c = 0 & \Longrightarrow c = 0. \end{array}$$

No solutions again.

3) $\lambda = \mu = 2$. Then, (22) implies $a = z = -2\beta$, $\gamma = -2b$, $\alpha = -2y$, and we have

$$\begin{array}{lcc} (15) & \Longrightarrow & 2\beta - 4y + 4\beta - 2p = 0, \\ (16) & \Longrightarrow & -2\beta + b + 2\beta - 2b - q = 0, \\ (17) & \Longrightarrow & 4b + c = 0, \\ (18) & \Longrightarrow & x + 4y = 0, \\ (19) & \Longrightarrow & -2y + 2\beta + p + y - 2\beta = 0, \\ (20) & \Longrightarrow & 4\beta - 4b + 2q + 2\beta = 0, \\ (21) & \Longrightarrow & 2p + 2q = 0. \end{array}$$

The solution of this system is

$$\beta = b = p = y = -q, \qquad c = x = 4q, \qquad a = \alpha = \gamma = z = 2q.$$

But having substituted $u_3 = 2q(02) - q(11) + 4q(20)$, and $u_7 = 4q(02) - 2q(11) + 4q(20)$ into (12) we get

$$\begin{gathered} 2(2l-1)q(02) - (2l+1)q(11) + 4(2l+3)q(20) + \\ 4(m+2)q(02) + 2(m+1)q(11) + 4mq(20) = 0 \end{gathered}$$

implying $q = 0$.

Now let $s = 1$. Then,

$$\begin{array}{lll} u_3 = a(01) + b(10), & u_7 = 2(\alpha(01) + \beta(10)) & u_{10} = x(01) + y(10) \\ u_2 = (\mu a + \lambda b)(00) & u_5 = (\mu\alpha + \lambda\beta - p)(00) & u_9 = (\mu x + \lambda y)(00) \\ & u_6 = (\mu\alpha + \lambda\beta + p)(00) & \end{array}$$

Let us substitute this into (7)–(8). We get

$$-a + \mu\alpha + \lambda\beta - p = 0 \quad \text{for } \mu \geq 1 \tag{23}$$

$$\mu a + (\lambda - 1)b = 0 \quad \text{for } \lambda \geq 1 \tag{24}$$

$$(\mu - 1)x + \lambda y = 0 \quad \text{for } \mu \geq 1 \tag{25}$$

$$\mu\alpha + \lambda\beta + p - y = 0 \quad \text{for } \lambda \geq 1 \tag{26}$$

The system (9)–(10) yields

$$\begin{array}{lll} \alpha = 0, & x = 0 & \text{for } \mu \geq 2 \\ a + \beta = 0, & x + y = 0 & \text{for } \lambda \geq 1,\ \mu \geq 1 \\ b = 0, & \beta = 0 & \text{for } \lambda \geq 2 \end{array} \tag{27}$$

1) Let $\lambda \geq 2$, $\mu \geq 2$. Then, equation (27) implies $a = b = \alpha = \beta = x = y = 0$ and (23) implies $p = 0$. No solutions.

2) $\lambda = 1$, $\mu \geq 2$. From (27) it follows that $\alpha = x = y = 0$, $\beta = -a$ and equations (24) and (23) imply that (24) $\Longrightarrow a = 0$ and (23) $\Longrightarrow p = 0$, respectively.

Having substituted $u_3 = b(10)$, $u_2 = b(00)$, and $u_5 = u_7 = 0$ into (11) and (12) we get $(l + 1)b = (2l - 1)b = 0$. Hence, $b = 0$. No solutions.

3) $\lambda = 0$, $\mu \geq 2$. Equation (27) implies $\alpha = x = 0$. Moreover, $b = \beta = y = 0$ (since there is no vector v_1). From (23) we get $p = a$. Having substituted $u_3 = a(01)$, $u_7 = 0$, $u_6 = a(00)$, $u_9 = 0$ into (12), (13) we get $(2l - 1)a = 0$, $la = 0 \Longrightarrow a = 0$. No solutions.

4) $\lambda = \mu = 1$. Equation (27) implies $\beta = -a$, $\alpha = -y$. Having substituted this into (23)–(26) we get

$$\begin{array}{ll} (23) \Longrightarrow & -a - y - a - p = 0, \\ (24) \Longrightarrow & a = 0, \text{ hence } a = y = p = 0, \\ (25) \Longrightarrow & y = 0, \\ (26) \Longrightarrow & -y - a + p - y = 0. \end{array}$$

Having substituted $u_3 = b(10)$, $u_2 = b(00)$, $u_{10} = x(01)$, $u_9 = x(00)$, $u_5 = u_6 = u_7 = 0$ into (11)–(14) we get

$$\left.\begin{array}{ll} (11) & lb(00) = 0 \\ (12) & (2l + 1)b(10) = 0 \\ (13) & mx(00) = 0 \\ (14) & (2m + 1)x(01) = 0 \end{array}\right\} \Longrightarrow b = x = 0.$$

5) $\lambda = 0$, $\mu = 1$. Then, $b = \beta y = 0$ (since there is no vector v_1). The system (27) does not give anything new. From (23)–(26) we deduce that $p = \alpha - a$. Thus, $u_3 = a(01)$, $u_2 = a(00)$, $u_7 = 2\alpha(01)$, $u_5 = a(00)$, $u_6 = (2\alpha - a)(00)$, $u_{10}x(01)$, $u_9 = x(00)$. Having substituted this into (11)–(14) we get

$$\begin{array}{ll} (11) & la(00) + ma(00) = 0, \\ (12) & (2l - 1)a(01) + 2(m + 1)\alpha(01) = 0, \\ (13) & l(2\alpha - a)(00) + mx(00) = 0, \\ (14) & 2l\alpha(01) + (2m + 1)(01) = 0. \end{array}$$

a) Let $a \neq 0$. Then, from the first equation, i.e., (11), we get $-l = m$; hence,

$$\begin{gathered}(2l-1)a + 2(-l+1)\alpha = 0,\\ l(2\alpha - a - x) = 0,\\ 2l\alpha + (-2l+1)x = 0.\end{gathered}$$

From the first equation we get $a = (l-1)c$, $\alpha = (l-\frac{1}{2})c$. From the second equation (or from the third one if $l = 0$) we get $x = lc$.

Thus,

$$\bar\lambda = (l, l), \quad \bar\mu = (-l+1, l), \quad \bar\nu = (0, -1),$$

i.e., the corresponding operator is of the form S_2^{*1}

b) $a = 0$. Then,

$$(m+1)\alpha = 0, \quad 2l\alpha + mx = 0, \quad 2l\alpha + (2m+1)x = 0.$$

If $\alpha = 0$, then from the second and the third equations we derive $x = 0$, i.e., $f = 0$. Hence, $\alpha \neq 0$, $m = -1$, $x = 2l\alpha$.

Thus,

$$\bar\lambda = (l, l), \quad \bar\mu = (0, -1), \quad \bar\nu = (l-1, l-2)$$

and the corresponding operator is of type S_2.

6) $\lambda = \mu = 0$. Then, $u_3 = u_7 = u_{10} = 0$ but $u_0 \neq 0$. The equations (23)–(27) do not give anything. Let us substitute $u_5 = -p(00)$, $u_6 = p(00)$ into (11)–(13):

$$\left.\begin{array}{ll}(11) & -mp(00) = 0\\ (13) & lp(00) = 0\end{array}\right\} \Longrightarrow l = m = 0.$$

Thus, $\bar\lambda = \bar\mu = (0, 0)$, $\bar\nu = (-1, -1)$, and the corresponding operator $B(f, g) = df \wedge dg$ is of type S_1.

There remains the case $s = 0$. In this case, $u_3 = a(00)$, $u_7 = 2\alpha(00)$, $u_{10} = x(00)$ all the other u_j being zero. From (9), (10) we deduce that $u_3 = u_{10} = 0$. Hence, there remains only $u_7 = 2\alpha(00)$. From (12), (14) we see that $m\alpha = k\alpha = 0$, and therefore

$$\bar\lambda = \bar\mu = (0, 0), \quad \bar\nu = (0, -2).$$

The corresponding operator $B(f, g) = Z(df; dg)$ is of type S_1.

8 The solutions of degree 3

The generic form of a homogeneous element of degree 3 is

$$\begin{aligned}&f = \partial_1'^3 u_1 + \partial_1'^2\partial_2' u_2 + \partial_1'\partial_2'^2 u_3 + \partial_2'^3 u_4 + \partial_1'^2\partial_1'' u_5 +\\ &\partial_1'\partial_2'\partial_1'' u_6 + \partial_2'^2\partial_1'' u_7 + \partial_1'^2\partial_2'' u_8 + \partial_1'\partial_2'\partial_2'' u_9 + \partial_2'^2\partial_2'' u_{10} + \partial_1'\partial_1''^2 u_{11} +\\ &\partial_1'\partial_1''\partial_2'' u_{12} + \partial_1'\partial_2''^2 u_{13} + \partial_2'\partial_1''^2 u_{14} + \partial_2'\partial_1''\partial_2'' u_{15} + +\partial_2'\partial_2''^2 u_{16} + \partial_1''^3 u_{17} +\\ &\partial_1''^2\partial_2'' u_{18} + \partial_1''\partial_2''^2 u_{19} + \partial_2''^3 u_{20}.\end{aligned}$$

The equation $X_+ f = 0$ yields

$$\begin{array}{cccc}
X_+u_1 = 0 & X_+u_5 = 0 & X_+u_{11} = 0 & X_+u_{17} = 0 \\
X_+u_2 = 3u_1 & X_+u_6 = 2u_5 & X_+u_{12} = 2u_{11} & X_+u_{18} = 3u_{17} \\
X_+u_3 = 2u_2 & X_+u_7 = u_6 & X_+u_{13} = u_{12} & X_+u_{19} = 2u_{18} \\
X_+u_4 = u_3 & X_+u_8 = u_5 & X_+u_{14} = u_{13} & X_+u_{20} = u_{19} \\
 & X_+u_9 = u_6 + 2u_8 & X_+u_{15} = u_{12} + 2u_{14} & \\
 & X_+u_{10} = u_7 + u_9 & X_+u_{16} = u_{13} + u_{15} &
\end{array}$$

The solution of this system is

$$\begin{array}{cccc}
u_1 = \frac{1}{6}(X_+)^3 u_4 & u_5 = \frac{1}{6}(X_+)^3 u_{10} & u_{11} = \frac{1}{6}(X_+)^3 u_{16} & u_{17} = \frac{1}{6}(X_+)^2 u_{20} \\
u_2 = \frac{1}{2}(X_+)^2 u_4 & u_6 = \frac{1}{3}(X_+)^2 u_{10} - X_+u' & u_{12} = \frac{1}{3}(X_+)^2 u_{16} - X_-u'' & u_{18} = \frac{1}{2}(X_+)^2 u_{20} \\
u_3 = X_+u_4 & u_7 = \frac{1}{3}X_+u_{10} - u' & u_{13} = \frac{1}{3}X_+u_{16} - u'' & u_{19} = X_+u_4 \\
 & u_8 = \frac{1}{6}(X_+)^2 u_{10} + X_+u' & u_{14} = \frac{1}{6}(X_+)^2 u_{16} + X_+u'' & \\
 & u_9 = \frac{2}{3}X_+u_{10} + u' & u_{15} = \frac{2}{3}X_+u_{16} + u'' &
\end{array}$$

where

$$(X_+)^4 u_4 = (X_+)^4 u_{10} = (X_+)^4 u_{16} = (X_+)^4 u_{20} = 0,$$
$$(X_+)^2 u' = (X_+)^2 u'' = 0.$$

From the equation $(x_2^2\partial_1)f = 0$ we get

$$x'_- u_2 - u_3 + x''_- u_8 = 0 \tag{28}$$

$$2x'_- u_3 - 3u_4 + x''_- u_9 = 0 \tag{29}$$

$$3x'_- u_4 + x''_- u_{10} = 0 \tag{30}$$

$$x'_- u_6 - u_7 + x''_- u_{12} - u_{13} = 0 \tag{31}$$

$$2x'_- u_7 + x''_- u_{15} - u_{16} = 0 \tag{32}$$

$$x'_- u_9 - u_{10} + 2x''_- u_{13} = 0 \tag{33}$$

$$x'_- u_{10} + x''_- u_{16} = 0 \tag{34}$$

$$x'_- u_{14} + x''_- u_{18} - u_{19} = 0 \tag{35}$$

$$x'_- u_{15} + 2x''_- u_{19} - 3u_{20} = 0 \tag{36}$$

$$x'_- u_{16} + 3x''_- u_{20} = 0 \tag{37}$$

From equations (30), (34), (37) thanks to Lemma 4.2 it follows that

$$u_4 = u_{10} = u_{16} = u_{20} = 0 \quad \text{for} \quad s \geq 4,\ \lambda \leq \mu \leq \nu.$$

Thanks to the same lemma equation (32) implies $u' = u'' = 0$.

From the equation $(x_2^2\partial_2)f = 0$ we get

$$h_2'u_2 + h_2''u_8 = 0 \tag{38}$$

$$(2h_2' - 1)u_3 + h_2''u_9 = 0 \tag{39}$$

$$(3h_2' - 3)u_4 + h_2''u_{10} = 0 \tag{40}$$

$$h_2'u_6 + h_2''u_{12} = 0 \tag{41}$$

$$(2h_2' - 1)u_7 + h_2''u_{13} = 0 \tag{42}$$

$$h_2'u_9 + (2h_2'' - 1)u_{15} = 0 \tag{43}$$

$$(2h_2' - 1)u_{10} + (2h_2'' - 1)u_{16} = 0 \tag{44}$$

$$h_2'u_14 + h_2''u_{18} = 0 \tag{45}$$

$$h_2'u_{15} + (2h_2'' - 1)u_{19} = 0 \tag{46}$$

$$h_2'u_{16} + (3h_2'' - 3)u_{20} = 0 \tag{47}$$

Let $s = 3$. Denote

$$u_{j_i} = a_i(03) + b_i(12) + c_i(21) + d_i(30) \text{ for } (j_1 = 4, j_2 = 10, j_3 = 16, j_4 = 20).$$

From (30), (34), (37) we get

$$\begin{array}{c} a_2 = a_3 = a_4 = 0 \text{for } \mu \geq 4 \\ 3a_1 + 3b_2 = a_2 + 3b_3 = a_3 + 9b_4 = 0 \text{for } \lambda \geq 1,\ \mu \geq 3 \\ 6b_1 + 2c_2 = 2b_2 + 2c_3 = 2b_3 + 6c_4 = 0 \text{for } \lambda \geq 2,\ \mu \geq 2 \\ 9c_1 + d_2 = 3c_2 + d_3 = 3c_3 + 3d_4 = 0 \text{for } \lambda \geq 3,\ \mu \geq 1 \\ d_1 = d_2 = d_3 = 0 \text{for } \lambda \geq 4 \end{array} \tag{48}$$

One more system is obtained from (28), (29), where

$$\begin{array}{c} u' = p(02) + q(11) + r(20), \\ u_4 = a(03) + b(12) + c(21) + d(30); \\ u_{10} = 3(\alpha(03) + \beta(12) + \gamma(21) + \delta(30)) - (\mu - 2)a - \lambda b + \\ (\mu - 1)(\mu - 2)\alpha + 2\lambda(\mu - 1)\beta + \lambda(\lambda - 1)\gamma + 2(\mu - 1)p + 2\lambda q = 0 \ \text{ for } \mu \geq 2 \end{array} \tag{49}$$

$$\begin{array}{c} \frac{1}{2}(\mu - 1)(\mu - 2)a + (\lambda - 1)(\mu - 1)b + \frac{1}{2}(\lambda - 1)(\lambda - 2)c + \\ \frac{1}{2}\mu(\mu - 1)\beta + (\lambda - 1)\mu\gamma + \frac{1}{2}(\lambda - 1)(\lambda - 2)\delta + \mu q + (\lambda - 1)r = 0 \text{ for } \lambda \geq 1, \mu \geq 1 \end{array} \tag{50}$$

$$\mu(\mu - 1)b + (2\lambda - 3)\mu c + (\lambda - 2)^2 d = 0 \text{ for } \lambda \geq 2 \tag{51}$$

$$-3a + 6(\mu - 2)\alpha + 6\lambda\beta + 3p = 0 \text{ for } \mu \geq 3 \tag{52}$$

$$2(\mu - 2)a + (2\lambda - 3)b + 4(\mu - 1)\beta + 4(\lambda - 1)\gamma + 2q = 0 \text{ for } \mu \geq 2, \lambda \geq 1 \tag{53}$$

$$4(\mu - 1)b + (4\lambda - 7)c + 2\mu\gamma + 2(\lambda - 1)\delta + r = 0 \text{ for } \mu \geq 1, \lambda \geq 2 \tag{54}$$

$$6\mu c + (6\lambda - 15)d = 0 \text{ for } \lambda \geq 3 \tag{55}$$

And one more equation from the condition $(X_-)^2 u' = 0$:

$$\mu(\mu - 1)p + 2\lambda\mu q + \lambda(\lambda - 1)r = 0. \tag{56}$$

Let us consider the corresponding cases.

