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An explicit bound of integral points on modular curves

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Abstract. In this paper, we give the constant C in [9, Theorem 1.2] by using an explicit Baker's inequality, hence we obtain an explicit bound for the heights of the integral points on modular curves.

1 Introduction

Let X be a smooth, connected projective algebraic curve defined over a number field K, and let $x \in K(X)$ be a non-constant rational function on X. If S is a finite set of places of K (including all the infinite places), we call a point $P \in X(K)$ an S-*integral point* if $x(P) \in \mathcal{O}_S$, where $\mathcal{O}_S = \mathcal{O}_{S,K}$ is the ring of S-integers in K. The set of S-integral points is denoted by $X(\mathcal{O}_S, x)$.

According to the classical theorem of Siegel [11] the set $X(\mathcal{O}_S, x)$ is finite if at least one of the following conditions is satisfied:

$$\text{the genus } g(X) \geq 1; \tag{1}$$

$$x \text{ admits at least 3 poles in } X(\bar{\mathbb{Q}}). \tag{2}$$

Unfortunately, the existing proofs of this theorem for general curves are not effective, that is they do not imply any explicit expression bounding the heights of integral points. But for many pairs (X, x), the effective proofs of this theorem were discovered by Baker's method, see [1], [2] and the references therein.

Sha [9] considered the case where $X = X_\Gamma$ is the modular curve corresponding to a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, and $x = j$ is the j-invariant.

To state his result, we introduce some notations. For a congruence subgroup Γ as above, the number of cusps on X_Γ is denoted by $v_\infty(\Gamma)$. For a number field K, let M_K be

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the set of all places of K, and $S \subseteq M_K$ a finite subset containing all infinite places. We put $d = [K : \mathbb{Q}]$ and $s = |S|$. Let $\mathcal{O}_K$ be the ring of integers of K. We define the following quantity

$$\Delta(N) := \sqrt{N^{dN}|D|^{\varphi(N)}}(\log(N^{dN}|D|^{\varphi(N)}))^{d\varphi(N)} \times \left(\prod_{\substack{v\in S\\ v\nmid\infty}} \log\mathcal{N}_{K/\mathbb{Q}}(v)\right)^{\varphi(N)}$$

as a function of $N \in \mathbb{N}^+$, where D is the absolute discriminant of K, $\varphi(N)$ is Euler's totient function, and the norm $\mathcal{N}_{K/\mathbb{Q}}(v)$ of a place v, by definition, is equal to $|\mathcal{O}_K/\mathfrak{p}_v|$ when v is finite and $\mathfrak{p}_v$ is its corresponding prime ideal, and is set to be 1 if v is infinite.

Sha [9] proved the following theorem.

Theorem 1.1 ([9] Theorem 1.2). *Let Γ be of level N. If $v_\infty(\Gamma) \geq 3$, then*

$$\mathrm{h}(j(P)) \leq (CdsM^2)^{2sM}(\log(dM))^{3sM}\ell^{dM}\Delta(M) \quad \text{for every} \quad P \in X_\Gamma(\mathcal{O}_S, j),$$

where C is an absolute effective constant, ℓ is the maximal prime such that there exists $v \in S$ with $v|\ell$, or $\ell = 1$ if S only contains infinite places, and M is defined as following:

$$M = \begin{cases} N & \text{if } N \text{ is not a power of any prime;} \\ 3N & \text{if } N \text{ is a power of } 2; \\ 2N & \text{if } N \text{ is a power of an odd prime.} \end{cases}$$

(Here $\mathrm{h}(\cdot)$ is the standard absolute logarithmic height defined on the set $\bar{\mathbb{Q}}$ of algebraic numbers.)

For certain applications it is useful to have an explicit value of the constant C from Theorem 1.1. In this note we prove the following result.

Theorem 1.2. *The constant C in Theorem 1.1 can be taken to be 2^{14}.*

In the proof, we follow the main lines of Sha's argument, with some minor modifications. We calculate explicitly the implicit constants occurring therein.

For a number field K, $v \in M_K$, we define the valuation $|\cdot|_v$ on K as following: for any $\alpha \in K$:

$$|\alpha|_v := |\sigma(\alpha)|, \text{ if } v \text{ is infinite with embedding } \sigma;$$
$$|\alpha|_v := \mathcal{N}_{K/\mathbb{Q}}(v)^{-\operatorname{ord}_v(\alpha)/[K_v:\mathbb{Q}_v]}, \text{ if } v \text{ is finite.}$$

2 Upper bound of S-regulator

As before, for a number field K, and a finite subset $S \subseteq M_K$ containing all infinite places, we put $d = [K : \mathbb{Q}]$, $s = |S|$ and $r = s - 1$. We fix $v_0 \in S$ and set

$$S' = S \setminus \{v_0\} = \{v_1, \ldots, v_r\}.$$

The S-regulator $R(S)$ is defined as

$$R(S) = |\det(d_{v_i} \log |\xi_k|_{v_i})_{1\le i,k\le r}|,$$

where $d_{v_i} = [K_{v_i} : \mathbb{Q}_{v_i}]$ is the local degree of v_i for each i, and $\{\xi_1, \cdots, \xi_r\}$ is a fundamental system of the S-units. It is independent of the choice of v_0 and of the fundamental system of S-units. We also denote by ω_K the number of roots of unity in K.

