DE GRUYTER
OPEN

Communications in Mathematics 24 (2016) 1–5

A note on the pp conjecture for sheaves of spaces of orderings

Paweł Gładki

Abstract. In this note we provide a direct and simple proof of a result previously obtained by Astier stating that the class of spaces of orderings for which the pp conjecture holds true is closed under sheaves over Boolean spaces.

The theory of abstract spaces of orderings was developed by Murray Marshall in a series of papers from the late 1970s, and provides an abstract framework for studying orderings of fields and the reduced theory of quadratic forms in general. We will refer to the monograph [5] as far as background, notation, and main results are concerned. A space of orderings is a pair (X, G) such that X is a nonempty set, G is a subgroup of $\{1, -1\}^X$, which contains the constant function -1, separates points of X, and satisfies some additional axioms (see [5] for details).

A space of orderings (X, G) has a natural topology introduced by the family of subbasic clopen Harrison sets:

$$H(a) = \{x \in X : a(x) = 1\},$$

for a given $a \in G$, which makes X into a Boolean space [5, Theorem 2.1.5].

For any multiplicative group G of exponent 2 with distinguished element -1, we set $X = \{x \in \chi(G) : x(-1) = -1\}$ and call the pair (X, G) a fan. A fan is also a space of orderings [5, Theorem 3.1.1]. We can also consider fans within a bigger space of orderings, and for this we need the notion of a subspace of a space (X, G) – a subset $Y \subseteq X$ is called a subspace of (X, G), if Y is expressible in the form $\bigcap_{a\in S} H(a)$, for some subset $S \subseteq G$. For any subspace Y we will denote by $G|_Y$ the group of all restrictions $a|_Y$, $a \in G$. The pair $(Y, G|_Y)$ is a space of orderings itself [5, Theorem 2.4.3]. Finally, if (X, G) is a space of orderings, by a fan in (X, G) we understand a subspace V such that the space $(V, G|_V)$ is a fan.

2010 MSC: Primary 11E10; Secondary 12D15
Key words: spaces of orderings, pp conjecture
DOI: 10.1515/cm-2016-0001

Sheaves of spaces of orderings had been defined in [4, Chapter 8], where the original results are phrased in terms of reduced Witt rings, and recently studied in [1] and [3]. The definition of the sheaf that we use here is exactly the one taken from [3]. Assume (X_i, G_i) is a space of orderings for each $i \in I$, where I is a Boolean space. Assume further that $X = \dot{\bigcup}_{i\in I} X_i$, the disjoint union of X_i's, is equipped with a topology such that

(1) X is a Boolean space,

(2) the inclusion map $\iota_i : X_i \hookrightarrow X$ is continuous, for each $i \in I$,

(3) the projection map $\pi : X \to I$ is continuous, and

(4) if $(i_\lambda)_{\lambda\in D}$ is any net in I converging to $i \in I$ and if $\sigma_1^\lambda, \sigma_2^\lambda, \sigma_3^\lambda, \sigma_4^\lambda$ is a 4-element fan in X_{i_λ} such that σ_j^λ converges to $\sigma_j \in X_i$ for each $j = 1, 2, 3, 4$, then $\sigma_1\sigma_2\sigma_3\sigma_4 = 1$.

Furthermore, let

$$G := \{\phi \in \mathrm{Cont}(X, \{\pm 1\}) : \phi|_{X_i} \in G_i \,\forall\, i \in I\}$$

where G_i is identified with its image in $\mathrm{Cont}(X_i, \{\pm 1\})$, $i \in I$, under the natural embedding. Then (X, G) is a space of orderings called the sheaf of spaces (X_i, G_i) over the Boolean space I.

The pp conjecture has been posed in [6], and, as proven in [2], has a negative solution in general, although it is still interesting to investigate numerous examples of spaces where it is valid. Recall that, for a space of orderings (X, G), a positive primitive (pp for short) formula $P(\underline{a})$ with n quantifiers and k parameters in G is of the form

$$P(\underline{a}) = \exists \underline{t} \bigwedge_{j=1}^{m} p_j(\underline{t}, \underline{a}) \in D_X(1, q_j(\underline{t}, \underline{a})),$$

where $\underline{t} = (t_1, \ldots, t_n)$, $\underline{a} = (a_1, \ldots, a_k)$, for $a_1, \ldots, a_k \in G$, and $p_j(\underline{t}, \underline{a})$, $q_j(\underline{t}, \underline{a})$ are $\pm$ products of some of the t_s's and a_r's, $s \in \{1, \ldots, n\}$, $r \in \{1, \ldots, k\}$, for $j \in \{1, \ldots, m\}$. The pp conjecture then asks whether it is true that if a pp formula holds in every finite subspace of a space of orderings, then it also holds in the whole space.