1) $\lambda \geq 4$, $\mu \geq 4$. From (48) we get $b_1 = c_1 = d_1 = a_2 = c_2 = d_2 = 0$, $b_2 = -a_1$, $u_4 = a(03)$, $u_{10} = -a(12)$. Having substituted this into the system (49)–(56) we get: $a = -3\beta$, $b = c = d = \alpha = \gamma = \delta = 0$ and

$$\begin{array}{lc}
(52) & 9\beta + 6\lambda\beta + 3p = 0 \\
(53) & -6(\mu - 2)\beta + 4(\mu - 1)\beta + 2q = 0 \\
(54) & r = 0 \\
(56) & \mu(\mu - 1)p + 2\lambda\mu q + \lambda(\lambda - 1)r = 0.
\end{array}$$

The determinant of the system is equal to $6\mu(6\lambda + 3\mu - 3)$. This determinant is $\neq 0$ for $\lambda \geq 4$, $\mu \geq 4$. Hence, $u_4 = u_{10} = u' = 0$. Similarly, $u_{16} = u_{20} = u'' = 0$. No solutions.

2) $\lambda = 3$, $\mu \geq 4$. Let us first consider the opposite case: $\lambda \geq 4$, $\mu = 3$. From (48) we deduce that $b_1 = c_1 = d_1 = c_2 = d_2 = d_3 = 0$, $a_1 = -b_3 = -3\beta$. Having substituted this into (49)–(56) we get

$$\begin{array}{lc}
(54) & r = 0 \\
(50) & -3\beta + 3\beta + 3q + (\lambda - 1)r = 0 \Longrightarrow q = 0 \\
(56) & 6p + 6\lambda q + \lambda(\lambda - 1)r = 0 \Longrightarrow p = 0 \\
(53) & -6\beta + 8\beta + 2q = 0 \Longrightarrow \beta = 0 \\
(52) & 9\beta + 6\alpha + 6\lambda\beta + 3p = 0 \Longrightarrow \alpha = 0.
\end{array}$$

Thus, $u_4 = u_{10} = u' = 0$.

Let us return to the case $\lambda = 3$, $\mu \geq 4$. By the just proved $u_{16} = u_{20} = u'' = 0$. From (48) we derive that $u_4 = c(21) + d(30)$, $u_{10} = -9c(30)$. Having substituted this into (49)–(56) we get

$$\begin{array}{lc}
(52) & p = 0 \\
(53) & q = 0 \\
(56) & \mu(\mu - 1)p + 6\mu q + 6r = 0 \Longrightarrow r = 0 \\
(54) & 5c - 12c + r = 0 \Longrightarrow c = 0 \\
(55) & 6\mu c + 3d = 0 \Longrightarrow d = 0
\end{array}$$

No nonzero solutions.

3) $\lambda = \mu = 3$. From system (48) we get $u_4 = a(03) + b(12) + c(21) + d(30)$ and

$u_{10} = 3(\alpha(03) - \frac{1}{3}a(12) - b(21) - 3c(30))$. Having substituted into (49)–(56) we get

$$
\begin{array}{ll}
(49) & -a - 3b + 2\alpha - 4a - 6b + 4p + 6q = 0 \\
(50) & a + 4b + c - a - 6b - 3c + 3q + 2r = 0 \\
(51) & 6b + 9c + d = 0 \\
(52) & -3a + 6\alpha - 6a + 3p = 0 \\
(53) & 2a + 3b - \frac{8}{3}a - 8b + 2q = 0 \\
(54) & 8b + 5c - 6b - 12c + r = 0 \\
(55) & 18c + 3d = 0 \\
(56) & 6p + 18q + 6r = 0
\end{array}
$$

The system is nondegenerate. No nonzero solutions.

The case $s = 3$ is exhausted since $\mu \geq \lambda \geq 2s - d = 3$.

Let now $s = 2$. In this case $\mu \geq \lambda \geq 1$. For the same j_i as for $s = 3$, define $u_{j_i} = a_i(02) + b_i(11) + c_i(20)$. Let us substitute this into equations (30), (34), (37). We get

$$
\begin{aligned}
a_2 = a_3 = a_4 = 0 \quad & \text{for } \mu \geq 3 \\
3a_1 + 2b_2 = a_2 + 2b_3 = a_3 + 6b_4 = 0 \quad & \text{for } \lambda \geq 1,\ \mu \geq 2 \\
6b_1 + c_2 = 2b_2 + c_3 = 2b_3 + 3c_4 = 0 \quad & \text{for } \lambda \geq 2,\ \mu \geq 1 \\
c_1 = c_2 = c_3 = 0 \quad & \text{for } \lambda \geq 3.
\end{aligned}
$$

Let us substitute

$$u_4 = a(02) + b(11) + c(20), \quad u_{10} = 3(\alpha(02) + \beta(11) + \gamma(20)), \quad u' = p(01) + q(10)$$

into equations (28), (29). We get

$$\frac{1}{2}\mu(\mu-1)a + (\lambda-1)\mu b + \frac{1}{2}(\lambda-1)(\lambda-2)c = 0 \text{ for } \lambda \geq 1 \tag{58}$$

$$-(\mu-1)a - \lambda b + \frac{1}{2}\mu(\mu-1)\alpha + \lambda\mu\beta + \frac{1}{2}\lambda(\lambda-1)\gamma + \mu p + \lambda q = 0 \text{ for } \mu \geq 1 \tag{59}$$

$$4\mu b + (4\lambda - 7)c = 0 \text{ for } \lambda \geq 2 \tag{60}$$

$$2(\mu-1)a + (2\lambda-3)b + 2\mu\beta + 2(\lambda-1)\gamma + q = 0 \text{ for } \lambda \geq 1,\ \mu \geq 1 \tag{61}$$

$$-3a + 4(\mu-1)\alpha + 4\lambda\beta + 2p = 0 \text{ for } \mu \geq 2. \tag{62}$$

Let us consider the corresponding cases.

1) $\lambda \geq 3$, $\mu \geq 3$. Then, (57) implies $u_4 = u_{10} = u_{16} = u_{20} = 0$. Therefore,

$$\left.\begin{array}{l} (61) \Longrightarrow q = 0 \\ (62) \Longrightarrow p = 0 \end{array}\right\} \Longrightarrow u' = 0.$$

Similarly, $u'' = 0$ no solutions.

2) $\lambda = 2$, $\mu \geq 3$. Again, let first $\lambda \geq 3$, $\mu = 2$. From (57) we deduce that $u_4 = 0$, $u_{10} = 3\alpha(02)$. Having substituted this into (58)–(62) we get

$$\left.\begin{array}{lc} (59) & \alpha + 2p + \lambda q = 0 \\ (61) & q = 0 \\ (62) & 4\alpha + 2p = 0 \end{array}\right\} \Longrightarrow u_4 = u_{10} = u' = 0.$$

Let us return to the case $\lambda = 2$, $\mu \geq 3$. In this case $u_{16} = u_{20} = u'' = 0$ and (57) implies that $u_4 = b(11) + c(20)$, $u_{10} = -6b(20)$. Having substituted this into (58)–(62) we get

$$\begin{array}{lc} (58) & \mu b = 0 \\ (59) & -2b - 2b + \mu p + 2q = 0 \\ (61) & 4\mu b + c = 0 \\ (62) & p = 0. \end{array}$$

No solutions.

3) $\lambda = \mu = 2$. From (57) it follows that

$$u_4 = a(02) + b(11) + c(20), \qquad u_{10} = 3(\alpha(02) - \frac{1}{2}a(11) - 2b(20)).$$

Having substituted this into the corresponding equations we get

$$\begin{array}{lc} (58) & a + 2b = 0 \\ (59) & -a - 2b + \alpha - 2a - 2b + 2p + 2q = 0 \\ (60) & 8b + c = 0 \\ (61) & 2a + b - 2a - 4b + q = 0 \\ (62) & -3a + 4\alpha - 4a + 2p = 0 \end{array}$$

The solution is

$$a = \alpha = 2x, \quad b = -x, \quad c = 8x, \quad p = 3x, \quad q = -3x;$$
$$u_4 = x(2(02) - (11) + 8(20)), \quad u_{10} = x(6(02) - 3(11) + 6(20)), \quad u' = 3x((01) - (10)).$$

Similarly,

$$u_{20} = y(8(02) - (11) + 2(02)), \quad u_{16} = 3y(2(02) - (11) + 2(20)), \quad u'' = 3y(-(01) + (10)).$$

Having substituted u_{10}, $u_9 = \frac{2}{3}X_+u_{10} + u' = u'$, $u_{13} = \frac{1}{3}X_+u_{16} - u'' = -u''$ into (33) we get

$$3x(11) - 6x(20) - 6x(02) + 3x(11) - 6x(20) + 12y(02) - 6y(11) = 0$$

implying $x = y = 0$.

4) $\lambda = 1$, $\mu \geq 3$. There is no vector v_2; hence, $c_1 = c_2 = c_3 = c_4 = 0$. From (57) we deduce $u_4 = a(02) + b(11)$, $u_{10} = -\frac{3}{2}a(11)$, $u_{16} = u_{20} = 0$. Having substituted this into (58)–(62) we get

$$\begin{array}{lc} (58) & \frac{1}{2}\mu(\mu - 1)a = 0 \\ (59) & -(\mu - 1)a - b - \frac{1}{2}\mu a + \mu p + q = 0 \\ (61) & 2(\mu - 1)a - b - \mu a + q = 0 \\ (62) & -3a - 2a + p = 0. \end{array}$$

The solution of this system is

$$a = p = 0, \quad q = b, \quad u_4 = q(11), \quad u_{10} = 0, \quad u' = q(10).$$

Having substituted u_4, u_{10}, $u_3 = X_+u_4 = q(01) + q(10)$, $u_9 = u'$ into (39), (40) we get

$$\left.\begin{array}{ll}(39) & (2l-1)q(01) + \mu q(2l+1)(10) + mq(10) = 0\\ (40) & 3lq(11) = 0\end{array}\right\} \Longrightarrow q = 0.$$

Moreover, from (31)–(32) it follows that $x''_-(X_+u'') = u''$, $x''_-(X_+u'') = 0$, i.e., $u'' = 0$.

5) $\lambda = 1$, $\mu = 2$. From (57) we get $u_4 = a(02) + b(11)$, $u_{10} = 3(\alpha(02) - \frac{1}{2}a(11))$ and (58)–(62) become

$$\begin{array}{ll}(58) & a = 0\\ (59) & -a - b + \alpha - a + 2p + q = 0\\ (61) & 2a - b - 2a + q = 0\\ (62) & -5a + 4\alpha + 2p = 0.\end{array}$$

The solution is

$$a = \alpha = p = 0, \qquad q = b.$$

To find u_{16}, u_{20}, u'', let us consider the case

$\lambda = 2$, $\mu = 1$. In this case we find u_4, u_{10}, u' with the help of equations (57)–(62).

From (57) we deduce that $u_4 = b(11) + c(20)$, $u_{10} = -6b(20)$ and having substituted this into (58)–(62) we get

$$\left.\begin{array}{ll}(58) & b = 0\\ (59) & -2b + p + 2q = 0\\ (60) & 4b + c = 0\\ (61) & b - 4b + q = 0\end{array}\right\} \Longrightarrow b = c = p = q = 0.$$

Thus, for $\lambda = 2$, $\mu = 1$ we have $u_4 = u_{10} = u' = 0$; hence, for $\lambda = 1$, $\mu = 2$ we have $u_{16} = u_{20} = u'' = 0$.

The solution $u_4 = b(11)$, $u' = b(10)$ does not satisfy equations (39), (40) as in the case $\lambda = 1$, $\mu \geq 3$.

6) $\lambda = \mu 1$. Then, $u_4 = b(11)$, $u_{10} = 3\beta(11)$ and we have

$$\begin{array}{ll}(58) & 0 = 0\\ (59) & -b + \beta + p + q = 0\\ (61) & -b + 2\beta + q = 0.\end{array}$$

The solution is $p = \beta$, $q = b - 2\beta$, $u_4 = b(11)$, $u_{10} = 3\beta(11)$, $u' = \beta(01) + (b - 2\beta)(10)$.

Similarly, $u_{20} = a(11)$, $u_{16} = 3\alpha(11)$, $u'' = (a - 2\alpha)(01) + \alpha(10)$.

