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Upgrading Probability via Fractions of Events

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Abstract. The influence of "Grundbegriffe" by A. N. Kolmogorov (published in 1933) on education in the area of probability and its impact on research in stochastics cannot be overestimated. We would like to point out three aspects of the classical probability theory "calling for" an upgrade: (i) classical random events are black-and-white (Boolean); (ii) classical random variables do not model quantum phenomena; (iii) basic maps (probability measures and observables – dual maps to random variables) have very different "mathematical nature". Accordingly, we propose an upgraded probability theory based on Łukasiewicz operations (multivalued logic) on events, elementary category theory, and covering the classical probability theory as a special case. The upgrade can be compared to replacing calculations with integers by calculations with rational (and real) numbers. Namely, to avoid the three objections, we embed the classical (Boolean) random events (represented by the $\{0,1\}$-valued indicator functions of sets) into upgraded random events (represented by measurable $[0,1]$-valued functions), the minimal domain of probability containing "fractions" of classical random events, and we upgrade the notions of probability measure and random variable.

1 Introduction

Our goal is to survey recent results related to upgrading the classical probability theory, CPT for short (cf. [2], [3], [13], [11], [12], [14], [15], [17]).

We start with two important initiatives to "modernize" CPT. In his pioneering paper [32], L. A. Zadeh has proposed to consider Borel fuzzy sets as fuzzy random events and probability integral of the corresponding measurable functions as their probabilities. Having in mind "soft computing" applications, he proposed fuzzy operations on fuzzy random events and introduced basic fuzzified probability notions

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(cf. [33]). More information about the resulting fuzzy probability and applications can be found at `http://people.eecs.berkeley.edu/~zadeh/papers`.

A thorough analysis of operations on fuzzy random events and generalized probability measures on fuzzy random events (based on T-norms and T-conorms) has been provided by R. Mesiar in [22] and by M. Navara in [23]. Fuzzy random events can be viewed as quantum structures (effect algebras, D-posets). A comprehensive treatise on quantum structures [6] by A. Dvurečenskij and S. Pulmannová serves as a standard reference.

It seems justified to call and understand by fuzzy probability what has resulted from Zadeh's approach and to find another name for the generalized probability motivated by quantum phenomena, as presented by S. Gudder in [17] and S. Bugajski in [2], [3]; while Gudder sticks with fuzzy probability, Bugajski has proposed to call it operational probability.

The theory as outlined by Gudder and Bugajski (many other authors have contributed, too, see [2], [3]) rests heavily on deep theorems of abstract analysis. We believe that basic category theory apparatus provides tools and language to reformulate and prove fundamental theorems and to present the resulting probability in a more transparent way.

2 Why we need "fractions" of random events?

Let (Ω, p) and (Ξ, q) be finite (discrete) probability spaces. Let T be a map of $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ into $\Xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$ such that $q(\xi_k) = \sum_{\omega_l \in T^{\leftarrow}(\xi_k)} p(\omega_l)$ for all $k \in \{1, 2, \ldots, m\}$ such that $q(\xi_k) > 0$. Then T is said to be a transformation of (Ω, p) to (Ξ, q) and (Ξ, q) is said to be the T-image of (Ω, p). If Ξ is a set of real numbers, then T becomes a random variable.

Let T be a transformation of (Ω, p) to (Ξ, q). Then T can be visualized as a system of n pipelines $\omega_l \mapsto T(\omega_l)$ through which $p(\omega_l)$ flows to $\xi_k = T(\omega_l)$. If ξ_k is the target of several pipelines, then $q(\xi_k)$ is the sum $\sum_{\omega_l \in T^{\leftarrow}(\xi_k)} p(\omega_l)$, i.e., the total influx through the pipelines in question. (See Figure 1.)

Question: Does there always exist a transformation of (Ω, p) to (Ξ, q)?

Answer: No.

Indeed, it is easy to see that if Ξ has more points than Ω, then there is no transformation of (Ω, p) to (Ξ, q).