We set

$$\zeta = \begin{cases} \dfrac{(\log 6)^3}{2} & \text{if } d = 2, \\ 4\left(\dfrac{\log d}{\log\log d}\right)^3 & \text{if } d \ge 3. \end{cases}$$

This ζ is better than the one in [9, Proposition 4.1 and Corollary 4.2], and can make these results valid, see [12, Theorem and Corollary 2].

Lemma 2.1. *We have*

$$0.1 \le R(S) \le h_K R_K \prod_{\substack{v\in S\\ v\nmid\infty}} \log \mathcal{N}_{K/\mathbb{Q}}(v),$$

$$R(S) \le \frac{\omega_K}{2}\left(\frac{2}{\pi}\right)^{r_2}\left(\frac{e\log|D|}{4(d-1)}\right)^{d-1}\sqrt{|D|}\prod_{\substack{v\in S\\ v\nmid\infty}} \log \mathcal{N}_{K/\mathbb{Q}}(v),$$

where e is the base of the natural logarithm, r_2 is the number of complex embeddings of K, and D is the absolute discriminant of K.

Proof. For the first inequality see [4, Lemma 3]. One may remark that the lower bound $R(S) \ge 0.1$ follows from Friedman's famous lower bound [5, Theorem B] for the usual regulator R_K. The second one follows from Siegel's estimate [10], or [6, Theorem 1]

$$h_K R_K \le \frac{\omega_K}{2}\left(\frac{2}{\pi}\right)^{r_2}\left(\frac{e\log|D|}{4(d-1)}\right)^{d-1}\sqrt{|D|}\prod_{\substack{v\in S\\ v\nmid\infty}} \log \mathcal{N}_{K/\mathbb{Q}}(v),$$

here, we replace $(1/(d-1))^{d-1}$ with 1 when $d = 1$. □

We will use the following lemma. For the convenience of the readers, we prove it here.

Lemma 2.2. $\omega_K \le 2d^2$. *Moreover,* $\omega_K \le d^2$ *if K contains a primitive n-th root of unity for some $n > 6$.*

Proof. It's sufficient to show that $\varphi(n) \geq \sqrt{n}$ for $n \neq 2, 6$.

For $k \geq 1$, set $f_k(x) := x^k - x^{k-1} - x^{k/2}$, $g_k(x) := x^k - x^{k-1} - \sqrt{2}x^{k/2}$. Then

$$f_k(x) = x^{(k-1)/2}(x^{(k-1)/2}(x-1) - x^{1/2}) \geq x - 1 - x^{1/2} > 0,$$

if $x \geq 3$. Similarly, $g_k(x) > 0$ if $x \geq 5$ or $k \geq 2, x \geq 3$.

Let $n = 2^m \prod_p p^{e_p}$, where p runs through all odd prime numbers. If $m = 0$, then

$$\varphi(n) = \prod_{e_p \geq 1} (p^{e_p} - p^{e_p - 1}) \geq \prod_{e_p \geq 1} p^{e_p/2} = \sqrt{n}.$$

It is similar for the case where $m \geq 2$.

If $m = 1$, then there exists a prime q such that $q \geq 5, e_q \geq 1$ or $q = 3, e_q \geq 2$. Hence

$$\varphi(n) = \prod_{e_p \geq 1} (p^{e_p} - p^{e_p - 1}) \geq \sqrt{2} q^{e_q/2} \prod_{\substack{p \neq q \\ e_p \geq 1}} p^{e_p/2} = \sqrt{n}. \qquad \square$$

3 Baker's inequality

In this section, we state Baker's inequality in an explicit form.

Theorem 3.1 (Baker's inequality). *Let n be an integer not less than 2, K be a number field of degree d, $\alpha_1, \cdots, \alpha_n \in K^*$, and $b_1, \cdots, b_n \in \mathbb{Z}$ such that $\alpha_1^{b_1} \cdots \alpha_n^{b_n} \neq 1$. We define $A_1, \cdots, A_n, B_0$ by*

$$\log A_i := \max\{\mathrm{h}(\alpha_i), 1/d\}, \quad 1 \leq i \leq n;$$

$$B_0 := \max\{3, |b_1|, \cdots, |b_n|\}.$$

Then for any $v \in M_K$, we have

$$|\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1|_v \geq \exp\{-\Upsilon \log A_1 \cdots \log A_n \log B_0\}, \tag{3}$$

where

$$\Upsilon = \begin{cases} 2^{8n+29} d^{n+2} \log(ed) & \text{if } v | \infty, \\ 2^{10n+10} \cdot e^{2n+2} d^{3n+3} p_v^d & \text{if } v | p_v < \infty. \end{cases} \tag{4}$$

The proof of this theorem is based on [7, Corollary 2.3] and [13, Main Theorem, page 190-191]. For the convenience of readers, we state their results here.