Let us substitute u_4, u_{10}, u_{16}, u_{20} as well as

$$\begin{aligned}
u_3 &= X_+u_4 = b(01) + b(10),\\
u_2 &= \tfrac{1}{2}X_+u_3 = b(00),\\
u_3 &= \tfrac{2}{3}X_+u_{10} + u' = 3\beta(01) + b(10),\\
u_7 &= \tfrac{1}{3}X_+u_{10} - u' = (3\beta - b)(10),\\
u_8 &= \tfrac{1}{6}(X_+)^2u_{10} + X_+u' = b(00),\\
u_6 &= X_+u_7 = (3\beta - b)(00),\\
u_{19} &= a(01) + a(10),\\
u_{18} &= a(00),\\
u_{15} &= a(01) + 3\alpha(00),\\
u_{14} &= a(00),\\
u_{13} &= (3\alpha - a)(01),\\
u_{12} &= (3\alpha - a)(00)
\end{aligned}$$

into the system (38)–(47). We get

$$\begin{aligned}
&(38)\quad lb(00) + mb(00) = 0,\\
&(39)\quad (2l-1)b(01) + (2l+1)b(10) + 3(m+1)\beta(01) + mb(10) = 0,\\
&(40)\quad 3lb(11) + 3(m+1)\beta(11) = 0,\\
&(41)\quad l(3\beta - b)(00) + m(3\alpha - a)(00),\\
&(42)\quad (2l+1)(3\beta - b)(10) + (m+1)a(01) + 3m\alpha(10) = 0,\\
&(43)\quad 3l\beta(01) + (l+1)b(10) + (2m+1)(3\alpha - a)(01) = 0,\\
&(44)\quad 3(2l+1)\beta(11) + 3(2m+1)\alpha(11) = 0,\\
&(45)\quad la(00) + ma(00) = 0,\\
&(46)\quad la(01) + 3(l+1)\alpha(10) + (2m+1)a(01) + (2m-1)a(10) = 0,\\
&(47)\quad 4(l+1)\alpha(11) + 3ma(11) \neq 0.
\end{aligned}$$

Consider the two cases:

a) $l + m \neq 0$. Then, (38) and (45) imply that $a = b = 0$.

$$\begin{aligned}
&(m+1)\beta = 0,\\
&l\beta + m\alpha = 0,\\
&(2l+1)\beta + m\alpha = 0,\\
&l\beta + (2m+1)\alpha = 0,\\
&(2l+1)\beta + (2m+1)\alpha = 0,\\
&(l+1)\alpha = 0,
\end{aligned}$$

where either $\beta \neq 0$ or $\alpha \neq 0$ (i.e., $f \neq 0$).

Let, for example, $\beta \neq 0$. Then, $m = -1$ and $\alpha = l\beta = (2l+1)\beta$ implying $l = -1$.

Thus, $\bar{\lambda} = \bar{\mu} = (0, -1)$ and $\bar{\nu} = (-2, -3)$ and the operator found is T_1.

b) $m = -l$. Then,

$$\begin{aligned}
(2l-1)b + 3(-l+1)\beta = &\ 0 & 3l\beta + (-2l+1)(3\alpha - a) = &\ 0\\
(2l+1)b - lb = &\ 0 & (l+1)b = &\ 0\\
lb + (-l+1)\beta = &\ 0 & (2l+1)\beta + (-2l+1)\alpha = &\ 0\\
l(3\beta - b - 3\alpha + a = &\ 0 & la + (-2l+1)a = &\ 0\\
(-l+1)a = &\ 0 & 3(l+1)\alpha + (-2l+1)a = &\ 0\\
(2l+1)(3\beta - b) - 3l\alpha = &\ 0 & (l+1)\alpha - la = &\ 0
\end{aligned}$$

If $l \neq \pm 1$, then $a = b = 0$. This case is already considered in heading a).

There remain cases $l = -1$, $m = 1$ and $l = 1$, $m = -1$. Since these cases are equivalent, let us consider only the first one.

We get

$$\begin{array}{cc}
-3b + 6\beta = 0 & -3\beta + 9\alpha - 3a = 0\\
0 = 0 & 0 = 0\\
-b + 2\beta = 0 & -\beta + 3\alpha = 0\\
3\beta - b - 3\alpha + a = 0 & 2a = 0\\
-2a = 0 & 3a = 0\\
-3\beta + b + 3\alpha = 0 & a = 0.
\end{array}$$

The solution of the system is $a = 0$, $b = 2\beta$, $\alpha = 4\beta$.

Thus, $\bar{\lambda} = (0, -1)$, $\bar{\mu} = (2, 1)$, $\bar{\nu} = (0, -1)$ and the corresponding operator is T_1^{*1}.

Let now $s = 1$. We get

$$\begin{array}{cc}
u_4 = a(01) + b(10) & u_{10} = 3(\alpha(01) + \beta(10))\\
u_3 = (\mu a + \lambda b)(00) & u_9 = (2\mu\alpha + 2\lambda\beta + p)(00)\\
u_2 = u_8 = 0, & u' = p(00).
\end{array} \tag{$*$}$$

Having substituted $(*)$ into the system (28)–(29) we get

$$-\mu a - \lambda b = 0 \tag{63}$$

$$-a + 2\mu\alpha + 2\lambda\beta + p = 0 \text{ for } \mu \geq 1 \tag{64}$$

$$2\mu a + (2\lambda - 3)b = 0 \text{ for } \lambda \geq 1 \tag{65}$$

and having substituted $u_{j_i} = a_i(01) + b_i(10)$ into the system (30), (34), (37) we get

$$\begin{array}{ll} a_2 = a_3 = a_4 = 0 & \text{for } \mu \geq 2 \\ 3a_1 + b_2 = a_2 + b_3 = a_3 + 3b_4 = 0 & \text{for } \lambda \geq 1,\ \mu \geq 1 \\ b_1 = b_2 = b_3 = 0 & \text{for } \lambda \geq 2. \end{array} \tag{66}$$

1) $\lambda \geq 2$, $\mu \geq 2$. From (66) we see that $u_4 = u_{10} = u_{16} = u_{20} = 0$ and from (64) we get $u' = 0$. Similarly, $u'' = 0$.

2) $\lambda = 1$ $\mu \geq 2$. From (66) we get $u_{16} = u_{20} = 0$, $u_4 = a(01) + b(10)$, $u_{10} = -3a(10)$. This gives

$$\left.\begin{array}{ll} (63) & -\mu a - b = 0 \\ (64) & -a - 2a + p = 0 \\ (65) & 2\mu a - b = 0 \end{array}\right\} \Longrightarrow a = b = p = 0.$$

Moreover, (31) implies $u_{13} = 0$, hence, $u'' = 0$.

3) $\lambda = \mu = 1$. From (66) we get $u_4 = a(01) + b(10)$, $u_{10} = 3(\alpha(01) - a(10))$. This gives

$$\left.\begin{array}{ll} (63) & -a - b = 0 \\ (64) & -a + 2\alpha - 2a + p = 0 \\ (65) & 2a - b = 0 \end{array}\right\} a = b = 0, p = -2\alpha.$$

The solution: $u_{10} = 3\alpha(01)$, $u_7 = 3\alpha(00)$. Similarly, $u_{16} = 3\beta(10)$, $u_{13} = 3\beta(00)$. And the other u_j vanish. Having substituted them into (31)-(32) we get

$$\left.\begin{array}{ll} (31) & -3\alpha(00) - 3\beta(00) = 0 \\ (32) & 6\alpha(10) - 3\beta(10) = a \end{array}\right\} \Longrightarrow \alpha = \beta = 0.$$

4) $\lambda = 0$, $\mu \geq 2$. From (66) we get $u_4 = a(01)$, $u_{10} = u_{16} = u_{20} = 0$. This gives

$$\left.\begin{array}{ll} (63) & -\mu a = 0 \\ (64) & -a + p = 0 \end{array}\right\} \Longrightarrow u_4 = u' = 0.$$

Moreover, equation (31) yields $u'' = 0$.

5) $\lambda = 0$, $\mu = 1$. From (66) we get $u_4 = a(01)$, $u_{10} = 3b(01)$, $u_{16} = 3c(01)$, $u_{20} = d(01)$. Having substituted u_4, u_{10}, u_{16}, u_2 as well as

$$\begin{array}{rc} u_3 = & X_+u_4 = a(00), \\ u_9 = & \frac{2}{3}X_+u_{10} + u' = (2b + p)(00), \\ u_7 = & (b - p)(00), \\ u_{19} = & d(00), \\ u_{15} = & (2c + q)(00), \\ u_{13} = & (c - q)(00) \end{array}$$

directly into (28)–(37) we get

$$\begin{array}{ll}
(28) & u_3 = 0 \Longrightarrow a = 0 \\
(35) & u_{19} = 0 \Longrightarrow d = 0 \\
(29) & x''_- u_9 \Longrightarrow p = -2b, u_9 = 0, u_7 = 3b(00) \\
(31) & u_7 + u_{13} = 0 \Longrightarrow u_{13} = -3b(00), c - q = -3b \\
(32) & x''_- u_{15} - u_{16} = 0 \Longrightarrow 2c + q - 3c = 0 \Longrightarrow q - c = 0 \Longrightarrow b = 0.
\end{array}$$

The solution is $u_{16} = 3c(01)$, $u_{15} = 3c(01)$. Having substituted it into (42)–(44) we get

$$\left.\begin{array}{ll}
(42) & m \cdot 3c(00) = 0 \\
(44) & (2m+1) \cdot 3c(01) = 0
\end{array}\right\} \Longrightarrow f = 0.$$

There are no nonzero solutions.

6) $\lambda = \mu = 0$. From (66) we get $u_4 = u_{10} = u_{16} = u_{20} = 0$, $u' = u_9 = -u_7 = p(00)$, $u'' = u_{15} = -u_{13} = q(00)$. From (31) we deduce that $u_7 + u_{13} = 0 \Longrightarrow q = -p$. Having substituted this into (39), (42), (46) we get

$$\begin{array}{ll}
(39) & lp = 0 \\
(42) & (2l-1)p - mp(00) = 0 \\
(46) & -mp = 0.
\end{array}$$

No nonzero solutions.

There remains the case $s = 0$. In this case $u_4 = a(00)$, $u_{10} = b(00)$, $u_{16} = c(00)$, $u_{20} = d(00)$; so equation (29), (32), (33) and (35) imply $u_4 = 0$, $u_{16} = 0$, $u_{10} = 0$, $u_{20} = 0$, respectively. No nonzero solutions. We have considered all the cases.

9 The general case ($n > 2$)

Recall that V and W are $\mathfrak{gl}(n)$-modules;

$$\begin{array}{l}
I(V^*) = \mathbb{K}[\partial_1, \dots, \partial_n] \otimes V^* \\
I(V^*, W^*) = \mathbb{K}[\partial'_1, \dots, \partial'_n] \otimes V^* \otimes \mathbb{K}[\partial''_1, \dots, \partial''_n] \otimes W^*
\end{array}$$

(the tensor product of the vector spaces, but not modules). Let $e_1, e_2, \dots, e_n$ be a basis in $\mathbb{K}^n$. Let $E = Span(e_{i_1}, e_{i_2}, \dots, e_{i_j}) \subset \mathbb{K}^n$. Denote by $\mathfrak{gl}(E) \subset \mathfrak{gl}(n)$ the Lie algebra of the operators that preserves $e_i \notin E$.

Let $\mathcal{L}_E = \mathbb{K}[\partial_{i_1}, \dots, \partial_{i_j}]$ be the subalgebra of $\mathcal{L}$. Set

$$I_E(V^*) = \mathbb{K}[\partial_{i_1}, \dots, \partial_{i_j}] \otimes V^*, \quad I_E(V^*, W^*) = I_E(V^*) \otimes I_E(W^*).$$

As $\mathfrak{gl}(E)$-modules, V^* and W^* split into the direct sum of irreducible submodules:

$$V^* = \oplus_\alpha V^*_\alpha, \quad W^* = \oplus_\beta W^*_\beta.$$

Hence, $V^* \otimes W^* = \oplus_{\alpha,\beta} V_\alpha \otimes V_\beta$. This implies the decomposition

$$I_E(V^*, W^*) = \oplus_{\alpha,\beta} I(V^*_\alpha \otimes I(W^*_\beta).$$

Denote by Π_E or $\Pi_{i,i_2\dots,i_j}$ the natural projection of $I(V^*, W^*)$ onto $I_E(V^*, W^*)$: we replace ∂'_i and ∂''_i for $i \notin (i_1, \dots, i_j)$ with zeros.

This projection commutes with the $\mathcal{L}_E$-action, and therefore it sends $\mathfrak{gl}(n)$-highest (hence, $\mathfrak{gl}(E)$-highest) singular (with respect to $\mathcal{L}$; hence, with respect to $\mathcal{L}_E$) vectors into the highest singular vectors.

By a natural basis in $I(V^*, W^*)$ we will mean a basis consisting of the elements

$$P(\partial'_1, \dots, \partial''_n) v \otimes w,$$

where P is a monomial and v, w are elements of the Gelfand–Tsetlin bases of V^* and W^*, respectively (we will also denote such elements by $P_1(\partial) v \otimes P_2(\partial) w$).

We will say that f contains a component $P(\partial'_1, \dots, \partial''_n) v \otimes w$ if the corresponding coordinate does not vanish in the natural basis. The *type of the component* in this case is the monomial $P(\partial'_1, \dots, \partial''_n)$.

Sometimes we will represent f in the form

$$f = \sum P_i(\partial'_1, \dots, \partial''_n) u_i,$$

where $u_i \in V^* \otimes W^*$ and $P_i(\partial'_1, \dots, \partial''_n) u_i$ is the sum of all the components of type P_i.

Recall that the weights of ∂'_i and ∂''_i are equal to $(0, \dots, 0, -1, 0, \dots, 0)$ with -1 on the i-th place. Among all the monomials that enter the decomposition of f, select the monomial $P_0(\partial'_1, \dots, \partial''_n)$ with the least (lexicographically) weight.

Let $f = P_0 u_0 + \sum P_i u_i$. Since f is a highest weight vector, then

$$(x_\alpha \partial_\beta) f = 0 \text{ for } \alpha < \beta$$

but

$$(x_\alpha \partial_\beta) f = P_0 \cdot (x_\alpha \partial_\beta) u_0 \quad \text{plus components of another type.}$$

Hence, $(x_\alpha \partial_\beta) u_0 = 0$, i.e., u_0 is a highest weight vector. Then $u_0 = v_0 \otimes w' + \dots + v' \otimes w_0$, where v_0, w_0 are the highest weight vectors of V^* and W^*, respectively. Thus we have proved the following lemma.

5.1 Lemma (On the highest component). *Let $f \in I(V_1^*, V_2^*)$ be a highest weight vector (not necessarily singular). Let $P_0(\partial_1, \dots, \partial_n)$ be one of the monomials with lexicographically lowest weight among the monomials that enter the decomposition of f. Then, f contains the components $P_0 v_0 \otimes w'$ and $P_0 v' \otimes w_0$, where v_0 and w_0 are the highest weight vectors of V^* and W^*, respectively.*

The component $P_0 v_0 \otimes w'$ will be called V-highest (or just the highest) while $P_0 v' \otimes w_0$ will be called W-highest one.

10 Second order operators

First, recall the list of the highest singular vectors or $n = 1$, 2 found earlier.