However, "est modus in rebus": instead of sending each $p(\omega_l)$ to some ξ_k via a simple "pipeline", $\omega_l \mapsto \xi_k = T(\omega_l)$, we can try to distribute $p(\omega_l)$ via a complex "upgraded pipeline", simultaneously sending to each ξ_k, $k \in \{1, 2, \ldots, m\}$, some fraction $w_{kl}p(\omega_l)$ of $p(\omega_l)$. Of course, not arbitrarily, but in such a way that the fractions sum up "properly", i.e., $\sum_{l=1}^{n} w_{kl}p(\omega_l) = q(\xi_k)$ and

$$\sum_{k=1}^{m} \sum_{l=1}^{n} w_{kl}p(\omega_l) = \sum_{l=1}^{n} p(\omega_l) \sum_{k=1}^{m} w_{kl} = \sum_{k=1}^{m} q(\xi_k) = 1 .$$

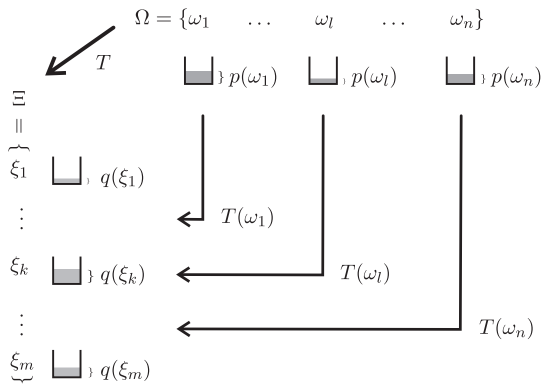

Figure 1

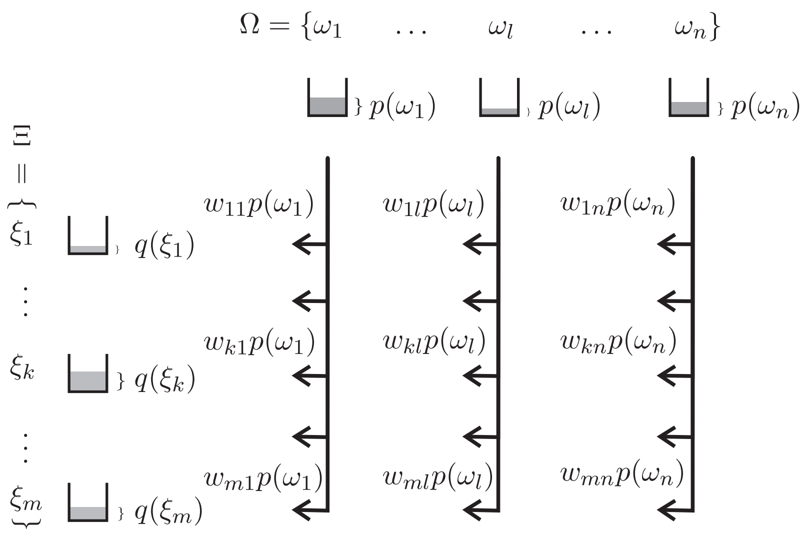

Figure 2

To comply with the second condition it suffices to guarantee that $\sum_{k=1}^{m} w_{kl} = 1$. In fact, this means that to each ω_l, $l \in \{1, 2, \dots, n\}$, we assign a suitable probability function $q_l = (w_{1l}, w_{2l}, \dots, w_{ml})$ on Ξ. (See Figure 2.)

The construction of an "upgraded pipeline" yields a generalized transformation of (Ω, p) to (Ξ, q); p flows trough the pipeline and it is transformed into q. The generalized transformation has a surprising background: upgraded probability.