As convention, for a nonzero element $z \in \mathbb{C}$ we set

$$\log z = \log |z| + \sqrt{-1} \arg z,$$

where $-\pi < \arg z \leq \pi$ is the principal argument of z.

Theorem 3.2 ([7, Corollary 2.3])**.** *Let $n \in \mathbb{N}^+$, let K be a number field of degree d, and let $\alpha_1, \cdots, \alpha_n \in K^*$. Let $b_1, \cdots, b_n \in \mathbb{Z}$ be such that $\Lambda := b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n \neq 0$. We define $A_1^*, \cdots, A_n^*, B$ by*

$$\log A_i^* = \max\{\mathrm{h}(\alpha_i), \frac{|\log \alpha_i|}{d}, \frac{0.16}{d}\}, \quad 1 \le j \le n,$$

$$B = \max\{3, \frac{|b_j| \log A_j^*}{\log A_n^*} : 1 \le j \le n\}.$$

Then

$$\log |\Lambda| \ge -C(n, \varkappa) d^{n+2} \log(ed) \log A_1^* \cdots \log A_n^* \log(eB),$$

where $C(n, \varkappa) = \min\{\frac{1}{\varkappa}(\frac{1}{2}en)^{\varkappa} 30^{n+3} n^{3.5}, 2^{6n+20}\}$,

$$\varkappa = \begin{cases} 1 & \text{if } \alpha_1, \cdots, \alpha_n \in \mathbb{R}, \\ 2 & \text{otherwise.} \end{cases}$$

Theorem 3.3 ([13] consequence of Main Theorem)**.** *Keep the notation of Theorem 3.2. We define $A_1, \cdots, A_n, B_0$ by*

$$\log A_i = \max\{\mathrm{h}(\alpha_i), \frac{1}{16e^2d^2}\}, \quad 1 \le i \le n,$$

$$B_0 = \max\{3, |b_1|, \cdots, |b_n|\}.$$

Then for any prime number p, and any prime ideal $\mathfrak{p}$ over p in the ring of integers of $\mathbb{Q}(\alpha_1, \cdots, \alpha_n)$, we have

$$\mathrm{ord}_{\mathfrak{p}}(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) < C_0(n, d, \mathfrak{p}) \log A_1 \cdots \log A_n \log B_0,$$

where $C_0(n, d, \mathfrak{p}) = (16ed)^{2(n+1)} n^{5/2} \log(2nd) \log(2d) \cdot e_{\mathfrak{p}}^n \frac{p^{f_{\mathfrak{p}}}}{(f_{\mathfrak{p}} \log p)^2}$, and $e_{\mathfrak{p}}, f_{\mathfrak{p}}$ are the ramification index and the residue degree at $\mathfrak{p}$ respectively.

Now we prove Theorem 3.1. The idea comes from [14, Section 9.4.4].

Proof of Theorem 3.1. If $v | p_v$ for some prime p_v, then from Theorem 3.3, we have

$$|\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1|_v > \exp\{-C_1(n, d, \mathfrak{p}) \log A_1 \cdots \log A_n \log B_0\},$$

where

$$C_1(n, d, \mathfrak{p}) = (\frac{\log p_v}{e_{\mathfrak{p}}}) C_0(n, d, \mathfrak{p}) = (16ed)^{2(n+1)} n^{5/2} \log(2nd) \log(2d) \cdot e_{\mathfrak{p}}^{n-1} \frac{p_v^{f_{\mathfrak{p}}}}{f_{\mathfrak{p}}^2 \log p_v}.$$

We have

$$\begin{aligned} C_1(n, d, \mathfrak{p}) &\le (16e)^{2(n+1)} d^{2n+2} n^{5/2} \cdot 2nd \cdot 2d \cdot d^{n-1} \cdot p_v^d \\ &\le 2^{10n+10} \cdot e^{2n+2} d^{3n+3} p_v^d, \end{aligned}$$

since $n^{7/2} \leq 4^n$.

If $v|\infty$, it is sufficient to bound $|\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1|$. When $|z| \leq 1/2$, the function $\frac{\log(1+z)}{z}$ is holomorphic, then by the maximal modulus principle, there exists z_0 with $|z_0| = 1/2$ such that

$$\left|\frac{\log(1+z)}{z}\right| \leq 2|\log(1+z_0)| \leq 2\log 2,$$

where we use the inequality $|\log(1+z_0)| = |\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z_0^n| \leq \sum_{n=1}^{\infty} \frac{1}{n}|z_0|^n = \log 2$. Hence, for $|z| \leq 1/2$,

$$|\log(1+z)| \leq 2\log 2|z| \leq 2|z|. \tag{5}$$

To prove Theorem 3.1, without loss of generality, we may assume that $b_i \neq 0$ for $1 \leq i \leq n$, and $A_1 \leq \cdots \leq A_n$, and set $\alpha = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1$. We need to consider three cases.