$n = 1$.

a) $\lambda = \mu = (0)$, $\nu = (-2)$.

$$f = \partial v \otimes w_0 \partial w.$$

b) $\lambda = (0)$, $\mu = (1)$, $\nu = (-1)$.

$$f = \partial^2 v \otimes w + \partial v \otimes \partial w.$$

b′) $\lambda = (1)$, $\mu = (0)$, $\nu = (-1)$.

$$f = \partial v \otimes \partial w + v \otimes \partial^2 w.$$

c) $\lambda = (0)$, $\mu = (m)$, $\nu = (m-2)$; $(m \neq 0, 1)$.

$$f = m\partial^2 v \otimes w + \partial v \otimes \partial w.$$

c′) $\lambda = (l)$, $\mu = (0)$, $\nu = (l-1)$.

$$f = \partial v \otimes \partial w + lv \otimes \partial^2 w.$$

d) $\lambda = (l)$, $\mu = (-l+1)$, $\nu = (-1)$; $(l \neq 0, -1)$

$$f = (l-1)\partial^2 v \otimes w + (2l-1)\partial v \otimes \partial w + lv \otimes \partial^2 w.$$

$n = 2$.

n	highest weights λ, μ and ν of V^*, W^* and f	type of $\Pi_1 f$	type of $\Pi_2 f$
1)	$\lambda = \mu = (0,0)$, $\nu = (-1,-1)$ $f = \partial_1 v_0 \otimes \partial_2 w_0 - \partial_2 v_0 \otimes \partial_1 w_0$	-	-
2)	$\lambda = \mu = (0,0)$, $\nu = (0,-2)$ $f = \partial_2 v_0 \otimes \partial_2 w_0$	-	a)
3)	$\lambda = (0,0)$, $\mu = (1,-1)$, $\nu = (-1,-1)$ $2\partial_1^2 v_0 \otimes w + 2\partial_1 v_0 \otimes \partial_1 w_0 + 2\partial_1\partial_2 v_0 \otimes w + \partial_1 v_0 \otimes \partial_2 w_1 +$ $\partial_2 v_0 \otimes \partial_1 w_1 + \partial_2^2 v_0 \otimes w_2 + \partial_2 v_0 \otimes \partial_2 w_2$	b)	b)
4)	$\lambda = (0,-1)$, $\mu = (m,m)$, $\nu = (m-1, m-2)$ $f = m\partial_1\partial_2 v_0 \otimes w_0 + \partial_1 v_0 \otimes \partial_2 w_0 +$ $m\partial_2^2 v_1 \otimes w_0 + \partial_2 v_1 \otimes \partial_2 w_0$	-	c) b) for m=1 a) for m=0
4′)	$\lambda' = (l,l)$, $\mu = (0,-1)$, $\nu = (l-1, l-2)$ $f = \partial_2 v_0 \otimes \partial_1 w_0 + lv_0 \otimes \partial_1\partial_2 w_0 +$ $\partial_2 v_0 \otimes \partial_2 w_1 + lv_0 \otimes \partial_2^2 w_1$	-	c) b') for l=1 a) for l=0
5)	$\lambda = (0,-1)$, $\mu = (m+1, m)$, $\nu = (m-1, m-1)$ $f = (m+1)\partial_1^2 v_0 \otimes w_0 + \partial_1 v_0 \otimes \partial_1 w_0 + (m+1)\partial_1\partial_2 v_0 \otimes w_1 +$ $(m+1)\partial_1\partial_2 v_1 \otimes w_0 + \partial_1 v_0 \otimes \partial_2 w_1 +$ $\partial_2 v_1 \otimes \partial_1 w_0 + (m+1)\partial_2^2 v_1 \otimes w_1 + \partial_2 v_1 \otimes \partial_2 w_1$	c) b) for m=0 a) for m=-1	c) b) a)
5′)	is similar to 5)		
6)	$\lambda = (l,l)$, $\mu = (-l+1, -l)$, $\nu = (0,-1)$ $f = (l-1)\partial_1\partial_2 v_0 \otimes w_0 + (l-1)\partial_1 v_0 \otimes \partial_2 w_0 + l\partial_2 v_0 \otimes \partial_1 w_0 +$ $lv_0 \otimes \partial_1\partial_2 w_0 + (l-1)\partial_2^2 v_0 \otimes w_1 + (2l+1)\partial_2 v_0 \otimes \partial_2 w_1 + lv_0 \otimes \partial_2^2 w_1$	-	d) b) for l=0 b') for l=1
6′)	is similar to 6)		

A) First, consider the case when $\Pi_i f = 0$ for any i. For $n = 2$ this happens in case 1). The singular vector is of the form

$$f = \partial_1 v_0 \otimes \partial_2 w_0$$

(here both v_0 and w_0 are highest weight vectors of weight $\lambda = \mu = (0, 0)$ and the weight of f is equal to $\nu = (-1, -1)$).

In the general case the highest component is of the form $\partial i_0 v_0 \otimes \partial_{j_0} w$ ($i_0 < j_0$). Hence, $\Pi_{i_0 j_0} f \neq 0$ and is equal to the sum of several vectors of the form 1), i.e.,

$$\Pi_{i_0 j_0} f = \sum_{\alpha,\beta} a_{\alpha\beta}(\partial i_0 v_\alpha \otimes \partial_{j_0} w_\beta - \partial j_0 v_\alpha \otimes \partial_{i_0} w_\beta),$$

where $v_\alpha \in V^*$, $w_\beta \in W^*$ are of weight $(\dots, 0_{i_0}, \dots, 0_{j_0}, \dots)$ with zeros on the i-th and j-th places.

Let $n = 3$ for the moment. The following three cases are possible:

1) $i_0 = 1$, $j_0 = 2$. In this case the weight of v_α is equal to $(0, 0, l)$ and the weight of w_β is equal to $(0, 0, m)$, where l and m do not depend on α and β because the sum of the coordinates of the weights are equal for all weight vectors.

In particular, the weight of v_0 is equal to $(0, 0, l)$ and the weight of w_0 is equal to $(0, 0, m)$. The weight of f is equal to $(-1, -1, l + m)$.

Since v_0, w_0 and f are highest weight vectors, it follows that

$$l \leq 0, \qquad m \leq 0, \qquad l + m \leq -1.$$

Since the multiplicity of the highest weight is equal to 1, it follows that α and β assume only one value, i.e.,

$$\Pi_{12} f = a(\partial_1 v_0 \otimes \partial_2 w_0 - \partial_2 v_0 \otimes \partial_1 w_0.$$

Observe that $\Pi_{12} f$ is not highest with respect to $\mathfrak{gl}(3)$ because

$$(x_1 \partial_3)(\Pi_{12} f) = -a(\partial_3 v_0 \otimes \partial_2 w_0 - \partial_2 v_0 \otimes \partial_3 w_0) \neq 0,$$

hence, $\Pi_{12} f \neq f$, i.e., either $\Pi_{13} f \neq 0$ or $\Pi_{23} f \neq 0$.

But $\Pi_{13} f$ and $\Pi_{23} f$ can be only of the form 1) (or the sum of several vectors of the form 1)), and therefore $\nu_3 = -1$, i.e., $l + m = -1$.

But $l \leq 0$, $m \leq 0$, hence, two cases are possible: $l = 0$ or $m = -1$. In both cases the operators exist:

$$S_1(\varphi, w) = d\varphi \wedge dw, \quad S_1(w, \varphi) = dw \wedge d\varphi.$$

Proof of uniqueness of the highest singular vector in each case. Let

$$f = a(\partial_1 v_0 \otimes \partial_2 w_2 - \partial_2 v_0 \otimes \partial_1 w_0) + \dots$$

and

$$g = b(\partial_1 v_0 \otimes \partial_2 w_0 - \partial_2 v_0 \otimes \partial_1 w_0) + \dots$$

be highest singular vectors. Then, $bf-ag$ is a highest singular vector but $\Pi_{12}(bf-ag)=0$.

Further in headings 2) and 3) we will see that this case corresponds to other weights of V^* and W^*. Hence, $bf-ag=0$, i.e., f and g are proportional. Almost in all the cases the uniqueness is also proved by this method.

Therefore, in what follows we will replace the proof with words "the uniqueness is proved routinely". To apply the routine method, it suffices to demonstrate that the highest component $P_0v_0\otimes w$ (or the w-highest component) is uniquely determined. In particular, the weight of w should be of multiplicity 1.

2) $i_0=1$, $j_0=3$. In this case the weight of v_α is equal to $(0,l,0)$ and the weight of w_β is equal to $(0,m,0)$; the weight of f is equal to $(-1,l+m,-1)$. Since the weight of f is a highest one, then $l+m=-1$ and this means that either the weight of v_α or the weight of w_β is not highest contradicting to Lemma 9.1. Hence, there are no singular vectors.

3) $i_0=2$, $j_0=3$. The weight of v_α is equal to $(l,0,0)$, the weight of w_β is equal to $(m,0,0)$ and the weight of f is equal to $(l+m,0,0)$. Lemma 9.1 implies that the weights $(l,0,0)$ and $(m,0,0)$ are highest ones, hence, of multiplicity 1, and therefore

$$\Pi_{23}f=a(\partial_2v_0\otimes\partial_3w_0-\partial_3v_0\otimes\partial_2w_0).$$

Since $\Pi_{13}f=\Pi_{12}f=0$, it follows that

$$f=\Pi_{23}f=a(\partial_2v_0\otimes\partial_3w_0-\partial_3v_0\otimes\partial_2w_0).$$

The condition $(x_3^2\partial_1)f=0$ yields

$$0=(x_3^2\partial_1)f=-2a(\partial_2v_0\otimes\partial_1((x_3\partial_1)w_0)-\partial_1((x_3\partial_1)v_0)\otimes\partial_3w_0)$$

implying $(x_3\partial_1)v_0=(x_3\partial_1)w_0=0$. This is true only for $l=m=0$. The corresponding operator exists. It is

$$S_1(\varphi,\psi)=d\varphi\wedge d\psi.$$

The case $n=3$ is considered completely.

Let us pass to the general case. First, let us prove that $j_0=i_0+1$. Indeed, otherwise $\Pi_{i_0,i_0+1,j_0}f$ should be of type 2) but there are no such vectors.

Let us show that the weight of f is equal to $\nu=(0,\dots,0,-1,-1,\dots,-1)$ (with (i_0-1)-many 0's); the weight of V^* is $\lambda=(0,\dots0,\underbrace{-1,\dots,-1}_{l})$; the weight of W^* is $\mu=(0,\dots,0,\underbrace{-1,\dots,-1}_{m})$.

Indeed, for $i<i_0$ the vector $\Pi_{ii_0j_0}f$ is the sum of several components of the form (3), hence, $\nu_i=0$.

Moreover, $\Pi_{ii_0j_0}f$ contains the highest component $l_{i_0}v_0\otimes l_{j_0}w'$ implying $\lambda_i=0$. For $i>j_0$ the vector $\Pi_{i_0j_0i}f$ is of the form 1), hence, $\nu_i=-1$. But $\Pi_{i_0j_0i}f$ again contains the highest component implying either $\lambda_i=0$ or $\lambda_i=-1$.

Similarly, the weight of W^* is equal to $(0,\dots,0,-1,\dots,-1)$. From the balance of the sum of weight coordinates it follows that $l+m+2=n-i_0+1$. Moreover, $l\leq n-1$, $m\leq n-1$ $\lambda_{i_0}=\mu_{i_0}=0$ occurs.

In all these cases the invariant operators exist. These operators are of the form

$$S_1(w_1, w_2) = dw_1 \wedge dw_2.$$

The uniqueness is proved routinely, since the highest component $l_{i_0}v_0 \otimes l_{j_0}w$ is uniquely determined (namely, $i_0 = n - l - m - 1$, $j_0 = n - l - m$, and the weight of w is equal to $(0, \dots, 0, \underbrace{-1, \dots, -1}_{m}, \underbrace{0, \dots, 0}_{l})$ and W^* has no multiple weights). Thus, case A) is considered completely.

B) Let now there exist an i such that $\Pi_i f \neq 0$. Let i_1 be the least such index. Let $\Pi_{i_1} f = \sum_\alpha f_\alpha$ be the sum of several vectors of types a)–d).

1) Let at least one of the summands be of type a). Then, $\nu_{i_1} = -2$, and therefore all of them are of type a). If $i > i_1$, then $\Pi_{i_1 i} f$ should be of type 5), that is $(m = -1)$, implying $\nu_i = -2$. For $i < i_1$ we see that $\Pi_{ii_1} f$ is of type 4), i.e. $(m = 0)$, or of type 2) implying either $\nu_i = 0$ or $\nu_i = 1$.

Thus, the weight of f is equal to $\nu = (0, \dots, 0, \underbrace{-1, \dots, -1}_{p}, \underbrace{-2, \dots, -2}_{q})$, where $q \geq 1$.

Now, let us show that if $\nu = (0, \dots, 0, -1, \dots, -1, -2, \dots, -2)$, then λ and μ are of the form $(0, \dots, 0, -1, \dots, -1)$ with l (resp. m) -1's.

First, let $n = 3$. The following cases are possible.

1) $\nu = (0, 0, 2)$. Observe that $\Pi_{12} f = 0$ because in the list of singular vectors for $n = 2$ there is no vector g such that $\nu_1 = \nu_2 = 0$ and $\Pi_1 g = \Pi_2 g = 0$. Moreover, $\Pi_{13} f = \Pi_3 f$ and $\Pi_{23} f = \Pi_3 f$ because they are of type 2).

Therefore, the V-highest and W-highest components are of the form $\partial_3 v \otimes \partial_3 w$, where the weights of v and w are equal to $(0, 0, 0)$ because $\Pi_{13} f$ and $\Pi_{23} f$ are of type 2).

2) $\nu = (0, -1, -2)$. Again $\Pi_{12} f = 0$, $\Pi_{13} f$ is of type 2), $\Pi_{23} f$ of type either 4) or 4′) $(m = 0)$. The highest component is of the form

$$\partial_2 v_0 \otimes \partial_3 w \text{ or } \partial_3 v_0 \otimes \partial_2 w.$$

In the first case $\Pi_{13} f = \partial_3 v \otimes \partial_3 w$ is of the form 2) and

$$\Pi_{23} f = \partial_2 v_0 \otimes \partial_3 w_0 + \partial_3 v \otimes \partial_3 w_0$$

is of type 4, hence, the weight of v is equal to $(0, -1, 0)$ and the weight of v_0 is equal to $(0, 0, -1)$; in the second case

$$\Pi_{23} f = \partial_3 v_0 \otimes \partial_2 w_0 + \partial_3 v_0 \otimes \partial_3 w$$

is of type 4′) and the weight of v_0 is equal to $(0, 0, 0)$.

Similarly, the weight of w_0 in the first case is equal to $(0, 0, 0)$ and in the second one is equal to $(0, 0, -1)$.

3) $\nu = (-1, -1, -2)$. The vectors $\Pi_{13} f$ and $\Pi_{23} f$ are of type 4) or 4′). There are three possibilities:

3.1) $\Pi_{13}f$ and $\Pi_{23}f$ are of type 4). Then, $\Pi_3 f$ consists of the components of the form $\partial_3 v\otimes\partial_3 w$ where the weight of v is equal to $(-1,-1,0)$ and the weight of w is equal to $(0,0,0)$. The image under Π_{13} consists of components

$$\partial_1 v_0\otimes\partial_3 w+\partial_3 v\otimes\partial_3 w,$$

where $v_0=(x_1\partial_3)v$. Therefore, the weight of v_0 is equal to $(0,-1,-1)$.

If we would have proven that $\partial_1 v_0\otimes\partial_3 w$ is the highest component this would have implied that $\lambda=(0,-1,-1)$, $\mu=(0,0,0)$.

But the component $\Pi_{12}f=\partial_1 v\otimes\partial_2 w+\dots$, of type 1) is also possible and then the weight of v could be equal to $(0,0,-2)$ while the weight of w to $(0,0,0)$.

But observe that

$$(x_1\partial_2)(\Pi_3 f)=a\partial_3(x_1\partial_2)v\otimes\partial_3 w$$

cannot cancel with other components of $(x_1\partial_2)f$. Hence, $(x_1\partial_2)(\Pi_3 f)=0$, i.e., $(x_1\partial_2)v=0$ which is impossible if the weight of V^* is equal to $(0,0,-2)$. Therefore, $\Pi_{12}f=0$ and the highest weight of V^* is equal to $(0,-1,-1)$.