Let

$$\{r_{kl};\, k \in \{1,2,\dots,m\},\, l \in \{1,2,\dots,n\}\,\}$$

be nonnegative numbers such that

$$\sum_{k=1}^{m} r_{kl} = p(\omega_l), \quad l \in \{1,2,\dots,n\}$$

and

$$\sum_{l=1}^{n} r_{kl} = q(\xi_k), \quad k \in \{1,2,\dots,m\}\,.$$

For $k \in \{1,2,\dots,m\}$ and $l \in \{1,2,\dots,n\}$ define $w_{kl} = \frac{1}{m}$ if $p(\omega_l) = 0$ (any choice such that $\sum\limits_{k=1}^{m} w_{kl} = 1$ does the same trick) and $w_{kl} = \frac{r_{kl}}{p(\omega_l)}$ otherwise. Clearly, $\sum\limits_{l=1}^{n} w_{kl} p(\omega_l) = q(\xi_k)$ for all $k \in \{1,2,\dots,m\}$ and $\sum\limits_{k=1}^{m} w_{kl} = 1$. This yields the "upgraded pipeline".

Observations

1. The "upgraded pipeline" is determined by a matrix W consisting of m rows $w_k = (w_{k1}, w_{k2}, \dots, w_{kn})$ and n columns $q_l = (w_{1l}, w_{2l}, \dots, w_{ml})$, where $w_{kl} \in [0,1]$ and each q_l is a suitable probability function on Ξ.

2. Each q_l distributes the content $p(\omega_l)$ among the points $\xi_k \in \Xi$ (respecting $\sum\limits_{l=1}^{n} w_{kl} p(\omega_l) = q(\xi_k)$).

3. For $k \in \{1,2,\dots,m\}$, $w_k \in [0,1]^{\Omega}$ represents the "fraction of sure event" determining what fraction of the total flow will reach ξ_k. Further, if Ξ' is a subset of Ξ, then "the sum" of corresponding fractions determines how much will flow into Ξ'.

4. The rows of W are special cases "fractions of events", i.e., random events in the upgraded probability. Further, W determines a map (morphism) from the upgrade of (Ξ, q) to the upgrade of (Ω, p); such maps (dual to random variables) play a central role in the resulting theory and are called observables.

3 From Boolean two-valued to Łukasiewicz multivalued logic

In [13], the notion of a probability domain has been introduced as a general construction to obtain various models of random events. Briefly, in a given model, a random event is a propositional function $u\colon X \longrightarrow [0,1]$, $u(x)$ is the "truth value" of u at x, and the events are equipped with some "logic" (Boolean, D-poset, effect algebra, Łukasiewicz, ...). From the viewpoint of category theory, the interval $I = [0,1]$ carries a suitable structural information about the model in question (it is a cogenerator) and the operations on events are inherited from the categorical (initial, or coordinatewise) structure of the power $[0,1]^X$.

This approach to probability domains can be summarized as follows (cf. [13]):

- Start with a "system $\mathcal{A}$ of events";
- Choose a "cogenerator" C – usually a structured set suitable for "measuring" (e.g., the two-element Boolean algebra {0,1}, the interval $I = [0, 1]$ carrying the Łukasiewicz MV-structure, D-poset structure, ...);
- Choose a set X of "properties" measured via C so that X separates $\mathcal{A}$;
- Represent each event $a \in \mathcal{A}$ via the "evaluation" of $\mathcal{A}$ into C^X sending $a \in \mathcal{A}$ to $a_X \in C^X$, $a_X \equiv \{x(a);\ x \in X\}$;
- Form the minimal "subalgebra" D of C^X containing $\{a_X;\ a \in \mathcal{A}\}$;
- The subalgebra forms a *probability domain* $D \subseteq C^X$ which has nice categorical properties.

Fields of sets, ID-posets and bold algebras can serve as typical examples of probability domains described above ([26], [27], [28]). Nontraditional cogenerators provide nontraditional models of probability theory ([29]).