(a) If $B_0 \leq 2nd$, with Liouville's inequality, we have

$$\mathrm{h}(\alpha) \leq \log 2 + \sum_{i=1}^{n} |b_i| \, \mathrm{h}(\alpha_i),$$

$$\log|\alpha| \geq -d\,\mathrm{h}(\alpha) \geq -d(\log 2 + nB_0 \log A_n).$$

Hence,

$$|\alpha| \geq \exp\{-(d\log 2 + 2n^2 d^2 \log A_n)\}.$$

Since $1 \leq d \log A_i$ for $1 \leq i \leq n$, and $\log 2 + 2n^2 \leq 2^{8n+29}\log(ed)$, we have

$$d\log 2 + 2n^2 d^2 \log A_n \leq (\log 2 + 2n^2) d^2 \log A_n \leq \Upsilon \log A_1 \cdots \log A_n \log B_0.$$

Hence we have inequality (3).

(b) If $B_0 > 2nd$, and $|\alpha| > 1/2$, since $\log 2 \leq 2^{8n+29}\log(ed)$, it is easy to deduce inequality (3) from this.

(c) If $B_0 > 2nd$, and $|\alpha| \leq 1/2$, this is the main part of the proof. By (5), we have

$$|\alpha| \geq \frac{1}{2}|\log(1+\alpha)| = \frac{1}{2}|\log(\alpha_1^{b_1} \cdots \alpha_n^{b_n})| = \frac{1}{2}|\Lambda|,$$

where $\Lambda = b_0 \log(-1) + b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n$, $b_0 = 2k$ for some integer k. Hence, it is sufficient to bound $|\Lambda|$.

To use Theorem 3.2, for $1 \leq i \leq n$ we set

$$\log A_i^* = \sqrt{\pi^2 + 1} \cdot \log A_i,$$

$$\log A_0^* = \frac{\pi}{d},$$

$$B = B_0^2.$$

We will show that for $1 \leq i \leq n$, we have

$$\log A_i^* \geq \max\{\mathrm{h}(\alpha_i), \frac{|\log \alpha_i|}{d}, \frac{0.16}{d}\},$$

$$\log A_0^* \geq \max\{\mathrm{h}(-1), \frac{|\log(-1)|}{d}, \frac{0.16}{d}\} = \frac{\pi}{d},$$

$$B \geq \max\{3, \frac{|b_j| \log |A_j^*|}{\log A_n^*} : 0 \leq j \leq n\}.$$

Indeed, notice that for $1 \leq i \leq n$, $\log A_i^* \geq \frac{0.16}{d}$, and we have

$$|\log \alpha_i|^2 \leq \pi^2 + (\log |\alpha_i|)^2,$$

$$\frac{\log |\alpha_i|}{d} \leq \mathrm{h}(\alpha_i) \leq \log A_i < \log A_i^*,$$

so

$$|\log \alpha_i| \leq (\pi^2 + d^2(\log A_i)^2)^{1/2} \leq \sqrt{\pi^2+1} \cdot d \log A_i.$$

For $\log A_0^*$, it's obvious. For B, obviously $B \geq 3$. Before showing that $B \geq \frac{|b_j| \log |A_j^*|}{\log A_n^*}$ for $0 \leq j \leq n$, we bound b_0 first. Since $|\alpha| \leq 1/2$, so $|\Lambda| \leq 1$ and

$$\begin{aligned}\pi|b_0| &\leq |\Lambda| + |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| \\ &\leq 1 + nB_0\sqrt{\pi^2+1}d \log A_n \\ &\leq 2\pi n d B_0 \log A_n,\end{aligned}$$

here we use the fact that $\sqrt{\pi^2+1} \leq \pi + 1$, $1 \leq (\pi - 1)ndB_0 \log A_n$. Since $B_0 > 2nd \geq 2n$, we have $B = B_0^2 > 2nB_0$,

$$\frac{|b_0| \log A_0^*}{\log A_n^*} = \frac{\pi |b_0|}{\sqrt{\pi^2+1} \cdot d \log A_n} \leq \frac{2\pi}{\sqrt{\pi^2+1}} nB_0 < 2nB_0 < B,$$

$$\frac{|b_i| \log A_i^*}{\log A_n^*} = \frac{|b_i| \log A_i}{\log A_n} \leq |b_i| \leq B_0 < B$$

for $1 \leq i \leq n$.

By Theorem 3.2, we have

$$\begin{aligned}\log |\Lambda| &\geq -C(n+1, \varkappa) d^{n+3} \log(ed) \log A_0^* \log A_1^* \cdots \log A_n^* \log(eB) \\ &\geq -3\pi(\pi^2+1)^{n/2} C(n+1, \varkappa) d^{n+2} \log(ed) \cdot \log A_1 \cdots \log A_n \log B_0,\end{aligned}$$

and

$$\begin{aligned}|\alpha| &\geq \frac{1}{2}|\Lambda| \\ &\geq \exp\{-(3\pi(\pi^2+1)^{n/2} C(n+1, \varkappa) + \log 2) d^{n+2} \log(ed) \cdot \log A_1 \cdots \log A_n \log B_0\}.\end{aligned}$$

Hence, it is sufficient to show that

$$3\pi(\pi^2+1)^{n/2} C(n+1, \varkappa) + \log 2 \leq 2^{2n+3} C(n+1, \varkappa) \leq 2^{8n+29}.$$

Indeed,

$$\begin{aligned}2(\pi^2+1)^{n/2}\left(4\cdot\left(\frac{4}{\sqrt{\pi^2+1}}\right)^n-\frac{3}{2}\pi\right)C(n+1,\varkappa)\\ &\geq 2(\pi^2+1)^{1/2}\left(\frac{16}{\sqrt{\pi^2+1}}-\frac{3}{2}\pi\right)C(2,\varkappa)\\ &\geq 0.92\cdot C(2,\varkappa)\\ &\geq \log 2,\end{aligned}$$

since $0.92\cdot C(2,\varkappa)\geq 0.92\cdot\min\{2^{2.5}e\cdot 30^5, 2^{32}\}\geq\log 2$. □

The following lemma will be used when we apply Theorem 3.1.