3.2) $\Pi_{13}f$ and $\Pi_{23}f$ are of type $4'$ is treated similarly.

3.3) Both types 4) and $4'$) are encountered. Then, $\Pi_3 f$ consists of components of the form $\partial_3 v\otimes\partial_3 w$, where either the weight of v is equal to $(0,-1,0)$ and the weight of w is equal to $(-1,0,0)$ or the other way round.

The components of type $\partial_1\partial_3$ are vectors $\partial_1 v\otimes\partial_3 w$, where the weight of v is equal to $(0,-1,0)$ and the weight of w is equal to $(0,0,-1)$ (if the corresponding summand $\Pi_{13}f$ is of type 4)) and $\partial_3 v\otimes\partial_1 w$, where the weight of v is equal to $(0,0,-1)$ and the weight of w is equal to $(0,-1,0)$ (if the corresponding summand is of type $4'$)).

The components of $\Pi_{12}f$ (if any) are of the form

$$\partial_1 v\otimes\partial_2 w \text{ and } \partial_2 v\otimes\partial_1 w,$$

where the weights of v and w are equal to $(0,0,-1)$ because $\Pi_{12}f$ is of type 1). Among the listed components there are highest ones, hence, the highest weights of V^* and W^* are equal to $(0,0,-1)$.

4) $\nu=(0,-2,-2)$. The V^*-highest and W^*-highest components are $\partial_2 v\otimes\partial_2 w$. The vector $\Pi_{12}f$ is of type 2) $\Pi_{23}f$ of type 5) ($m=-1$), hence, the weights of v and w are equal to $(0,0,-1)$.

5) $\nu=(-1,-2,-2)$. The vector $\Pi_{23}f$ is of type 5) (if $m=-1$), $\Pi_{12}f$ of type 4) or $4'$) (if $m=0$).

In the first case the weight of v is equal to $(-1,0,-1)$, the weight of w is equal to $(0,0,-1)$ while in the second case it is the other way round.

In the first case

$$\Pi_{12}f=\partial_2 v_0\otimes\partial_2 w+\partial_1 v\otimes\partial_2 w,$$

where $v_0=(x_1\partial_2)v$, hence, the weight of v_0 is equal to $(0,-1,-1)$ and the weight of w is equal to $(0,0,-1)$.

In the second case the weights are transposed.

6) The $\nu = (-2,-2,-2)$. The V^* -highest and W^*-highest components are of the form $\partial_1 v \otimes \partial_1 w$; the vectors $\Pi_{12} f$ and $\Pi_{13} f$ are of type 5), hence, the weights of v and w are equal to $(0,-1,-1)$.

Let now $n \geq 4$. Let $P(\partial_{i_0}\partial_{j_0}) v \otimes w$ be the highest component, the weight of v be equal to $(\lambda_1, \dots, \lambda_n)$. Then, $\Pi_{i_0 j_0 i} f$ is of one of the types indicated, hence, either $\lambda_i = 0$ or $\lambda_i = -1$.

Similarly, the highest weight of W^* is equal to $(0, 0, \dots, -1)$. Thus,

$$\lambda = (0 \dots, 0, \underbrace{-1, \dots, -1}_{l},\ \mu = (0, \dots, 0 \underbrace{-1, \dots, -1}_{m},\ \nu = (0, \dots, \underbrace{-1, \dots, -1}_{p}, \underbrace{-2, \dots, -2}_{q}.$$

The corresponding operators should be conjugate to those we will consider in the next heading. Therefore, we will not prove neither their existence nor their uniqueness.

2. The second case: $\Pi_{i_1} f$ contains a summand of the form

$$\partial_{i_1}^2 v \otimes w + \partial_{i_1} v \otimes \partial_{i_1} w$$

but does not contain any summands of type 2). Then, $\nu = -1$ and all the summands are of the form indicated; the i_1-th coordinate of the weight v vanishes while that of w is equal to 1. $\Pi_{i_1 i} f$ for $i > i_1$ is either of type 3) or of type 5) $(m = 0)$, hence, $\nu_i = 1$ and $\Pi_i f$ is also of type b). $\Pi_i f = 0$ for $i < i_1$ and, therefore, $\Pi_{i i_1} f$ is either of type 4) $(m = 1)$ or 6) $(l = 0)$ implying $\nu_i = 0$.

Thus, $\nu = (0, \dots, 0, \underbrace{-1, \dots, -1}_{r})$. If $i, j < i_1$, then $\Pi_{ij} f = 0$ (the list of singular vectors for $n = 2$ does not contain any vector g such that $\nu_1 = \nu_2 = 0$ and $\Pi_1 g = \Pi_2 g = 0$).

Therefore, the V-highest and W-highest components are $\partial_1 \partial_{i_1} v_0 \otimes w$ and $\partial_1 \partial_{i_1} v \otimes w_0$, respectively (perhaps, $i_1 = 1$).

Denote the weights of v_0, v, w_0, w by

$$(\lambda_1, \dots, \lambda_n), \quad (\lambda'_1, \dots, \lambda'_n), \quad (\mu_1, \dots, \mu_n), \quad (\mu'_1, \dots, \mu'_n).$$

Since $\Pi_{i i_1} f$ is either of the form 4) or 6), it follows that $\lambda_1 = \lambda'_1 = 0$ and $\mu_1 = \mu'_1 = 1$ (if $i_1 = 1$, then this is also true because $\Pi_{i_1} f$ is of type b)).

But $\lambda_1 = \max\{\lambda_i, \lambda'_i\}$ and $\mu_1 = \max\{\mu_i, \mu'_i\}$, hence, $\lambda_i \leq 0$, $\mu_i \leq 1$, $\lambda'_i \leq 0$, $\mu'_1 \leq 1$. Observe that

$$\lambda'_i + \mu_i = \begin{cases} \nu_i, \text{ for } i \neq 1, i \neq i_1 \\ \nu_i + 1, \text{ for } i = 1 \text{ or } ii = i_1,\ i_1 \neq 1 \\ \nu_i + 2, \qquad\qquad \text{for } ii = i_1 = 1 \end{cases}$$

But either $\nu_i = 0$ or $\nu_i = 1$, hence, $\lambda'_i + \mu_i \geq -1$.

But $\lambda'_i \leq 0$, hence, $-1 \leq \mu_i \leq 1$. Thus, $(\underbrace{-1, \dots, -1}_{p}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{q})$ is the highest weight of W^*, and therefore the highest weight of $(W^+)^*$ is equal to

$$(0, \dots, -1, \dots, -1, -2, \dots, -2).$$

So every operator encountered in this heading is conjugate to the operator from the previous heading.

In particular, this implies that the highest weight of V^* is $(0,\dots,0,\underbrace{-1,\dots,-1}_{l})$.

5.2. Statement. *1)* $l \le n-1$; *2)* $p \ge 1$; *3)* $r \ge 1$; *4)* $l+1 \ge p$; *5)* $l-p+q+2=r$.

Proof. 1) It follows from the fact that $\lambda_1 = 0$.

2) It follows from the fact that $\mu_1 = 1$.

3) It follows from the fact that $\nu_{i_1} = -1$.

4) Let us consider the W-highest component $\partial_1\partial_{i_1} v \otimes w_0$. We have $\mu_i = 1$ and $\lambda_i' + \mu_i \le 0$ for $i = 2,3,\dots,p$; hence, $\lambda_i' = -1$. Therefore, the number of -1's among the coordinates of the weight of v is equal to $p-1$, i.e., $l \ge p-1$.

5) It is verified by the direct comparison of the sums of coordinates of the weights. □

5.3. Statement. For any collection of l, p, q, r satisfying the conditions of Statement 10.1, there exists an invariant operator and this operator is unique.

Proof. **Existence**. Observe that in the tensor product of $\Lambda^{l+1}(\mathbb{K}^n)$ and the representation with highest weight $(\underbrace{1,\dots,1}_{p},0,\dots,0,\underbrace{-1,\dots,-1}_{q})$ contains an irreducible component isomorphic to $\Lambda^{r-1}(k^n)$. Therefore, the operator $S_1^{*-1}(w,s) = dZ(dw,s)$ exists.

Uniqueness. Observe that the weights of V^* are multiplicity free hence, the W-highest component $\partial_1\partial_{i_1} v \otimes w_0$ is uniquely determined. Apply the standard method. □

3) $\Pi_{i_1} f$ contains the summands of the form

$$m\partial_{i_1}^2 v \otimes w + \partial_{i_1} v \otimes \partial_{i_1} w (m \ne 0,1).$$

Then, $\nu_{i_1} = m-2$, and therefore all the summands in $\pi_{i_1} f$ are of this form.

The vector $\Pi_{i_1 i} f$ for $i > i_1$ is of type 5) implying $\nu_i = m-2$.

The vector $\Pi_{i i_1} f$ for $i < i_1$ is of type 4), hence, $\nu_i = m-1$ and the component $\partial_i\partial_{i_1} v \otimes w$ does not vanish.

We have $\Pi_{ij} f = 0$ for $i,j < i_1$ because for $n=2$ there is no such a vector. Hence, the V-highest and W-highest components are of the form $\partial_1\partial_{i_1} v \otimes w$. The vector $\Pi_{i i_1} f$ is of type 4), and therefore contains components

$$\partial_1\partial_{i_1} v \otimes w + \partial_{i_1}^2 v' \otimes w,$$

where $v = (x_1\partial_{i_1})v'$. Since $\Pi_{i i_1} f$ for $i > i_1$ is of type 5) and $i < i_1$ of type 4) for v, it follows that the weight of v is equal to $(-1,\dots,-1,0,-1,\dots,-1)$ while the weight of w is equal to $(\underbrace{m,\dots,m}_{i_1},m-1\dots,m-1)$.

Since $v = (x_1\partial_{i_1})v'$, it follows that the weight of v is equal to $(0, -1, \dots, -1)$. Thus,

$$\lambda = (0, -1, \dots, -1), \quad \mu = (\underbrace{m, \dots, m}_{i_1}, m-1, \dots, m-1),$$

$$\nu = (\underbrace{m-1, \dots, m-1}_{i_1-1}, m-2, \dots, m-2).$$

The corresponding operator is $S_2(w, s) = P_4(dw, s)$ and its uniqueness is proved routinely.

4) The vector $\Pi_{i_1} f$ contains a component of type d), $\nu_{i_1} = -1$. The vector $\Pi_{ii_1} f$ vanishes for $i > i_1$; hence, $i = n$.

The vector $\Pi_{in} f$ is of type 6) for $i < n$, hence, $\nu_i = 0$. Thus, $\nu = (0, \dots, 0, -1)$. This already shows that the operator conjugate to this one is of different type. But since all the other cases are already considered, it follows that in this case all the operators are conjugate to the ones already considered.

We have considered all the possibilities for the singular vectors of degree 2.

11 Third order operators

First, let us recall the list of singular highest weight vectors of degree 3 for $n = 2$.

1) $\lambda = \mu = (0, -1)$, $\nu = (-2, -3)$.

$$f = \partial_1\partial_2 v_0 \otimes \partial_1 w_0 - \partial_1 v_0 \otimes \partial_1\partial_2 w_0 + \partial_2^2 v_1 \otimes \partial_1 w_0 +$$
$$\partial_1\partial_2 v_0 \otimes \partial_2 w_1 - \partial_1 v_0 \otimes \partial_2^2 w_1 - \partial_2 v_1 \otimes \partial_1\partial_2 w_1 - \partial_2 v_1 \otimes \partial_2^2 w_1,$$

where $v_0 = X_+ v_1$ and $w_0 = X_+ w_1$; the weights of v_0 and w_0 are equal to $(0, -1)$; the weights of v_1 and w_1 are equal to $(-1, 0)$.

2) $\lambda = \nu = (0, -1)$, $\mu = (2, 1)$. The singular vector is of the form

$$f = 2\partial_1^2\partial_2 v_0 \otimes w_0 + \partial_1\partial_2 v_0 \otimes \partial_1 w_0 + 2\partial_1^2 v_0 \otimes \partial_2 w_0 +$$
$$\partial_1 v_0 \otimes \partial_1\partial_2 w_0 + 2\partial_1\partial_2^2 v_0 \otimes w_1 + 3\partial_1\partial_2 v_0 \otimes \partial_2 w_1 + \partial_1 v_0 \otimes \partial_2^2 w_1 +$$
$$\partial_1\partial_2^2 v_1 \otimes w_0 + \partial_2^2 v_1 \otimes \partial_1 w_0 + 2\partial_1\partial_2 v_1 \otimes \partial_2 w_0 - \partial_2 v_1 \otimes \partial_1\partial_2 w_0 + 2\partial_2^3 v_1 \otimes w_1 +$$
$$3\partial_2^2 v_1 \otimes \partial_2 w_2 + \partial_2 v_1 \otimes \partial_2^2 w_1,$$

where $v_0 = X_+ v_1$ and $w_0 = X_+ w_1$; the weights of v_0, v_1, w_0, w_1 are equal to $(0, -1)$, $(-1, 0)$, $(2, 1)$, $(1, 2)$, respectively.

$\lambda = (2, 1)$, $\mu = \nu = (0, -1)$. This case is similar to 2). Consider two subcases:

a) We have $\Pi_{ij} = 0$ for any i, j. Then, there exists a space $E = \langle e_{i_1}, e_{i_2}, e_{i_3} \rangle$ such that

$$\Pi_i f =$$
$$\partial_1'\partial_2'\partial_3' u_1 + \partial_1'\partial_2'\partial_3'' u_2 + \partial_1'\partial_2''\partial_3' u_3 + \partial_1''\partial_2'\partial_3' u_4 +$$
$$\partial_1'\partial_2''\partial_3'' u_5 + \partial_1''\partial_2'\partial_3'' u_6 + \partial_1''\partial_2''\partial_3' u_7 + \partial_1''\partial_2''\partial_3'' u_8$$

But this vector cannot be a highest weight one. Indeed,

$$0 = (x_1\partial_2)f =$$
$$-\partial_2'^2\partial_3'u_1 - \partial_2'^2\partial_3 u_2 - \partial_2'\partial_2''\partial_3'(u_3+u_4) - \partial_2'\partial_2''\partial_3''(u_5+u_6) - \partial''^2_2\partial_3'u_7 - \partial''^2_2\partial_3''u_8$$

+ components of the same form as in f.

Hence, $u_1 = u_2 = u_7 = u_8 = u_3 + u_4 = u_5 + u_6$ and

$$0 = (x_2\partial_3)f = -\partial_1'\partial_3'\partial_3''u_3-$$
$$\partial_1''\partial_3'^2u_4 - \partial_1'\partial''^2_3u_5 - \partial_1''\partial_3'\partial_3''u_6+$$
$$\text{components of the same form as in } f;$$

hence, $u_3 = u_4 = u_5 = u_6 = 0$, i.e., $f = 0$.

B) There exist i, j such that $\Pi_{ij}f \neq 0$. Then, $\Pi_{ij}f$ is the sum of several vectors of types 1), 2) and 2'). Moreover, if at least one of the summands is of type 1), then $\Pi_{i'j}f$ and, therefore, all the summands are of the same type.