D-posets have been introduced in [19] in order to model events in quantum probability. They generalize Boolean algebras, MV-algebras and other probability domains and enable us to construct a category in which observables and states become morphisms ([4], [31]). Recall that a D-poset is a partially ordered set X with the greatest element 1_X, the least element 0_X, and a partial binary operation called *difference*, such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1) $a \ominus 0_X = a$ for each $a \in X$;

(D2) If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

Fundamental to applications ([8], [27]) are D-posets of fuzzy sets, i.e. systems $\mathcal{X} \subseteq I^X$ carrying the coordinatewise partial order, coordinatewise convergence of sequences, containing the top and bottom elements of I^X, and closed with respect to the partial operation difference defined coordinatewise. We always assume that $\mathcal{X}$ is *reduced*, i.e., for $x, y \in X$, $x \neq y$, there exists $u \in \mathcal{X}$ such that $u(x) \neq u(y)$. A D-homomorphism is a map preserving the D-poset structure (partial order, constants, difference). Denote $\mathbb{ID}$ the category having (reduced) D-posets of fuzzy sets as objects and having sequentially continuous D-homomorphisms as morphisms. Objects of $\mathbb{ID}$ are subobjects of the powers I^X. Concerning the undefined notions, the reader is referred to [6], [5] and [1].

Recall ([6], [7]) that a *bold algebra* (also *T_L-clan*, see [25]) is a system $\mathcal{X} \subseteq [0, 1]^X$ containing the constant functions 0_X, 1_X and closed with respect to the usual (Łukasiewicz) operations: for $u, v \in \mathcal{X}$ put

$$(u \oplus v)(x) = u(x) \oplus v(x) = \min\{1, u(x) + v(x)\}, \quad u^*(x) = 1 - u(x), \quad x \in X\,.$$

Bold algebras are MV-algebras representable as $[0, 1]$-valued functions, MV-algebras generalize Boolean algebras and bold algebras generalize in a natural way fields of sets (viewed as indicator functions). Each bold algebra can be considered as on object of $\mathbb{ID}$. More information concerning MV-algebras and probability on MV-algebras can be found in [30].

Each bold algebra $\mathcal{X} \subseteq [0,1]^X$ is a lattice, where for $u, v \in \mathcal{X}$ we have

$$(u \vee v)(x) = u(x) \vee v(x) \quad \text{and} \quad (u \wedge v)(x) = u(x) \wedge v(x)\,, \quad x \in X\,,$$

and, in fact, bold algebras are lattice (defined coordinatewise) ID-posets. If a bold algebra $\mathcal{X} \subseteq [0,1]^X$ is sequentially closed in $[0,1]^X$ (with respect to the coordinatewise sequential convergence), then $\mathcal{X}$ is a *Łukasiewicz tribe* ($\mathcal{X}$ is closed not only with respect to finite, but also with respect to countable Łukasiewicz sums, cf. Corollary 2.8 in [7]). Let $\mathcal{X} \subseteq [0,1]^X$ be a bold algebra. Then the smallest sequentially closed subset of $[0,1]^X$ containing $\mathcal{X}$ is a Łukasiewicz tribe; it is the intersection of all Łukasiewicz tribes $\mathcal{Y} \subseteq [0,1]^X$ such that $\mathcal{X} \subseteq \mathcal{Y}$.

Observations

1. ID-posets provide a minimal model of random events: sure and impossible event, complement. Further, bold algebras model disjunction and conjunction, and Łukasiewicz tribes model closedeness with respect limits (remember, stochastics is about limit properties).

2. What is missing? Fractions! Fortunately, Łukasiewicz tribes closed with respect to fractions are exactly what is needed. Let $\mathcal{X} \subseteq [0,1]^X$ be a Łukasiewicz tribe. Then there is a unique σ-field $\mathbf{A}$ of subsets of X such that (identifying sets and their indicator functions) $\mathbf{A} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbf{A})$, where $\mathcal{M}(\mathbf{A})$ is the set of all $\mathbf{A}$-measurable functions ranging in $[0,1]$. Further, if $\mathcal{X}$ contains fractions, then $\mathcal{X} = \mathcal{M}(\mathbf{A})$. So, the inclusion of fraction events results in upgrading σ-fields to $[0,1]$-valued measurable functions.

3. As proposed by L. Zadeh, probability measures are to be upgraded to probability integrals (integrals with respect to probability measures). What about sequential continuity of D-homomorphisms? If X is a singleton $\{a\}$ and $\mathbf{T} = \{\emptyset, \{a\}\}$ is the σ-field of all subsets of X, then $\mathcal{M}(\mathbf{T})$ and $[0,1]$ can be identified and, surprisingly, sequentially continuous D-homomorphisms on $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$ are exactly probability integrals ([9]). Finally, due to the Lebesgue Dominated Convergence Theorem, probability integrals are sequentially continuous.