Lemma 3.4 ([8, Lemma 2.2]). *Let $b\geq 0, h\geq 1, a>(e^2/h)^h$, and let $x\in\mathbb{R}^+$ be such that*

$$x-a(\log x)^h-b\leq 0,$$

then $x<2^h(b^{1/h}+a^{1/h}\log(h^h a))^h$. In particular, if $h=1$, then $x<2(b+a\log a)$.

4 Proof of Theorem 1.2

We only consider the case of mixed level, i.e Theorem 1.1, since if N is a power of some prime p, we can replace N by $3N$ if $p=2$, and by $2N$ if $p\neq 2$. From the assumption, we have that $N\geq 6$.

We consider the case where $\mathbb{Q}(\zeta_N)\subset K$ at first, then consider the general case.

For $P\in X_\Gamma(\mathcal{O}_S,j)$, since $j(P)\in\mathcal{O}_S$, we have

$$\mathrm{h}(j(P))=d^{-1}\sum_{v\in S}d_v\log^+|j(P)|_v\leq\sum_{v\in S}\log^+|j(P)|_v\leq s\log|j(P)|_w,$$

for some $w\in S$. Hence, it suffices to bound $\log|j(P)|_w$.

If $|j(P)|_w\leq 3500$, then $\mathrm{h}(j(P))\leq 16s$, which is a better bound than that given in Theorem 1.1 when $C=2^{14}$.

If $|j(P)|_w>3500$, then by [9, Proposition 3.3] or [3, Proposition 3.1], we have $P\in\Omega_{c,w}$ for some cusp c, and $|j(P)|_w\leq 2|q_w(P)^{-1}|_w$, where $\Omega_{c,w}$ and q_w are defined in [3, Section 3]. Hence, we only need to bound $\log|q_w(P)^{-1}|_w$.

Notice that, if $|q_w(P)|_w>10^{-N}$, then $\log|j(P)|_w\leq 2N\log 10$ and $\mathrm{h}(j(P))<6sN$, which is better than that given in Theorem 1.1 when $C=2^{14}$.

In the sequel, we consider the case where $P\in\Omega_{c,w}$ and $|q_w(P)|_w\leq 10^{-N}$.

By the statements in [9, Page 4507-4508], there exists a modular unit W on X_Γ which is integral over $\mathbb{Z}[j]$, and a constant $\gamma_w\in\mathbb{Q}(\zeta_N)$ such that

$$|\gamma_w^{-1}W(P)-1|_w\leq 4^{24N^7}|q_w(P)|_w^{1/N},$$

$$\mathrm{h}(\gamma_w) \le 24N^7 \log 2,$$

and $W(P)$ is a unit of $\mathcal{O}_S$. Hence $W(P) = \omega\eta_1^{b_1}\cdots\eta_r^{b_r}$ for some $b_1,\cdots,b_r \in \mathbb{Z}$, where ω is a root of unity and $\{\eta_1,\cdots,\eta_r\}$ is a fundamental system of S-units from [9, Proposition 4.1]. We set

$$\Lambda = \gamma_w^{-1}W(P) = \eta_0\eta_1^{b_1}\cdots\eta_r^{b_r},$$

where $\eta_0 = \omega\gamma_w^{-1}$. Then we have

$$|\Lambda - 1|_w \le 4^{24N^7}|q_w(P)|_w^{1/N}. \tag{6}$$

If $\Lambda \ne 1$, we will use this upper bound and the lower bound from Theorem 3.1 to get a bound of $|q_w(P)|_w$ which gives an upper bound of $\mathrm{h}(j(P))$. For the case where $\Lambda = 1$, see [9, Section 8].

To state the following lemma, we set $r^r = 1$ when $r = 0$, i.e $s = 1$.

Lemma 4.1. *If $\mathbb{Q}(\zeta_N) \subset K$ and $\Lambda \ne 1$, then we have*

$$\mathrm{h}(j(P)) \le 40dsr^{2r}\zeta^r N^8\tilde{\Upsilon}R(S)\log(d^2sr^{4r}\zeta^s N^{16}\tilde{\Upsilon}R(S)),$$

where $\tilde{\Upsilon} = 2^{13s+22}d^{2s+3}\ell^d$, and ζ has been defined in Section 2.