Moreover, since $\nu_j = -3$ and $\Pi_j f \neq 0$, then $\Pi_{i'j}f$ is also of type 1) for $i' < j$. For $j' > j$ the vector $\Pi_{jj'}f$ vanishes implying that $j = n$.

Since $\Pi_{1n}f$ is of type 1), the weights of the vectors v and w that enter $\partial_n^2 v \otimes \partial_n w$ are equal to $(-1, \dots, -1, 0)$ and the weight of $(-2, \dots, -2, -3)$ is equal to f.

So far, the highest of the components found is $\partial_1\partial_n v' \otimes \partial_1 w'$. It enters the vector

$$\Pi_{1n}f = \partial_1\partial_n v' \otimes \partial_1 w' + \dots + \partial_n^2 v \otimes \partial_n w,$$

where $v' = (x_1\partial_n)v$ and $w' = (x_1\partial_n)w$, and therefore the weights of v' and w' are equal to $(0, -1, \dots, -1)$.

But $\Pi_{1j} = 0$ for $j < n$ because $\nu_1 = \nu_j = -2$ and no such operator exists for $n = 2$. Therefore, the V-highest and W-highest components are of the form $\partial_1\partial_n v' \otimes \partial_1 w'$ implying that $\lambda = \mu = (0, -1, \dots, -1)$. Such an operator exists:

$$T(w_1, w_2) = P_4(dw_1, dw_2).$$

If $\Pi_{ij}f$ does contain a summand of type 2) or $2'$), then $\nu_j = -1$ and all the other summands are of the same form.

We similarly prove that $j = n$ and the weight of f is equal to $(0, \dots, 0, -1)$. But either the highest weight of V^* or the highest weight of W^* is of different form because $\Pi_{in}f$ is either of type 2 or $2'$). Therefore, in this case the operators obtained are conjugate to the operators of different types, i.e., to $T(w_1, w_2)$. The proof is completed.

12 Operators in the spaces of twisted forms

Recall that the space twisted of p-forms with twist l is $T(l+1, \dots, l+1, l, \dots, l)$ with p-many $l+1$'s. Denote the space $\Omega^p_l := \Omega^p \otimes Vol^l$. In particular, p-forms with twist 0 are the usual differential forms, p-forms with twist -1 are polyvector fields, any volume form on the n-dimensional manifold M can be considered as a 0-forms with twist 1 or as an n-form with twist 0.

Having selected a nondegenerate volume form δ on M, any element from Ω^p_l can be represented either in the form

$$s = \omega \cdot \delta^l, \text{ where } \omega \in \Omega^p,$$

thanks to the existence of the nonzero 0-order operator $Z : \Omega^p \otimes Vol^l \longrightarrow \Omega^p_l$, or, if alternatively, in the form

$$s = \xi \cdot \delta^{l+1}, \text{ where } \xi \in \Omega^p_{-1} \text{ is a polyvector of degree } n-p.$$

Let $x = (x_1, \dots, x_n)$ be local coordinates in a neighborhood of a point $P \in M$. Select $\delta = vol$. Obviously, volume preserving diffeomorphisms transform twisted forms by the same formulas as the usual forms: they consider Vol^l as the space of functions.

Zero order differential operators

Let the weight of the $\mathfrak{gl}(n)$-module V be

$$\lambda = (l+1, \dots, l+1, l, \dots, l)$$

with p-many $l+1$'s; that of the $\mathfrak{gl}(n)$-module W be

$$\mu = (m+1, \dots, m+1, m, \dots, m)$$

with q-many $m+1$'s. Then, $V \otimes W$ splits into the direct sum of irreducible modules of which exactly one has the highest weight of the same form:

$$\nu = \begin{cases} (l+m+1, \dots, l+m+1, l+m, \dots, l+m) & \text{with } p+q\text{-many } l+m+1\text{'s} \\ & \text{if } p+q \leq n \\ (l+m+2, \dots, l+m+2, l+m+1, \dots, l+m+1) & \text{with } p+q-n\text{-many } l+m+2\text{'s} \\ & \text{if } p+q \geq n \end{cases}$$

The corresponding invariant operators are the exterior product of twisted differential forms and the exterior product of twisted polyvector fields; in both cases, the "twists" are considered as coefficients:

$$Z_1(\omega_1 \cdot \delta^l, \omega_2 \cdot \delta^m) = \omega_1 \wedge \omega_2 \cdot \delta^{l+m}$$

$$Z_1(\omega_1 \cdot \delta^l, \omega_2 \cdot \delta^m) = \omega_1 \wedge \omega_2 \cdot \delta^{l+m} \text{ for } p+q \leq n;$$

$$Z_2(\xi_1 \cdot \delta^{l+1}, \xi_2 \cdot \delta^{m+1}) = \xi_1 \wedge \xi_2 \cdot \delta^{l+m+2} \text{ for } p+q \geq n.$$

From multiplicity-free occurrence of the target space, it follows that Z_1 and Z_2 are proportional if $p+q=n$.

I hope the reader can forgive me that here I skip verification of invariance of these operators. Actually, to prove it correctly with all details takes some space and arguments.

First order differential operators

On the spaces of twisted forms, there are only the following invariant bilinear operators:

1) $P_6 : \Omega^p_l \times \Omega^q_m \longrightarrow \Omega^{p+q+1}_{l+m}$ for $p+q \leq n-2$ and $(l,m) \neq (0,0)$;

2) $P_7 : \Omega^p_l \times \Omega^q_m \longrightarrow \Omega^{p+q+1-n}_{l+m+1}$ for $p+q \geq n-1$ and $(l,m) \neq (0,0),(0,-1),(-1,0)$;

in the exceptional cases we have

3) $P_1(\omega_1,\omega_2) = ad\omega_1 \wedge \omega_2 + b\omega_1 \wedge d\omega_2$ for $p+q \leq n-2$ and $(l,m) = (0,0)$;

4) $P_1(\omega_1,\omega_2) = aZ_2(d\omega_1,\omega_2) + bZ_2(\omega_1,d\omega_2)$ for $p+q \geq n-1$ and $(l,m) = (0,0)$;

5) $P(\omega_1,\omega_2\delta^{-1}) = aZ_2(d\omega_1,\omega_2\delta^{-1}) + bdZ_2(\omega_1,\omega_2\delta^{-1})$ for $p+q \geq n-1$ and $(l,m) = (0,-1)$; for $(l,m) = (-1,0)$ *mutatis mutandis*;

additionally, if the result is not a twisted form, there exist the following operators:

6) $P_1(\omega,s) = Z(d\omega,s)$ for $l=0$;

7) $P_1(s,\omega) = Z(s,d\omega)$ for $m=0$.

I hope that the reader forgives me not retyping one more page of formulas in order to prove the more or less obvious.

Let us prove that the above list exhausts all the operators. To this end, let us prove that the highest singular vectors exist only in these cases.

For $n=1$ the singular vectors are:

1) $\lambda = (l)$, $\mu = (m)$, $\nu = (l+m-1)$ and $(l,m) \neq (0,0)$:

$f = m\partial v \otimes w - lv \otimes \partial w$,

2) if $(l,m) = (0,0)$, then $f = a\partial v \otimes w + bv \otimes \partial w$ for any a, b.

For $n=2$ the singular vectors are:

1) $l=m=0$. In particular, $\lambda = (l,l-1)$, $\mu = (m,m-1)$, $\nu = (l+m-1,l+m-2)$,

$$f = (l+m-1)(m\partial_1 v_0 \otimes w_0 - lv_0 \otimes \partial_1 w_0) + l(m\partial_2 v_0 \otimes w_1 - (l-1)v_0 \otimes \partial_2 w_1) + m((m-1)\partial_2 v_1 \otimes w_0 - lv_1 \otimes \partial_2 w_0),$$

$f = a(\partial_1 v_0 \otimes w_0 + \partial_2 v_1 \otimes w_0) + b(v_0 \otimes \partial_2 w_0 + v_0 \otimes \partial_2 w_1)$,

$f = (a+b)v_0 \otimes \partial_1 w_0 - a\partial_2 v_0 \otimes w_1) + bv_0 \otimes \partial_2 w_1 + a\partial_2 v_1 \otimes w_0 + av_1 \otimes \partial w_0$

2) $l=m=0$. In particular, $\lambda = (l,l)$, $\mu = (m,m-1)$, $\nu = (l+m-1,l+m-1)$,

$f = m\partial_1 v_0 \otimes w_0 - lv_0 \otimes \partial_1 w_0 + m\partial_2 v_0 \otimes w_1 - lv_0 \otimes \partial_2 w_1$,

$f = a\partial_1 v_0 \otimes w_0 + bv_0 \otimes \partial_1 w_0 + a\partial_2 v_0 \otimes w_1 + bv_0 \otimes \partial_2 w_1$

2′) $\lambda = (l,l-1)$, $\mu = (m,m)$, $\nu = (l+m-1),l+m-1)$ is similar.

3) $l=m=0$. In particular, $\lambda = (l,l)$, $\mu = (m,m)$, $\nu = (l+m,l+m-1)$,

$f = m\partial_2 v_0 \otimes w_0 - lv_0 \otimes \partial_2 w_0$,

$f = a\partial_2 v_0 \otimes w_0 + bv_0 \otimes \partial_2 w_0$

4) $\lambda = (0,0)$, $\mu = (m,m-1)$, $\nu = (m,m-2)$,

$f = \partial_2 v_0 \otimes w_0$.

4′) $\lambda = (l,l-1)$, $\mu = (0,0)$, $\nu = (l,l-2)$,

$f = v_0 \otimes \partial_2 w_0$.

$n \in \mathbb{Z}_+$. Let us prove that $\Pi_n f \neq 0$. Indeed, $\Pi_{in} f$ is of one of the types 1)–4), hence, $\Pi_n f \neq 0$. The vector $\Pi_{1n} f$ is also of one of these types, hence, $\nu_1 - \nu_n \leq 2$.

Consider the case $\nu_1 - \nu_n = 2$. Then, $\Pi_{1n} f$ is the sum of the summands of type 4) or 4′). Let at least one of the summands, $\partial_n v \otimes w$, be for example of type 4).

Let the weights of v and w be equal to $(\lambda_1,\dots,\lambda_n)$ and $(\mu_1,\dots,\mu_n)$, respectively. Then, $\lambda_n=0$, $\mu_1=m$, $\mu_n=m-1$. Let f be the weight of

$$(t,\dots,t,\underbrace{t-1,\dots,t-1}_{a},\underbrace{t-2,\dots,t-2}_{b}).$$

The balance of sums of the coordinates of weights implies $n(l+m-t)=p+q-a-2b$ wherefrom either $t=l+m$, $a+2b=p+q$ or $l=l+m-1$, $a+2b=p+q-n$.

But the second case is impossible because in this case $\nu_n=l+m-3$; hence, $\lambda_n=l-1$, and therefore $\Pi_i f=0$ for $i<n$ (since the last coordinate of the weight of $\partial_i v'\otimes w'$ cannot be equal to $l+m-3$). Therefore, $\Pi_{in}f$ is of type 3 or 4) implying $\lambda_i=\lambda_n=0$, i.e., $\lambda=(0,\dots,0,0)$ and $\lambda_n=l$. This is a contradiction.

In the first case $t-2=\nu_n=\lambda_n+\mu_n-1=m-2$, hence, $t=m$, $l=0$. Thus, the first multiple is a twist-free form. Such operators will be considered in the next section. Therefore, let us pass to another case.

Let now $\nu_1-\nu_n\leq 1$, i.e., the image of the operator is also a twisted form. The fact that such operators exist only in the cases listed easily follows from the balance of the sums of coordinates: let $\nu=(t,\dots,t,\underbrace{t-1,\dots,t-1}_{r})$.

Then, $0\leq r\leq n-1$. But

$$\sum\lambda_i=nl-p,\quad \sum\mu_i=nm-q,\quad \sum\nu_i=nt-r,\quad \sum\lambda_i+\sum\mu_i-1=\sum\nu_i,$$

i.e., $n(l+m-t)=p+q+1-r$, where $-n+2\leq p+q+1-r\leq 2n-1$, and therefore either

I) $t=l+m$, $r=p+q+1$ if $p+q+1<n$, or

II) $t=l+m-1$, $r=p+q+1-n$ if $p+q+1\geq n$.

We have to prove that in each case the operator is unique. To this end, by the routine method we have to indicate the component which the singular highest weight vector must contain. If there are two non-proportional highest weight singular vectors, then certain linear combination of them does not contain the component indicated which is impossible.

Let us assume that $l,m\neq 0$ (the opposite case is considered in the next section). Observe that $\Pi_n f$ must be highest with respect to $\mathfrak{gl}(E)$, where $E=\langle e_1,\dots,e_{n-1}\rangle$ because the components $(x_i\partial_j)(\Pi_n f)$, where $i<j<n$ cannot cancel with the other components of $(x_i\partial_j)f$, and therefore by Lemma 9.1 the vector $\Pi_n f$ contains the component with vector v highest with respect to $\mathfrak{gl}(E)$. In other words, the weight of is equal to either $(l,\dots,l,l-1,\dots,l-1)$ or $(l,\dots,l,l-1,\dots,l-1,l)$.

I) $\nu_n=l+m-1$, hence, $\Pi_n f$ contains summands of the form $m\partial_n v\otimes w-lv\otimes\partial_n w$, where the n-th coordinate of the weight of v is equal to l that of w is equal to m. Therefore, there should be a component $lv_0'\otimes\partial w'$, where the weight of v_0' is equal to $(l,\dots,l,l-1,\dots,l-1,l)$. The presence of this component is compulsory.

II) $\nu_n l+m-2$, hence, $\Pi_n f$ may contain summands of the form $(m-1)\partial_n v\otimes w-lv\otimes\partial_n w$ (hence, the n-th coordinate of the weight of v is equal to l that of w is equal to $w-m-1$)

and of the form $m\partial_n v' \otimes w' - (l-1)v' \otimes \partial_n w'$ (hence, the n-th coordinate of the weight of v' is equal to $l-1$ that of w' is equal to m).

Let us establish when only the summands of the second type may occur. In this case if $\lambda_i = l$ (such i exist since $p < n$) the vector $\Pi_{in} f$ has no components of the form $p(\partial)v_1 \otimes w_0$ (see the list of singular vectors for $n = 2$), but for $l \neq 0$, $m \neq 0$ this is impossible.

Therefore, this case is excluded and $lv \otimes \partial_n w$ is the "compulsory" component; the weight of v is equal to $(l, \dots, l, l-1, \dots, l-1, l)$. The uniqueness is proved.

13 The case $T(V_i) = \Omega^p$

5.1. Lemma. *Let $B : (\Omega^p, T(V_1)) \to T(V_2)$ be an invariant differential operator of order d. Then,*

1) If there exist $\omega \in \Omega^p$ such that $d\omega = 0$ and $B(\omega, s) \not\equiv 0$, then: when $p = 0$, there exists an operator $U : T(V_1) \to T(V_2)$ of order d such that $B(1, s) = U$; and when $p \geq 1$, there exists an invariant differential operator $B^+ : (\Omega^{p-1}, T(V_1)) \to T(V_2)$ of order $d + 1$ such that $B^+(\omega, s) = B(d\omega, s)$.