4. To sum up, if we incorporate fractions of random events into an upgraded probability, then $(\Omega, \mathcal{M}(\mathbf{A}), \int(\cdot)\,\mathrm{d}p)$ is the minimal upgrade of the classical probability space $(\Omega, \mathbf{A}, p)$.

4 From analysis to arrows

In this section we illustrate some benefits of the categorical approach to probability. First, taking into account new results and the categorical background, some original notions "deserve an upgrade". Secondly, arrows and commutative diagrams visualize deep theorems and help to understand their proofs ([16]).

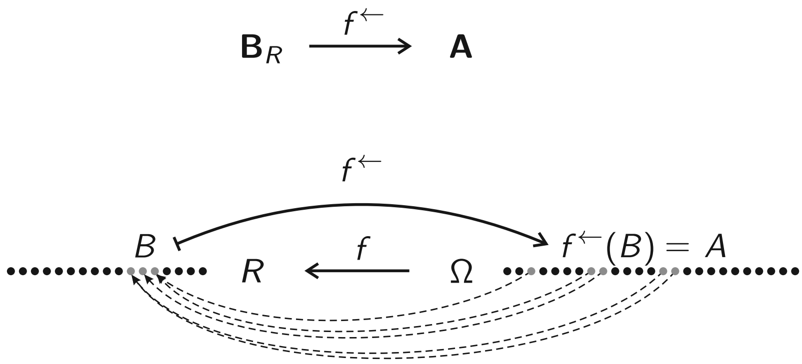

Figure 3

For example, the categorical approach to probability "forces" us to reconsider the notion of a probability measure. A probability measure p on random events from $\mathbf{A}$ is a map of $\mathbf{A}$ into $[0, 1]$ by which we usually measure the relative frequency of the events. The domain and the range of p have very different mathematical nature. We avoid this by upgrading p to $\int (\cdot)\,\mathrm{d}p$ and the upgraded map becomes a morphism (of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$). Observe that there is a one-to-one correspondence between probability measures and probability integrals, hence no classical information is lost.

The notion of a classical random variable (see Figure 3) calls for an upgrade, too. Its dual, the preimage map, is a sequentially continuous Boolean homomorphism on random events and, despite an implicit exploitation, in the classical probability it does not have a name ([18], [21]). The duals to upgraded random variables are called observables and play a central role in the upgraded probability (see also [20], [24]).

Let $(\Omega, \mathbf{A}, p)$ ba a classical probability space and let f be a random variable, i.e., a measurable map of Ω into the real line R. Denote $\mathbf{B}_R$ the real Borel sets. Then f induces a map D_f of the set $\mathcal{P}(\mathbf{A})$ of all probability measures on $\mathbf{A}$ into the set $\mathcal{P}(\mathbf{B}_R)$ of all probability measures on $\mathbf{B}_R$: for $q \in \mathcal{P}(\mathbf{A})$ we define $D_f(q) = q \circ f^{\leftarrow}$, where $f^{\leftarrow}$ is the preimage map ($f^{\leftarrow}(B) = \{\omega \in \Omega; f(\omega) \in B\}$, $B \in \mathbf{B}_R$); we say that f pushes forward q to $D_f(q)$. Observe that $f^{\leftarrow}$ is a sequentially continuous Boolean homomorphism on $\mathbf{B}_R$ into $\mathbf{A}$. If we identify each $\omega \in \Omega$ and the Dirac point-probability δ_ω and, similarly, each $r \in R$ and δ_r, then a straightforward calculation shows that $D_f(\delta_\omega) = \delta_{f(\omega)}$. Consequently, D_f can be considered as an extension of f (mapping $\Omega \subseteq \mathcal{P}(\mathbf{A})$ into $R \subseteq \mathcal{P}(\mathbf{B}_R)$) to D_f (mapping $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B}_R)$). Hence D_f is a *channel through which the stochastic information about* $\mathbf{A}$ *is transported to the stochastic information about* $\mathbf{B}_R$. (See Figure 4.)