Our main goal in this note is the following:

Theorem 1. *If the pp conjecture holds in (X_i, G_i), for all $i \in I$, then it also holds in the sheaf (X, G) of (X_i, G_i) over the Boolean space I.*

This is essentially Theorem 2.2 in [1], where the author uses the language and methods from model theory. Here we provide the proof that uses only some simple topology and set theory along with the standard notation used in the theory of spaces of orderings. We begin with a series of lemmas that come from [4], where they are stated for abstract Witt rings, which are basically the same things as abstract spaces of orderings; nonetheless, we shall provide proofs here for the sake of completeness of presentation, as well as the language used.

Lemma 1. *For all $i \in I$, and for every $f \in G_i$, there exists $\phi \in \mathrm{Cont}(X, \{\pm 1\})$ such that $\phi|_{X_i} = f$.*

Proof. Fix $i \in I$ and $f \in G_i$. Consider the subsets $A := \iota_i(H(f))$ and $B := \iota_i(H(-f))$ of X. Since $H(f)$ and $H(-f)$ are clopen subsets of a compact space, they are compact, and since ι_i is continuous, A and B are compact. Moreover, since X is Hausdorff, for every $x \in A$ and $y \in B$ there exist clopen neigborhoods U_x of x and $V_{x,y}$ of y such that $U_x \cap V_{x,y} = \emptyset$. Thus $V_x = \bigcup_{y \in B} V_{x,y}$ forms an open cover of B such that $U_x \cap V_x = \emptyset$. Since B is compact, there exists a finite subcover of V_x, and thus V_x is clopen. Further, $C = \bigcup_{x \in A} U_x$ forms an open cover of A that, again, can be assumed finite, so that C is clopen. Let $D = \bigcap_{x \in A} V_x$. Then $C \cap D = \emptyset$ and D is clearly closed. Since X is normal, it follows that there are disjoint open sets U and V such that $C \subset U$, $D \subset V$ and $U \cap V = \emptyset$.

U and V as open sets are unions of clopen sets covering A and B, respectively. Using compactness one more time we may assume U and V to be clopen. The fuction $\phi : X \to \{\pm 1\}$ given by

$$\phi(x) = \begin{cases} 1 & \text{if } x \in U \\ -1 & \text{if } x \in X \setminus U \end{cases}$$

is clearly continuous and, as $V \subset X \setminus U$, $\phi|_{X_i} = f$. □

Lemma 2. *(cf. [4, Lemma 8.6]) For all $i \in I$, and for every $\phi \in \mathrm{Cont}(X, \{\pm 1\})$ such that $\phi|_{X_i} \in G_i$ there exists a clopen neigborhood J_i of i such that $\phi|_{X_j} \in G_j$, for $j \in J_i$.*

Proof. Fix $i \in I$ and $\phi \in \mathrm{Cont}(X, \{\pm 1\})$. Suppose, *a contrario,* that for every clopen neigborhood J of i there exists $j_J \in J$ such that $\phi|_{X_{j_J}} \notin G_{j_J}$. By [5, Corollary 3.2.4], for each j_J there exists a 4-element fan $\sigma_1^{j_J}, \sigma_2^{j_J}, \sigma_3^{j_J}, \sigma_4^{j_J}$ in (X_{j_J}, G_{j_J}) such that

$$\phi(\sigma_1^{j_J}) \cdot \phi(\sigma_2^{j_J}) \cdot \phi(\sigma_3^{j_J}) \cdot \phi(\sigma_4^{j_J}) = -1.$$