2) If $d\omega = 0$ implies $B(\omega, s) = 0$, then there exists $B^- : (\Omega^{p+1}, T(V_1)) \to T(V_2)$ an invariant operator of order $d - 1$ such that $B(\omega, s) = B^-(d\omega, s)$ (if $d = 0$, this case is excluded).

Proof. 1) By definition $B^+(\omega, s) = B(d\omega, s)$ is invariant since it is expressed in terms of invariant operators. The operator B^+ does not vanish identically since there exists $\omega_1 \in \Omega^p$ such that $d\omega_1 = 0$ but $B(\omega_1, s)(x_0) \neq 0$ and there exists $\omega_0 \in \Omega^{p-1}$ such that $d\omega_0$ in a neighborhood of point x_0 such that $B^+(\omega_0, s)(x_0) = B(\omega_1, s)(x_0) \neq 0$.

2) Let $\omega_1, \omega_2 \in \Omega^p$ be such that $d\omega_1 = d\omega_2 = \omega \in \Omega^{p+1}$. Then, $d(\omega_1 - \omega_2) = 0$, and therefore $B(\omega_1 - \omega_2, s) = 0$, that is, $B(\omega_1, s) = B(\omega_2, s)$. This shows that: if $\omega = d\omega'$ in a neighborhood of x, then $B^-(\omega, s)(x) = B(\omega', s)(x)$ is well-defined; and that, if $d\omega \neq 0$, then $B^-(\omega, s) = 0$.

The invariance is obvious:

$$gB^-(\omega, s) = gB(\omega', s) = B(g\omega', gs) = B^-(g\omega, gs).$$

Let us prove that B^- is local. Let $x_0 \notin \operatorname{supp}\omega$. Then, $x_0 \in \operatorname{supp} \int_{x_0}^x \omega$, where $\int_{x_0}^x \omega$ is determined in a neighborhood of x_0 because ω is closed.

Hence, $B(\int_{x_0}^x \omega, s)(x_0) = 0$, i.e., $B^-(\omega, s)(x_0) = 0$. If $x_0 \notin \operatorname{supp} s$, then

$$B^-(\omega, s)(x_0) = B(\omega' s)(x_0) = 0.$$

The locality is proved. Therefore, B^- is a differential operator.

But $B(\omega, s) = B^-(d\omega, s)$, and therefore the order of B^- is equal to $d - 1$. □

13.2. Corollary. *In §1, there are listed all the first order operators of the form*

$$B : (\Omega^p, T(V_1)) \to T(V_2)$$

$$B : (T(V_1), \Omega^q) \to T(V_2)$$

$$B : (T(V_1), T(V_2)) \to \Omega^r.$$

Proof. Indeed: let $B : (\Omega^p, T(V_1)) \to T(V_2)$ be a first order operator. The following cases are possible:

1) There exists a second order operator $B^+(\Omega^{p-1}, T(V_1)) \to T(V_3)$ such that $B^+(\omega, s)$ is equal to $B(d\omega, s)$. But all such second order operators are known; these are

$$\begin{aligned} B_1^+(\omega_1, \omega_2) \quad &= Z(d\omega_1, d\omega_2),\\ B_2^+(\omega, s) \quad &= dZ(d\omega, s),\\ B_3^+(\omega, s) \quad &= P_4(d\omega, s) \text{ for } p = n-1). \end{aligned}$$

The corresponding first order operators are

$$\begin{aligned} B_1(\omega_1, \omega_2) &= Z(\omega_1, d\omega_2),\\ B_2(\omega, s) &= dZ(\omega, s),\\ B_3(\delta, s) &= P_4(\delta, s). \end{aligned}$$

If $B(\omega, s) - B_j(\omega, s) \neq 0$, then go to case 3).

2) $p = 0$ and $B(1, s)U(s) \not\equiv 0$. But all the unary operators are known: $U(\omega) = d\omega$, $B(1, \omega) = d\omega$, $B(\varphi, \omega) - \varphi d\omega = 0$ if $d\varphi = 0$, and therefore $B(\varphi, \omega) - \varphi d\omega$ which corresponds to case 3).

3) There exists an operator $B^-(\Omega^{p+1}, T(V_1)) \to T(V_2)$ of order zero such that $B(\omega, s)$ is equal to $B^-(d\omega, s)$. But the zero order operator is $Z(\omega, s)$, hence,

$$B(\omega, s) = Z(d\omega, s) = P_1(\omega, s).$$

In the proof of the existence of the operators $B : (T(V_1), \Omega) \to T(V_2)$ the arguments are the same and as in that of the operators of the form $B(T(V_1), T(V_2)) \to \Omega$, these operators are conjugate to the already considered operators, and therefore all of them are listed. Proof is completed. $\square$

14 Higher order operators

In this section we will prove that there are no invariant differential operators of order ≥ 4.

Let

$$f = \sum P_i(\partial'_1, \ldots, \partial''_n)U_i, \quad f \in I(V_1^* W^*).$$

Let $E = \langle E_{i_i}, \ldots, e_{i_j} \rangle \subset \mathbb{K}^n$ and $\mathfrak{gl}(E) \subset \mathfrak{gl}(n)$. Recall that

$$I_E(V^*) = \mathbb{K}[\partial'_{i_1}, \ldots, \partial'_{i_\gamma}] \otimes V^*; \quad I_E(V^*, W^*) = I_E(V^*) \otimes I_E(W^*).$$

The vector f can be represented as a polynomial in

$$\partial'_{j_i}, \ldots, \partial'_{i_{\bar\gamma}}, \partial''_{j_1}, \ldots, \partial''_{j_{\bar\gamma}}, \text{ where } \{j_i, \ldots, j_{\bar\gamma}\} = \{1, \ldots, n\} \setminus \{i_j, \ldots, i_\gamma\}$$

with coefficients from $I_E(V^*)$, namely,

$$f = \sum P_i(\partial'_{j_1}, \ldots, \partial''_{j_{\bar\gamma}} f_i, \quad f_i \in I_E(V^*).$$

In particular, the constant term of this polynomial is $\Pi_E f$.

If f is $\mathfrak{gl}(n)$-highest, then it is also $\mathfrak{gl}(E)$-highest, i.e., $x_\alpha\partial_\beta f = 0$ for $\alpha < \beta$ and $\alpha, \beta \in \{i_1, \dots, i_\gamma\}$. From the equation

$$x_\alpha\partial_\beta(\sum P_i(\partial'_{j_i}, \dots, \partial''_{j_{\bar\gamma}})f_i = \sum P_i(\partial'_{j_i}, \dots, \partial''_{j_\gamma})(x_\alpha\partial_\beta)f_i = 0$$

it follows that $(x_\alpha\partial_\beta)f_i = 0$, hence, all the vectors f_i are $\mathfrak{gl}(E)$-highest ones.

Similarly, if f is singular with respect to $\mathcal{L}$, then all the coefficients f_i are singular with respect to $\mathcal{L}_E$.

Now let us pass to the proof (of the fact that there are no invariant operators of order > 3).

For $n = 1$ the proof was carried out in §3.

Let now $n = 2$. The highest singular vector cannot contain components in which either ∂_1 or ∂_2 enters otherwise the above decomposition contains a highest weight vector of degree $d \leq 4$ in dimension $n = 1$.

Here is the generic form of the vector f of degree 4 without components of the form $\partial_i^4 U$:

$$f = f_1 + f_2 + f_3 + f_4 + f_5,$$

where

$$f_1 = \partial'^3_1\partial'_2 u_1 + \partial'^2_1\partial'^2_2 u_2 + \partial'_1\partial'^3_2 u_3,$$

$$\begin{aligned} f_2 = {} & \partial'^3_1\partial''_2 u_4 + \partial'^2_1\partial'_2\partial''_1 u_5 + \partial'^2_1\partial'_2\partial''_1 u_6 + \\ & \partial'_1\partial'^2_2\partial''_1 u_7 + \partial'_1\partial'^2_2\partial''_2 u_8 + \partial'^3_2\partial''_1 u_9, \end{aligned}$$

$$\begin{aligned} f_3 = {} & \partial'^2_1\partial''_1\partial''_2 u_{10} + \partial'_1\partial'_2\partial''^2_1 u_{11} + \partial'^2_1\partial''^2_2 u_{12} + \\ & \partial'_1\partial'_2\partial''_1\partial''_2 u_{13} + \partial'^2_2\partial''^2_1 u_{14} + \\ & \partial'_1\partial'_2\partial''^2_2 u_{15} + \partial'^2_2\partial''_1\partial''_2 u_{16} \end{aligned}$$

The form of f_4 and f_5 is similar to that f_2 and f_1.

Thus, we have

$$\begin{aligned} X_+ f_1 = {} & -3\partial'^2_1\partial'^2_2 u_1 - 2\partial'_1\partial'^3_2 u_2 - \partial'^4_2 u_3 + \partial'^3_1\partial'_2(X_+ u_1) + \\ & \partial'_1\partial'^3_2(X_+ u_3); \end{aligned}$$

$$\begin{aligned} X_+ f_2 = {} & -3\partial'^2_1\partial'_2\partial''_2 u_4 - 2\partial'_1\partial'^2_2\partial''_1 u_5 - \partial'^2_1\partial'_2\partial''_2 u_5 - \\ & 2\partial'_1\partial'^2_2\partial''_2 u_6 - \partial'^3_2\partial''_1 u_7 - \partial'_1\partial'^2_2\partial''_2 u_7 - \partial'^3_2\partial''_2 u_8 - \partial'^3_2\partial''_2 u_9 + \\ & \partial'^3_1\partial''_2 X_+ u_4 + \partial'^2_1\partial'_2\partial'_1 X_+ u_5 + \partial'^2_1\partial'_2\partial''_2 X_+ u_6 + \\ & \partial'_1\partial'^2_2\partial''_1 X_+ u_7 + \partial'_1\partial'^2_2\partial''_2 X_+ u_8 + \partial'^3_2\partial''_1 X_+ u_9 \end{aligned}$$

$$X_+f_3 = -2\partial'_1\partial'_2\partial''_1\partial''_2u_{10} - \partial'^2_1\partial''^2_1u_{10} - \partial'^2_2\partial''^2_1u_{11} - 2\partial'_1\partial'_2\partial''_1\partial''_2u_{11}-$$
$$2\partial'_1\partial'_2\partial''^2_2u_{12} - \partial'^2_2\partial''_1\partial''_2u_{13} - \partial'_1\partial''_2\partial''^2_2u_{13}-$$
$$2\partial'^2_2\partial''_1\partial''_2u_{14} - \partial'^2_2\partial''^2_2u_{15} - \partial'^2_2\partial''^2_2u_{16} + \partial'^2_1\partial''_1\partial''_2X_+u_{10}+$$
$$\partial'_1\partial'_2\partial''^2_1X_+u_{11} + \partial'^2_1\partial''^2_2X_+u_{12} + \partial'_1\partial'_2\partial''_1\partial''_2X_+u_{13}+$$
$$\partial'^2_2\partial''^2_1X_+u_{14} + \partial'_1\partial'_2\partial''^2_2X_+u_{15} + \partial'^2_2\partial''_1\partial''_2X_+u_{16}$$

The form of X_+f_4 and X_+f_5 is similar to that of X_+f_2 and X_+f_1.

Since $X_+f = 0$, it follows that

$$\left.\begin{array}{c} u_3 = 0 \\ 2u_2 = X_+u_3 \\ 3u_1 = X_+u_2 \\ 0 = X_+u_1 \end{array}\right\} \Longrightarrow u_1 = u_2 = u_3 = 0, quad f_1 = 0.$$

Similarly, $f_5 = 0$.

Further, it follows that

$$\left.\begin{array}{c} u_8 + u_9 = 0 \\ u_7 = X_+u_9 \\ 2u_6 + u_7 = X_+u_8 \\ 2u_5 = X_+u_7 \\ 3u_4 + u_5 = X_+u_6 \\ 0 = X_+u_5 \\ o = X_+u_4 \end{array}\right\} \Longrightarrow \begin{array}{c} u_8 = A, \\ u_9 = -A \\ u_7 = -X_+A, \\ u_6 = X_+A, \\ u_5 = -\frac{1}{2}(X_+)^2A, \\ u_4 = \frac{1}{2}(X_+)^2A \\ (X_+)^3A = 0. \end{array}$$

Hence, $f_4 = 0$.

Finally, it follows that

$$\left.\begin{array}{c} u_{15} + u_{16} = 0 \\ u_{13} + 2u_{14} = X_+u_{16} \\ 2u_{12} + u_{13} = X_+u_{15} \\ u_{11} = X_+u_{14} \\ 2u_{10} + 2u_{11} = X_+u_{13} \\ u_{10} = X_+u_{12} \\ 0 = X_+u_{11} \\ 0 = X_+u_{10} \end{array}\right\} \Longrightarrow \begin{array}{c} u_{15} = B, \quad u_{16} = -B, \quad u_{13} = C \\ u_{14} = \frac{1}{2}(-X_+B - C), \quad u_{12} = \frac{1}{2}(X_+B - C) \\ u_{11} = -\frac{1}{2}(X_+)^2B, \quad u_{10} = \frac{1}{2}(X_+)^2B \\ (X_+)^3B = 0, \quad X_+C = 0. \end{array}$$

The singularity condition reads as

$$(x_2^2\partial_1)f_2 = -2\partial'^3x''_-u_4 - 2\partial'^2_1\partial''_1x'_-u_5 - 2\partial'^2_1\partial''_2x'_-u_6-$$
$$2\partial'^2_1\partial'_2x''_-u_6 - 4\partial'_1\partial'_2\partial''_1x'_-u_7 + 2\partial'^2_2\partial''_1u_7-$$
$$4\partial'_1\partial'\partial''_2X_-u_8 + 2\partial'^2_1\partial''u_8 - 2\partial'_1\partial'^2_2x''_-u_8-$$
$$6\partial'^2_2\partial''_1X_-u_9 + 6\partial'_1\partial'_2\partial''_1u_9$$

The form of $(x_2^2)f_4$ is similar while

$$(x_2^2\partial_1)f_3 = -2\partial'^2_1\partial''_1 x''_- u_{10} - 2\partial'_1\partial''^2_1 x'_- u_{11} - 4\partial'^2_1\partial''_2 x''_- u_{12} + 2\partial'_1\partial''_1\partial''_2 x'_- u_{13} -$$
$$2\partial'_1\partial'_2\partial''_1 x''_- u_{13} - 4\partial'_2\partial''^2_1 x'_- u_{14} + 2\partial'_1\partial''^2_1 u_{14} - 2\partial'_1\partial''^2_2 X_- u_{15} -$$
$$4\partial'_1\partial'_2\partial''_1 x''_- u_{15} + 2\partial'_1\partial'_2\partial''_1 u_{15} - 4\partial'_2\partial''_1\partial''_2 u_{16} +$$
$$2\partial'_1\partial''_1\partial''_2 u_{16} - 2\partial'^2_2\partial''_1 x''_- u_{16}.$$

Since $(x_2^2\partial_1)f = 0$, it follows that

$$\begin{aligned}
x''_- u_4 = x''_- u_6 = x''_- u_8 &= 0\\
x'_- u_5 - u_7 + x''_- u_{10} - u_{12} &= 0\\
x'_- u_6 - u_8 + 2x''_- u_{12} &= 0\\
2x'_- u_7 - 3u_9 + x'_- u_{13} - u_{15} &= 0\\
2x'_- u_8 + 2x''_- u_{15} &= 0\\
3x'_- u_9 + x''_- u_{16} &= 0
\end{aligned}$$

and similar equations that relate f_3 with f_4.