Usually (cf. [6]), an observable is a map of the real random events $\mathbf{B}_R$ into fuzzy random events $\mathcal{M}(\mathbf{A})$ which preserves suitable operations on events. Again, in order to get a morphism, the domain and the range of an upgraded observable should

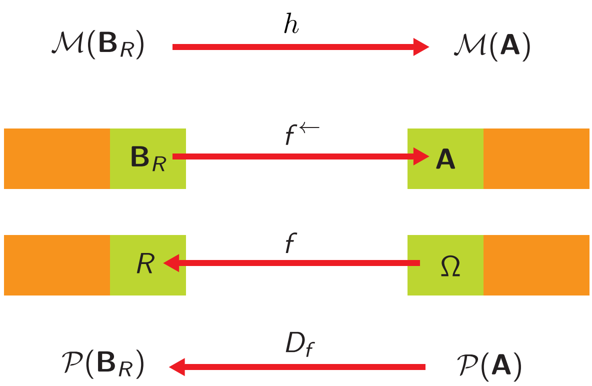

Figure 4

be of the same type. Fortunately, each such map of $\mathbf{B}_R$ into $\mathcal{M}(\mathbf{A})$ can uniquely extended to a sequentially continuous D-homomorphism $h\colon\ \mathcal{M}(\mathbf{B}_R) \longrightarrow \mathcal{M}(\mathbf{A})$ (cf. Theorem 4.1 in [10]). This leads to the following upgrade of a "categorical" notion of observable.

Definition 1. A sequentially continuous D-homomorphism $h\colon\ \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ is said to be an *observable*. Moreover, if $h(B) \in \mathbf{A}$ for all $B \in \mathbf{B}$, then h is said to be *conservative*.

Example 1. Observe that if $\mathbf{A} = \mathbf{T}$ and $p\colon\ \mathbf{B} \longrightarrow [0,1]$ is a nondegenerated probability measure, then the corresponding integral $\int(\cdot)\,\mathrm{d}p$, viewed as an observable mapping $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{T}) = [0,1]$, fails to be conservative.

Each observable h defines a map T_h of the set $\mathcal{IP}(\mathbf{A})$ of all probability integrals on $\mathcal{M}(\mathbf{A})$ into the set $\mathcal{IP}(\mathbf{B})$ of all probability integrals on $\mathcal{M}(\mathbf{B})$. Indeed, consider each $\int(\cdot)\,\mathrm{d}p$ as a sequentially continuous D-homomorphism of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T}) = [0,1]$. The composition of h and $\int(\cdot)\,\mathrm{d}p$ is a sequentially continuous D-homomorphism of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{T}) = [0,1]$, hence a probability integral on $\mathcal{M}(\mathbf{B})$.

Definition 2. Let h be an observable, then T_h is said to be a *statistical map*. Moreover, if T_h maps degenerated integrals into degenerated integrals, then T_h is said to be *conservative*.

Theorem 1. *A statistical map T_h is conservative iff the dual observable h is conservative.*

Proof. Both implications follow by a straightforward calculations based on the definition of a statistical map. $\square$

Example 2. (i) A classical degenerated random variable is defined as follows. Fix $r \in R$ and define $f : \Omega \longrightarrow R$ by putting $f(\omega) = r$ for all $\omega \in \Omega$. Then $D_f(p) = \delta_r$ for all $p \in \mathcal{P}(\mathbf{A})$.