Proof. We define $A_0,\cdots,A_r,B_0$ by

$$\log A_i := \max\{\mathrm{h}(\eta_i), 1/d\}, 0 \le i \le r;$$

$$B_0 := \max\{3, |b_1|,\cdots,|b_r|\}.$$

Since $\Lambda = \eta_0\eta_1^{b_1}\cdots\eta_r^{b_r} \ne 1$, by Theorem 3.1, we have

$$|\Lambda - 1|_w \ge \exp\{-\Upsilon \log A_0 \cdots \log A_r \log B_0\},$$

where

$$\Upsilon = \begin{cases} 2^{8s+29}d^{s+2}\log(ed), & \text{if } w|\infty, \\ 2^{10s+10}\cdot e^{2s+2}d^{3s+3}p_w^d, & \text{if } w|p_w < \infty. \end{cases} \tag{7}$$

Obviously $2^{10s+19}\cdot 2^{3s+3}d^{3s+3}\ell^d = 2^{13s+22}d^{3s+3}\ell^d$ is larger than Υ in each one of the cases since $d \ge 2, s \ge 1$, so we can take $\Upsilon = 2^{13s+22}d^{3s+3}\ell^d$.

By (6), we have

$$\exp\{-\Upsilon \log A_0 \cdots \log A_r \log B_0\} \le 4^{24N^7}|q_w(P)|_w^{1/N},$$

that is

$$\log|q_w(P)^{-1}|_w \le N\Upsilon \log A_0 \cdots \log A_r \log B_0 + 48N^8 \log 2. \tag{8}$$

By [9, Proposition 4.1], we have $\zeta\,\mathrm{h}(\eta_k) \ge 1/d$ and $\zeta \ge 1$, so

$$\log A_k \le \zeta\,\mathrm{h}(\eta_k), \quad k = 1,\cdots,r,$$

$$\log A_1 \cdots \log A_r \leq d^{-r} r^{2r} \zeta^r R(S).$$

Notice that the both sides are 1 when $r = 0$. On the other hand, since

$$\mathrm{h}(\eta_0) = \mathrm{h}(\gamma_w) \leq 24N^7 \log 2,$$

we have

$$\log A_0 \leq 24N^7 \log 2.$$

For B_0, we set $B^* = \max\{|b_1|, \cdots, |b_r|\}$ if $r \geq 1$, and $B^* = 0$ if $r = 0$. By [9, Corollary 4.2 and Proposition 6.1] we have

$$\begin{aligned} B^* &\leq 2dr^{2r}\zeta \mathrm{h}(W(P)) \\ &\leq 2dr^{2r}\zeta(2sN^8 \log |q_w^{-1}(P)|_w + 94sN^8 \log N), \end{aligned} \tag{9}$$

so

$$B_0 \leq 2dr^{2r}\zeta(2sN^8 \log |q_w^{-1}(P)|_w + 94sN^8 \log N).$$

We write

$$\alpha = 4dsr^{2r}\zeta N^8,$$

$$\beta = 188dsr^{2r}\zeta N^8 \log N = 47\alpha \log N,$$

$$C_1 = \alpha N \Upsilon \log A_0 \cdots \log A_r,$$

$$C_2 = 48\alpha N^8 \log 2 + \beta.$$

Hence, inequalities (8) and (9) yield

$$\alpha \log |q_w(P)^{-1}|_w + \beta \leq C_1 \log(\alpha \log |q_w(P)^{-1}|_w + \beta) + C_2.$$

By Lemma 3.4, we obtain

$$\alpha \log |q_w(P)^{-1}|_w + \beta \leq 2(C_1 \log C_1 + C_2).$$

Hence,

$$\log |q_w(P)^{-1}|_w \leq 2\alpha^{-1} C_1 \log C_1 + \alpha^{-1}(2C_2 - \beta),$$

$$\log |j(P)|_w \leq \log 2 |q_w(P)^{-1}|_w \leq 2\alpha^{-1} C_1 \log C_1 + \alpha^{-1}(2C_2 - \beta) + \log 2,$$

so we have

$$\mathrm{h}(j(P)) \leq 2s\alpha^{-1} C_1 \log C_1 + s\alpha^{-1}(2C_2 - \beta) + s \log 2.$$

Next we bound each term on the right-hand side:

$$\begin{aligned} &2s\alpha^{-1} C_1 \log C_1 \\ &\qquad = 2sN\Upsilon \log A_0 \cdots \log A_r \log(4dsr^{2r}\zeta N^9 \Upsilon \log A_0 \cdots \log A_r) \\ &\qquad \leq 48 \log 2 \cdot d^{-r} sr^{2r} \zeta^r N^8 \Upsilon R(S) \log(96 \log 2 \cdot d^{-r+1} sr^{4r} \zeta^{r+1} N^{16} \Upsilon R(S)) \\ &\qquad \leq 39 d^{-r} sr^{2r} \zeta^r N^8 \Upsilon R(S) \log(d^{-r+1} sr^{4r} \zeta^{r+1} N^{16} \Upsilon R(S)), \end{aligned}$$

here we use the fact that $48\log 2 \times \log(96\log 2) \le 140 < 5\log(d^{-r+1}\Upsilon)$; we also have

$$\begin{aligned} s\alpha^{-1}(2C_2-\beta)+s\log 2 &= 96\log 2\cdot sN^8+47s\log N+s\log 2\\ &\le 98\log 2\cdot sN^8. \end{aligned}$$