Direct the family $\mathcal{J}$ of all clopen neigborhoods J of i by inclusion. The net $(i_J)_{J \in \mathcal{J}}$ is then convergent to i. The net $(\sigma_1^{i_J})_{J \in \mathcal{J}}$ is then a net in the compact space X and thus has a cluster point σ_1. Let $(\sigma_1^{i_J})_{J \in \mathcal{J}_1}$ be a cofinal subnet of $(\sigma_1^{i_J})_{J \in \mathcal{J}}$ that converges to σ_1, $\mathcal{J}_1 \subset \mathcal{J}$. Similarly, the net $(\sigma_2^{i_J})_{J \in \mathcal{J}_1}$ is then a net in the compact space X and thus has a cluster point σ_2. Let $(\sigma_2^{i_J})_{J \in \mathcal{J}_2}$ be a cofinal subnet of $(\sigma_2^{i_J})_{J \in \mathcal{J}_1}$ that converges to σ_2, $\mathcal{J}_2 \subset \mathcal{J}_1$. Repeating the argument two more times we eventually get nets $(\sigma_3^{i_J})_{J \in \mathcal{J}_3}$ convergent to σ_3, $\mathcal{J}_3 \subset \mathcal{J}_2$, and $(\sigma_4^{i_J})_{J \in \mathcal{J}_4}$ convergent to σ_4, $\mathcal{J}_4 \subset \mathcal{J}_3$. Then, as each constructed net is a subnet of the previous one, $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is the limit of the net

$$(\sigma_1^{j_J}, \sigma_2^{j_J}, \sigma_3^{j_J}, \sigma_4^{j_J})_{J \in \mathcal{J}_4}.$$

But then

$$\phi(\sigma_1^{j_J}) \cdot \phi(\sigma_2^{j_J}) \cdot \phi(\sigma_3^{j_J}) \cdot \phi(\sigma_4^{j_J}) = -1,$$

for $J \in \mathcal{J}_4$, and by continuity of ϕ also

$$\phi(\sigma_1) \cdot \phi(\sigma_2) \cdot \phi(\sigma_3) \cdot \phi(\sigma_4) = -1,$$

which contradicts axiom (4) of a sheaf. □

Corollary 1. *(cf. [4, Lemma 8.7]) For all $i \in I$ the restriction map $G \to G_i$ given by $f \mapsto f|_{X_i}$ is surjective.*

Proof. This follows immediately from Lemmas 1 and 2 □

Lemma 3. *(cf. [4, Lemma 8.8]) For all $i \in I$, and for every open neigborhood U of X_i there exist a clopen neigborhood K_i of i such that $X_i \subset \pi^{-1}(K_i) \subset U$.*

Proof. This follows directly from the fact that $X \setminus U$ is compact. □

We now proceed to the proof of Theorem 1

Proof. As in the discussion preceeding the proof, let

$$P(\underline{a}) = \exists \underline{t} \bigwedge_{j=1}^{m} p_j(\underline{t}, \underline{a}) \in D_X(1, q_j(\underline{t}, \underline{a}))$$

be a pp formula that holds on every finite subspace of (X, G). Here $\underline{t} = (t_1, \ldots, t_n)$, $\underline{a} = (a_1, \ldots, a_k)$, for $a_1, \ldots, a_k \in G$, and $p_j(\underline{t}, \underline{a})$, $q_j(\underline{t}, \underline{a})$ are $\pm$ products of some of the t_s's and a_r's, $s \in \{1, \ldots, n\}$, $r \in \{1, \ldots, k\}$, for $j \in \{1, \ldots, m\}$. Our goal is to show that $P(\underline{a})$ holds in (X, G). Let $a_r^i := a_r|_{X_i}$, for $r \in \{1, \ldots, k\}$ and $i \in I$ and define

$$P^i(\underline{a}^i) = \exists \underline{t}^i \bigwedge_{j=1}^{m} p_j(\underline{t}^i, \underline{a}^i) \in D_{X_i}(1, q_j(\underline{t}^i, \underline{a}^i)),$$

where $\underline{t}^i = (t_1^i, \ldots, t_n^i)$, $\underline{a}^i = (a_1^i, \ldots, a_k^i)$.