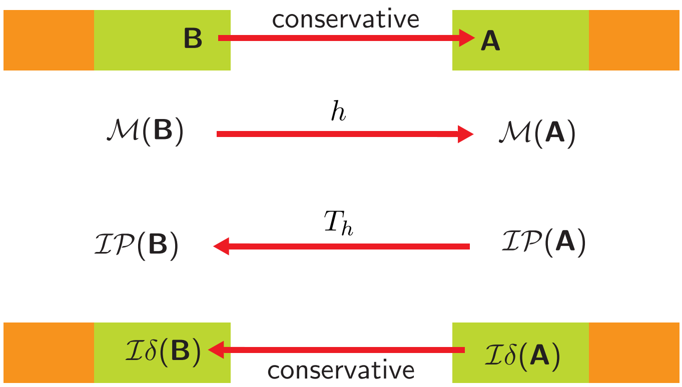

Figure 5

(ii) A degenerated statistical map is defined analogously: For a fixed $q \in \mathcal{P}(\mathbf{B})$ we need an observable $h: \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ such that the corresponding T_h maps each probability integral $\int (\cdot)\, \mathrm{d}p$, $p \in \mathcal{P}(\mathbf{A})$, on $\mathcal{M}(\mathbf{A})$ to the same probability integral $\int (\cdot)\, \mathrm{d}q$ on $\mathcal{M}(\mathbf{B})$. It suffices to define h as follows: for $u \in \mathcal{M}(\mathbf{B})$ let $h(u)$ be the constant function v_q on Ω the value of which is $\int u \, \mathrm{d}q$. Then, for all $p \in \mathcal{P}(\mathbf{A})$, we have $\int h(u)\, \mathrm{d}p = \int v_q \, \mathrm{d}p = \int u \, \mathrm{d}q$, and hence $T_h(\int (\cdot)\, \mathrm{d}p) = \int (\cdot)\, \mathrm{d}q$. If $B \in \mathbf{B}$, $0 < q(B) < 1$, and χ_B is the indicator function of B, then $0 < (h(\chi_B))(\omega) < 1$ for all $\omega \in \Omega$, hence h and T_h fail to be conservative. (See Figure 5.)

Observe that one of the possible "upgraded pipelines" (cf. Figure 2) can be constructed via the following nonconservative statistical map: it is the unique "upgrading" of a map mapping each $\omega \equiv \delta_\omega$ to the probability q on the subsets of Ξ.

5 Duality between observables and statistical maps

An observable h can be viewed as a channel through which the information concerning operations on random events in $\mathbf{B}$ is transported to the information concerning operations on random events in $\mathbf{A}$. A statistical map T_h can be viewed as another channel, going the opposite way from the set $\mathcal{IP}(\mathbf{A})$ of all probability integrals on $\mathcal{M}(\mathbf{A})$ to the set $\mathcal{IP}(\mathbf{B})$ of all probability integrals on $\mathcal{M}(\mathbf{B})$. The relationship between observables and statistical maps can be described in terms of a categorical duality.

In category theory, an equivalence of categories is a relation between two categories which establishes that these categories are "essentially the same". If a category is equivalent to the opposite (or dual) of another category then one speaks of a duality of categories, and says that the two categories are dually equivalent. An equivalence of categories consists of a functor between the involved categories, which is required to have an "inverse" functor. However, in contrast to the situation common for isomorphisms in an algebraic setting, the composition of the functor and its "inverse" is not necessarily the identity mapping. Instead it is sufficient that each object be naturally isomorphic to its image under this composition. Thus one may describe the functors as being "inverse up to isomorphism".

Formally, an equivalence of categories $\mathbb{C}$ and $\mathbb{D}$ consists of a functor $F\colon \mathbb{C} \longrightarrow \mathbb{D}$, a functor $G\colon \mathbb{D} \longrightarrow \mathbb{C}$, and two natural isomorphisms $\epsilon\colon FG \longrightarrow \mathrm{id}_{\mathbb{D}}$ and $\eta\colon GF \longrightarrow \mathrm{id}_{\mathbb{C}}$. Here FG and GF denote the respective compositions of F and G, and $\mathrm{id}_{\mathbb{C}}$ and $\mathrm{id}_{\mathbb{D}}$ denote the identity functors on $\mathbb{C}$ and $\mathbb{D}$. If F and G are contravariant functors one speaks of a duality of categories instead.