After replacing $d^{-s}\Upsilon = 2^{13s+22}d^{2s+3}\ell^d$ by $\tilde{\Upsilon}$, we have

$$\mathrm{h}(j(P)) \le 40dsr^{2r}\zeta^r N^8\tilde{\Upsilon}R(S)\log(d^2sr^{4r}\zeta^s N^{16}\tilde{\Upsilon}R(S)). \qquad \square$$

We will use the bound $\zeta \le 2^{13}(\log d)^3$ subsequently. If $d=2$,

$$\zeta = \frac{(\log 6)^3}{2} = \frac{(\log_2 6)^3}{2}(\log d)^3 \le 2^4(\log d)^3;$$

if $d \ge 3$, then

$$\zeta = 4\left(\frac{\log d}{\log\log d}\right)^3 \le 4\left(\frac{\log d}{\log\log 3}\right)^3 \le 4809(\log d)^3 \le 2^{13}(\log d)^3.$$

By Lemma 2.1 and Lemma 2.2, we have

$$R(S) \le \frac{\omega_K}{2}\frac{1}{(d-1)^{d-1}}(\log|D|)^{d-1}\sqrt{|D|}\prod_{\substack{v\in S\\ v\nmid\infty}}\log\mathcal{N}_{K/\mathbb{Q}}(v),$$

$$\omega_K \le 2d^2,$$

$$\log R(S) \le \log(\frac{\omega_K}{2}) + d\log|D| + s\log(d\ell) \le 2\log d + d\log|D| + s\log(d\ell).$$

We have $d \le 2s$ and $\log s \le s/2$. Then we have

$$\begin{aligned} \log\tilde{\Upsilon} &= (13s+22)\log 2 + (2s+3)\log d + d\log\ell\\ &\le (15s+25)\log 2 + (2s+3)\log s + d\log\ell\\ &\le 28s + (s+2)s + s\ell\\ &\le 32s^2\ell \end{aligned}$$

and

$$\begin{aligned} &\log(d^2sr^{4r}\zeta^{r+1}N^{16}\tilde{\Upsilon}R(S))\\ &\quad\le 2\log d + 4s\log s + 13s\log 2 + 3s\log\log d + 16\log N\\ &\qquad + \log\tilde{\Upsilon} + 2\log d + d\log|D| + s\log(d\ell)\\ &\quad\le 2s + 2s^2 + 10s + 2s^2 + 16\log N + 32s^2\ell + 2s + 2s\log|D| + s^2\ell\\ &\quad\le 8N + 2s\log|D| + 51s^2\ell\\ &\quad\le 61s^2\ell N\log|D|\\ &\quad\le 2^6s^2N\ell\log|D|. \end{aligned}$$

Hence combining with Lemma 4.1, we have

$$\begin{aligned}
&\mathrm{h}(j(P))\\
&\quad\le 2^6\cdot ds^{2s-1}\zeta^r N^8\tilde{\Upsilon}R(S)\log(d^2sr^{4r}\zeta^{r+1}N^{16}\tilde{\Upsilon}R(S))\\
&\quad\le 2^{26s+15}\cdot d^{2s+4}(\log d)^{3r}s^{2s-1}N^8\ell^d\frac{\omega_K}{2}\frac{1}{(d-1)^{d-1}}(\log|D|)^{d-1}\sqrt{|D|}\\
&\qquad\cdot\prod_{\substack{v\in S\\ v\nmid\infty}}\log\mathcal{N}_{K/\mathbb{Q}}(v)\cdot(2^6s^2N\ell\log|D|)\\
&\quad=2^{26s+20}d^{2s+4}(\log d)^{3r}s^{2s+1}N^9\ell^{d+1}\omega_K\frac{1}{(d-1)^{d-1}}(\log|D|)^d\sqrt{|D|}\prod_{\substack{v\in S\\ v\nmid\infty}}\log\mathcal{N}_{K/\mathbb{Q}}(v). \qquad (10)
\end{aligned}$$

Next we deal with the general case. Set $\widetilde{K}=K\cdot\mathbb{Q}(\zeta_N)=K(\zeta_N)$. Let $\widetilde{S}$ be the set consisting of the extensions of the places from S to $\widetilde{K}$, that is,

$$\widetilde{S}=\{\tilde{v}\in M_{\widetilde{K}}:\tilde{v}|v,v\in S\}.$$

Then $P\in X_\Gamma(\mathcal{O}_{\widetilde{S}},j)$. Put $\tilde{d}=[\widetilde{K}:\mathbb{Q}]$, $\tilde{s}=|\widetilde{S}|$, $\tilde{r}=\tilde{s}-1$, and let $\widetilde{D}$ be the absolute discriminant of $\widetilde{K}$.