$P^i(\underline{a}^i)$ holds on every space (X_i, G_i), $i \in I$. Indeed, fix $i \in I$ and fix a finite subspace $\{y_1, \ldots, y_l\}$ of X_i. Let Y be the finite subspace of (X, G) generated by $y_1, \ldots, y_l$ in the sense of [5, p. 33]. Then $P(\underline{a})$ holds on Y with some $u_1, \ldots, u_n \in G$ verifying it, so let $u_s^i := u_s|_{X_i}$, $s \in \{1, \ldots, n\}$. Then $P^i(\underline{a}^i)$ holds on $\{y_1, \ldots, y_l\}$ with $u_1^i, \ldots, u_n^i \in G_i$ verifying it. Since $\{y_1, \ldots, y_l\}$ was choosen arbitrarily and since the pp conjecture holds true for the space (X_i, G_i), $P^i(\underline{a}^i)$ holds on (X_i, G_i) with some $t_1^i, \ldots, t_n^i \in G_i$ verifying it.

By Corollary 1 we may assume that $t_1^i, \ldots, t_n^i \in G$, for $i \in I$. Let

$$U_i = \bigcap_{j=1}^{m} \left[H\left(p_j\left(\underline{t}^i, \underline{a}\right)\right) \cup H\left(-q_j\left(\underline{t}^i, \underline{a}\right)\right)\right], \text{ for } i \in I.$$

Then $X_i \subset U_i$ and $P(\underline{a})$ holds true on the open set U_i with $t_1^i, \ldots, t_n^i$ verifying it. By Lemma 3, let K_i be a clopen neigborhood of i such that $X_i \subset \pi^{-1}(K_i) \subset U_i$, for $i \in I$. By Lemma 2, let J_s^i be a clopen neigborhood of i such that $t_s^i|_{X_j} \in G_j$, for $j \in J_s^i$, $s \in \{1, \ldots, n\}$, $i \in I$. Let $J_i = J_1^i \cap \cdots \cap J_n^i$, $i \in I$. Define the clopen neigborhoods $I_i = J_i \cap K_i$, $i \in I$.

Now $\bigcup_{i \in I} I_i$ is an open cover of the compact space I, so we may choose $I_1, \ldots, I_\ell$ among I_i, $i \in I$, so that $I = I_1 \cup \cdots \cup I_\ell$. Replacing $I_1, \ldots, I_\ell$ by

$$I_1, I_2 \setminus I_1, I_3 \setminus (I_1 \cup I_2), \ldots, I_\ell \setminus (I_1 \cup \cdots \cup I_{\ell-1}),$$

if necessary, we might as well assume $I_1, \dots, I_\ell$ are pairwise disjoint. Define the functions $t_s : X \to \{\pm 1\}$ by

$$t_s(x) = \begin{cases} t_s^1(x) & \text{if } x \in X_i \text{ for } i \in I_1, \\ t_s^2(x) & \text{if } x \in X_i \text{ for } i \in I_2, \\ & \vdots \\ t_s^\ell(x) & \text{if } x \in X_i \text{ for } i \in I_\ell, \end{cases}$$

for $s \in \{1, \dots, n\}$. Clearly t_s is continuous, and $t_s|_{X_i} \in G_i$, for $i \in I$, so that $t_s \in G$, $s \in \{1, \dots, n\}$. Moreover, $P(\underline{a})$ holds on (X, G) with $t_1, \dots, t_n$ verifying it, which concludes the proof. □

Acknowledgement

The author would like to express his thanks to the referee for his careful reading and useful improvements that made the presentation more comprehensible.

References

[1] V. Astier: On some sheaves of reduced special groups. *Arch. for Math. Logic* 46 (5) (2007) 481–488.

[2] P. Gładki, M. Marshall: The pp conjecture for spaces of orderings of rational conics. *J. Algebra Appl.* 6 (2) (2007) 245–257.

[3] P. Gładki, M. Marshall: Quotients of index two and general quotients in a space of orderings. *Fund. Math.* 229 (2015) 255–275.

[4] M. Marshall: *Abstract Witt rings.* Queen's University, Kingston, Ontario (1980).

[5] M. Marshall: Spaces of orderings and abstract real spectra. *Lecture Notes in Mathematics* 1636 (1996).

[6] M. Marshall: Open questions in the theory of spaces of orderings. *J. Symbolic Logic* 67 (1) (2002) 341–352.

E-mail: `pawel.gladki@us.edu.pl`

Author's addresses:
Institute of Mathematics, University of Silesia, ul. Bankowa 14, Katowice, Poland, 40-007

Department of Computer Science, AGH University of Science and Technology, al. Mickiewicza 30, Kraków, Poland, 30-059

Received: 24 August, 2015
Accepted for publication: 2 May, 2016
Communicated by: Karl Dilcher