Since $u_8 = -u_9 = A$ and $u_{15} = -u_{16} = B$, then the last two equations imply that

$$x'_- A = x''_- B = 0.$$

But the first equation implies $x''_- A = 0$. Moreover, the last two equations (connecting f_3 and f_4) imply that $x'_- A = x''_- B = 0$. But the first of the equations implies $x'_- B = 0$. Thus,

$$x'_- A = x''_- A = 0, \quad x'_- B = x''_- B = 0.$$

Hence,

$$A = av_\lambda \otimes w_\mu, \quad B = bv_\lambda \otimes w_\mu,$$

where v_λ and w_μ are the lowest weight vectors in V^* and W^*, respectively.

But since $(X_+)^3 A = 0$ and $(X_+)^3 B = 0$ we deduce that $\lambda + \mu \leq 2$.

Consider the following cases:

1) $f_2 \neq 0$ and $f_4 \neq 0$. Since $A \neq 0$, the equation $x''_- u_6 = x''_- X_+ A$ implies that $\mu = 0$.

Similarly, $f_4 \neq 0$ implies that $\lambda = 0$. But then $x'_- u_6 - u_8 + 2x''_- u_{12} = 0$ implies that $u_8 = 0$, hence, $A = 0$. This is a contradiction.

2) $f_2 = 0$. We have

$$\begin{aligned}
u_{12} &= x''_- u_{10} & 0 &= x''_- u_{12}\\
B &= u_{15} = x''_- u_{13}, & B &= x''_- C.\\
B &= (a(02) + b(11) + c(20), & C &= x(01) + y(10).
\end{aligned}$$

a) $\lambda = 0$, $\mu = 2$. $a(02) = 2x(02)$, $a = 2x$. Then,

$$u_{12} = \frac{1}{2}(X_+ 2x(02) - x(01)) = \frac{1}{2}x(01), \quad u_{10} = x(00).$$

The coefficients do not match.

b) $\lambda = \mu = 1$, then $B = b(11) = y(11)$ and

$$\left.\begin{array}{c} u_{15} = -u_{16} = b(11) \\ u_{12} = b(01), \\ u_{14} = -b(10), \\ u_{13} = -b(01) + b(10) \\ u_{10} = -u_{11} = b(00) \end{array}\right\} \Longrightarrow \begin{array}{c} u_{12} = \frac{1}{2}(b - x)(01), \quad u_{10} = b(00); \\ \frac{1}{2}(b - x) = b \Longrightarrow x = -b. \end{array}$$

We get a fourth order operator $c(\omega_1, \omega_2) = d^2\omega_1 \wedge d^2\omega_2$ which is invariant only with respect to $\mathfrak{svect}(2)$, not $\mathfrak{vect}(2)$.

c) $\lambda = 2$, $\mu = 0$. Then, $c = y(10)$, $u_{12} = 0 \Longrightarrow c = 0$. The generic vector of degree 5 is of the form

$$f = f_1 + f_2 + f_3 + f_4 + f_5,$$

where

$$f_1 = \partial'^3_1\partial'^2_2 u_1 + \partial'^2_1\partial'^3_2 u_2,$$

$$\begin{aligned} f_2 = {} & \partial'^3_1\partial'_2\partial''_2 u_3 + \partial'^2_1\partial''_1\partial'^2_2 u_4 + \\ & \partial'^2_1\partial'^2_2\partial''_2 u_5 + \partial'_1\partial''_1\partial'^3_2 u_6 \end{aligned}$$

$$\begin{aligned} f_3 = {} & \partial'^3_1\partial''^2_2 u_7 + \partial'^2_1\partial''_1\partial'_2\partial''_2 u_8 + \partial'_1\partial''^2_1\partial'_2 u_9 + \\ & \partial'^2_1\partial'_2\partial''^2_2 u_{10} + \partial'_1\partial''_1\partial'^2_2\partial''_2 u_{11} + \partial''^2_1\partial'^3_2 u_{12} \end{aligned}$$

and the form of f_4, f_5 and f_6 is similar to that of f_5, f_2 and f_1, respectively.

We have

$$\begin{aligned} X_+ f_1 = {} & -3\partial'^2_1\partial'^2_2 u_1 - 2\partial'_1\partial'^4_2 u_2 + \\ & \partial'^3\partial'^2_2(X_+ u_1) + \partial'^2_1\partial'^3_2(X_+ u_2); \end{aligned}$$

$$\begin{aligned} X_+ f_2 = {} & -3\partial'^2_1\partial'^2_2\partial''_2 u_3 - 2\partial'_1\partial'^3_2\partial''_1 u_4 - \partial'^2_1\partial'^2_2\partial''_2 u_4 - \\ & 2\partial'_1\partial'^3_2\partial''_2 u_5 - \partial'^4_2\partial''_1 u_6 + \text{ components with the } X_+ u_i. \end{aligned}$$

Since $X_+ f_i = 0$, it follows that $u_1 = u_2 = 0$, $u_6 = 0$, $2u_5 + u_7 = 0$, $2u_4 = X_+ u_6$, $(= 0)$, $3u_3 + u_4 = X_+ u_5$, $X_+ u_3 = X_+ u_4 = 0$, $u_5 = A$, $u_7 = -2A$, $u_3 = \frac{1}{3}X_+ A$, $(X_+)^2 A = 0$. Consider

$$\begin{aligned} X_+ f_3 = {} & -3\partial'^2_1\partial'_2\partial''^2_2 u_7 - 2\partial'_1\partial'^2_2\partial''_1\partial''_2 u_8 - \partial'^2_1\partial'_2\partial''^2_2 u_8 - \\ & \partial'^3_2\partial''^2_1 u_9 - 2\partial'_1\partial'^2_2\partial''_1\partial''_2 u_9 - 2\partial'_1\partial'^2_2\partial''^2_2 u_{10} - \\ & \partial'^3_2\partial''_1\partial''_2 u_{11} - \partial'_1\partial'^2_2\partial''^2_2 u_{11} - 2\partial'^3_2\partial''_1\partial''_2 u_{12} + \dots, \end{aligned}$$

From $X_+ f_3 = 0$ we deduce that

$$\begin{aligned}
&2u_{10} + u_{11} = 0,\\
&u_{11} + 2u_{12} = 0,\\
&u_{10} = u_{12} = B,\\
&u_{11} = -2B,\\
&3u_7 + u_8 = X_+ u_{10},\\
&2u_8 + 2u_9 = X_+ u_{11},\\
&u_9 = X_+ u_{12},\\
&u_9 = X_+ B,\\
&u_8 = -2X_+ B,\\
&u_7 = X_+ b,\\
&(X_+)^2 B = 0,\\
&X_+ u_7 = X_+ u_8 = X_+ u_9 = 0.
\end{aligned}$$

Finally, $f_1 = f_2 = f_3 = 0$ and, similarly, $f_4 = f_5 = f_6 = 0$.

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References

[A] Arnold V., *Mathematical methods of classical mechanics.* Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein. Corrected reprint of the second (1989) edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, 199?. xvi+516 pp.

[BL] Bernstein J.N., Leites D.A., Invariant differential operators and irreducible representations of Lie superalgebra of vector fields. (in Russian) Serdica 7 (1981), no. 4, 320–334 (1982); translated in: Selecta Math. Soviet. 1 (1981), no. 2, 143–160.

[BoLe] Bouarroudj S. and D. Leites D., Invariant differential operators in positive characteristic. J. Algebra. V. 499, (2018), 281–297; `https://arxiv.org/abs/1605.09500`

[Bu] Buttin C., Theéorie des opérateurs differentiels gradué sur les formes différentielles. Bull. Soc. Math. France, v. 102, 1974, no. 1, 49–73

[E] Epstein D. B. A., Natural tensors on riemannian manifolds, J. Diff. Geometry, v. 10, 1975, no. 4, 631–645

[FF1] Feigin B.L., Fuchs D.B., Skew symmetric invariant differential operators on the line. Functional Anal. Appl. 13 (1979), no. 4, 314–315 (1980).

[FF2] Feigin B. L., Fuchs D. B., Skew-symmetric invariant differential operators on the line and Verma modules over the Virasoro algebra. Functional Anal. Appl. 16 (1982), no. 2, 114–126

[FN] Frölicher A., Nijenhuis A., Theory of vector-valued differential forms. I. Derivations of the graded ring of differential forms. Nederl. Akad. Wetensch. Proc. Ser. A. 59 = Indag. Math. 18 (1956), 338–359; II. Almost-complex structures. Nederl. Akad. Wetensch. Proc. Ser. A 61 = Indag. Math. 20 1958 414–429; id., Some new cohomology invariants for complex manifolds. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 59 = Indag. Math. 18 (1956), 540–552, 553–564; id., A theorem on stability of complex structures. Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 239–241

[G1] Grozman P.Ya., Classification of bilinear invariant operators on tensor fields. (in Russian) Funktsional. Anal. i Prilozhen. 14 (1980), no. 2, 58–59

[G2] Grozman P.Ya., Local invariant bilinear operators over tensor fields on the plane. (in Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1980, no. 6, 3–6, 116.

[G3] Grozman P.Ya., On bilinear invariant differential operators acting on tensor fields on the symplectic manifold. J. Nonlinear Math. Phys. 8 (2001), no. 1, 31–37; `https://arxiv.org/abs/math.SG/0101266`

[GLS] Grozman P., Leites D., Shchepochkina I., Invariant differential operators on supermanifolds and The Standard Model. In: M. Olshanetsky, A. Vainstein (eds.) *Multiple facets of quantization and supersymmetry. Michael Marinov Memorial Volume*, World Sci. Publishing, River Edge, NJ, 2002, 508–555. `https://arxiv.org/abs/math.RT/0202193`; [ESI preprint 1111 (2001)]

[IoMa] Iohara K., Mathieu O., A Global version of Grozman's theorem. Math. Zeitschrift, v. 274, no. 3, (2013) 955–992; `https://arxiv.org/abs/1204.0695`

[Kac] Kac V.G., Lie superalgebras, Adv. Math. 26 (1977), 8–96.

[Ki1] Kirillov A.A., Invariant differential operators on geometric quantities. (Russian) Funkcional. Anal. i Prilozhen. 11 (1977), no. 2, 39–44, 95.

[Ki2] Kirillov A.A., Natural differential operations on tensor fields. (Russian) Akad. Nauk SSSR Inst. Prikl. Mat. Preprint No. 56 (1979), 28 pp.

[Ki3] Kirillov A.A., Invariant operators over geometric quantities. (Russian) Current problems in mathematics, Vol. 16 (Russian), pp. 3–29, 228, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1980; English translation in J. Sov. Math. v. 18, 1992, no. 1, 1–21

[KMS] Kolář, Ivan; Michor, Peter W.; Slovák, Jan, *Natural operations in differential geometry.* Springer-Verlag, Berlin, 1993. vi+434 pp.

[KoI] Kostrikin, I. A. Representations of height 1 of infinite-dimensional Lie algebras of the series K_n. (Russian) Uspekhi Mat. Nauk 34 (1979), no. 1(205), 229–230; id., Irreducible graded representations of Lie algebras of Cartan type. Soviet Math. Dokl. 19 (1978), no. 6, 1369–1371 (1979)

[Ko] Kochetkov Yu., Singular vectors and invariant operators. Russian Math. Surveys 43 (1988), no. 5, 242–243

[Ko1] Kochetkov Yu., Discrete modules for Lie superalgebras of the series Le. In: Onishchik A. (ed.) *Problems in group theory and homological algebra*, Yaroslav. Gos. Univ., Yaroslavl, 1989, 142–148

[Ko2] Kochetkov Yu., Singular vectors, invariant operators and discrete modules for the Lie algebra of Hamiltonian vector fields $H(n)$. Russian Math. Surveys 44, 1989, no. 5, 201–202

[L] Leicher H., Natural operations on covariant tensor fields. J. Differential Geometry 8 (1973), 117–123

[LKW] Leites D.,; Kochetkov, Yu., Weintrob A., New invariant differential operators on supermanifolds and pseudo-(co)homology. , 217–238. In: Andima et. al. (eds.) *General topology and applications* (Staten Island, NY, 1989) Lecture Notes in Pure and Appl. Math., 134, Dekker, New York, 1991

[L1] Leites D. A., Lie superalgebras. J. Soviet Math., v. 30, 1984, no. 6, 2481–2513

[OV] Onishchik A., Vinberg È., *Algebraic groups and Lie groups*, Translated from the Russian and with a preface by D. A. Leites. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. xx+328 pp.

[N1] Nijenhuis A., Jacobi-type identities for bilinear differential concomitants of certain tensor fields. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17 (1955), 390–397, 398–403

[N2] Nijenhuis A., Geometric aspects of formal differential operations on tensors fields. 1960 Proc. Internat. Congress Math. 1958 pp. 463–469 Cambridge Univ. Press, New York

[R1] Rudakov A.N., Irreducible representations of infinite-dimensional Lie algebras of the Cartan type, Mathematics of the USSR-Izvestia 8 (1974), 835–866.

[R2] Rudakov A.N., Irreducible representations of infinite-dimensional Lie algebras of types S and H, Mathematics of the USSR-Izvestia 9 (1975), 465–480.

[P] Palais R. S., Natural operations on differential forms. Trans. Amer. Math. Soc. 92 1959 125–141

[PT] Palais R. S., Terng Chuu Lian, Natural bundles have finite order. Topology 19 (1977), no. 3, 271–277

[S1] Schouten J. A., Über Differentialkomitanten zweier kontravarianter Grössen. (German) Nederl. Akad. Wetensch., Proc. 43, (1940), 449–452

[S2] Schouten J. A., *Ricci-calculus. An introduction to tensor analysis and its geometrical applications.* 2d. ed. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd X. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. xx+516 pp.; id., On the differential operators of first order in tensor calculus. Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 1–7. Edizioni Cremonese, Roma, 1954; id., *Tensor Analysis for Physicists.* Oxford, At the Clarendon Press, 1951. x+275 pp.

[SS] Schouten I.A., Struik D.J., Einfuhrung in die neueren Methoden der Differentialgeometrie, Noordhoff, 1935.

[T] Terng Chuu Lian, Natural vector bundles and natural differential operators. Amer. J. Math. 100 (1978), no. 4, 775–828

[Tu] Tulczyjew W. M., The graded Lie algebra of multivector fields and the generalized Lie derivative of forms. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 22 (1974), 937–942

[V] Veblen O., *Differential invariants and geometry*, Atti del Congresso Internazionale dei Matematici, Bologna, 1928.

[WB] Wess J., Bagger J., *Supersymmetry and supergravity*, Princeton Univ. Press, 1983

[W] Weyl H., *The classical groups. Their invariants and representations.* Fifteenth printing. Princeton Landmarks in Mathematics. Princeton Paperbacks. Princeton University Press, Princeton, NJ, 1997. xiv+320 pp.

[Zh] Zhelobenko, D. P. *Compact Lie groups and their representations* (Russian), Nauka, Moscow, 1970. 664 pp. [Translated from the Russian by Israel Program for Scientific Translations. Translations of Mathematical Monographs, Vol. 40. American Mathematical Society, Providence, R.I., 1973. viii+448 pp.]