Lemma 4.2.

$$\begin{gathered}
N-\varphi(N)\ge 4,\\
\tilde{s}\le s\varphi(N),\\
\tilde{d}\le d\varphi(N),\\
\omega_{\widetilde{K}}\le 2d^2\varphi(N)^2,\\
|\widetilde{D}|\le N^{dN}|D|^{\varphi(N)},\\
\prod_{\substack{v\in\widetilde{S}\\ v\nmid\infty}}\log\mathcal{N}_{\widetilde{K}/\mathbb{Q}}(v)\le 4^{s\varphi(N)}\left(\prod_{\substack{v\in S\\ v\nmid\infty}}\log\mathcal{N}_{K/\mathbb{Q}}(v)\right)^{\varphi(N)}.
\end{gathered}$$

Proof. The first three inequalities come directly from the definition of $\widetilde{K}$ and $\widetilde{S}$ and $N\ge 6$ has at least two prime factors. The fourth inequality comes from $\omega_{\widetilde{K}}\le 2\tilde{d}^2\le 2d^2\varphi(N)^2$.

Let $D_{\widetilde{K}/K}$ be the relative discriminant of $\widetilde{K}/K$. We have

$$\widetilde{D}=\mathcal{N}_{K/\mathbb{Q}}(D_{\widetilde{K}/K})D^{[\widetilde{K}:K]}.$$

We denote by $\mathcal{O}_K$ and $\mathcal{O}_{\widetilde{K}}$ the ring of integers of K and $\widetilde{K}$, respectively. Since $\widetilde{K}=K(\zeta_N)$, we have

$$\mathcal{O}_K\subset\mathcal{O}_K(\zeta_N)\subset\mathcal{O}_{\widetilde{K}}.$$

Note that the absolute value of the discriminant of the polynomial $x^N - 1$ is N^N, we obtain

$$D_{\widetilde{K}/K} | N^N,$$

so

$$|\mathcal{N}_{K/\mathbb{Q}}(D_{\widetilde{K}/K})| \leq N^{dN}.$$

Hence,

$$|\widetilde{D}| \leq N^{dN} |D|^{\varphi(N)}.$$

Notice that $\widetilde{K}/K$ is Galois. Let v be a non-Archimedean place of K, and let $v_1, \ldots, v_g$ be all its extensions to $\widetilde{K}$ with residue degree f over K. Then $gf \leq [\widetilde{K} : K] \leq \varphi(N)$, which implies $g \log_2 f \leq gf \leq \varphi(N)$, i.e. $f^g \leq 2^{\varphi(N)}$. Note that $2 \log \mathcal{N}_{K/\mathbb{Q}}(v) > 1$ and $\mathcal{N}_{\widetilde{K}/\mathbb{Q}}(v_k) = \mathcal{N}_{K/\mathbb{Q}}(v)^f$ for $1 \leq k \leq g$, $g \leq \varphi(N)$, we have

$$\begin{aligned} \prod_{k=1}^{g} \log \mathcal{N}_{\widetilde{K}/\mathbb{Q}}(v_k) &\leq 2^{\varphi(N)} (\log \mathcal{N}_{K/\mathbb{Q}}(v))^g \\ &\leq 2^{\varphi(N)} (2 \log \mathcal{N}_{K/\mathbb{Q}}(v))^g \\ &\leq 4^{\varphi(N)} (\log \mathcal{N}_{K/\mathbb{Q}}(v))^{\varphi(N)}. \end{aligned}$$

Hence

$$\prod_{\substack{v \in \widetilde{S} \\ v \nmid \infty}} \log \mathcal{N}_{\widetilde{K}/\mathbb{Q}}(v) \leq 4^{s\varphi(N)} \left(\prod_{\substack{v \in S \\ v \nmid \infty}} \log \mathcal{N}_{K/\mathbb{Q}}(v) \right)^{\varphi(N)}. \qquad \square$$

Combine the lemma above with the bound (10), we have

$$\begin{aligned} &\mathrm{h}(j(P)) \\ &\quad \leq 2^{26\tilde{s}+20} \tilde{d}^{2\tilde{s}+4} (\log \tilde{d})^{3\tilde{r}} \tilde{s}^{2\tilde{s}+1} N^9 \ell^{\tilde{d}+1} \omega_{\widetilde{K}} \frac{1}{(\tilde{d}-1)^{\tilde{d}-1}} (\log |\widetilde{D}|)^{\tilde{d}} \sqrt{|\widetilde{D}|} \prod_{\substack{v \in \widetilde{S} \\ v \nmid \infty}} \log \mathcal{N}_{\widetilde{K}/\mathbb{Q}}(v) \\ &\quad \leq 2^{28s\varphi(N)+21} d^{2s\varphi(N)+6} (\log d\varphi(N))^{3s\varphi(N)} s^{2s\varphi(N)+1} \varphi(N)^{4s\varphi(N)+7} N^9 \ell^{d\varphi(N)+1} \Delta(N) \\ &\quad \leq 2^{28sN} d^{2sN} (\log dN)^{3sN} s^{2sN} N^{4sN} \ell^{dN} \Delta(N) \\ &\quad \leq (2^{14} dsN^2)^{2sN} (\log dN)^{3sN} \ell^{dN} \Delta(N). \end{aligned}$$

This completets the proof of Theorem 1.2.

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