One can show that a functor $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ yields an equivalence of categories if and only if it is simultaneously:

– full, i.e. for any two objects c_1 and c_2 of $\mathbb{C}$, the map

$$\mathrm{Hom}_{\mathbb{C}}(c_1, c_2) \longrightarrow \mathrm{Hom}_{\mathbb{D}}(Fc_1, Fc_2)$$

induced by F is surjective,

– faithful, i.e. for any two objects c_1 and c_2 of $\mathbb{C}$, the map

$$\mathrm{Hom}_{\mathbb{C}}(c_1, c_2) \longrightarrow \mathrm{Hom}_{\mathbb{D}}(Fc_1, Fc_2)$$

induced by F is injective, and

– essentially surjective (dense), i.e. each object d in $\mathbb{D}$ is isomorphic to an object of the form Fc, for c in $\mathbb{C}$.

Let $\mathbb{C}$ be the category the objects of which are of the form $(\Omega, \mathcal{M}(\mathbf{A}))$ and the morphisms are observables and let $\mathbb{D}$ be the category the objects of which are of the form $\mathcal{IP}(\mathbf{A})$ and the morphisms are statistical maps.

Theorem 2. *The categories $\mathbb{C}$ and $\mathbb{D}$ are dual.*

Proof. Hint. Denote $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ the contravariant functor sending $\mathcal{M}(\mathbf{A})$ to $\mathcal{IP}(\mathbf{A})$ and sending an observable h to the statistical map T_h. A straightforward calculation shows that F is full, faithful and essentially surjective. □

Observe that each upgraded pipeline is determined by the matrix W, the rows of which "code" an observable and the columns of which "code" the corresponding dual statistical map, hence W "codes" the duality.

The duality reveals a surprising fact: in the upgraded probability, the fuzzy character of random events and the quantum character of statistical maps are dual! (Remember, an elementary random event, i.e., an outcome of a random experiment, is mapped to a probability measure.)

6 The upgrading of CPT as an epireflection

Let $\mathbb{B}$ be the category the objects of which are of the form $(\Omega, \mathbf{A})$ and the morphisms are sequentially continuous Boolean homomorphisms. Let $\mathbb{L}$ be the category the objects of which are of the form $(\Omega, \mathcal{M}(\mathbf{A}))$ and the morphisms are observables. Consider the following covariant functor $U\colon \mathbb{B} \longrightarrow \mathbb{L}$ which sends

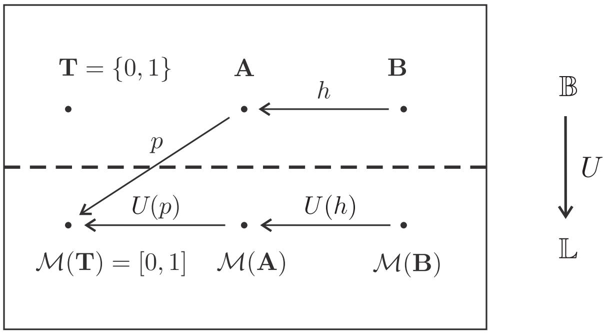

Figure 6

$(\Omega, \mathbf{A})$ to $(\Omega, \mathcal{M}(\mathbf{A}))$ and sends a sequentially continuous Boolean homomorphism $h\colon \mathbf{B} \longrightarrow \mathbf{A}$ to its unique extension (an observable) $U(h)\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$. The relationship between the "Boolean" category $\mathbb{B}$ and the "Łukasiewicz" category $\mathbb{L}$ characterizes the upgrading of CPT: $\mathbb{L}$ is an epirelective subcategory of $\mathbb{B}$, there is a one-to-one correspondence between objects of $\mathbb{B}$ and $\mathbb{L}$, $\mathbb{L}$ has "more" morphisms than $\mathbb{B}$ (remember, there are nonconservative observables), and each probability measure on $\mathbf{A}$ is a "shadow" of an observable on $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$. (See Figure 6.)

7 Concluding remarks

In this survey we have presented an attempt to "upgrade" the classical probability theory. Accordingly, instead of "definition, lemma, theorem, proof" style, we have rather included motivation, comments, and schemes. We hope that a categorical approach and exploitation of basic categorical constructions will help the reader to appreciate our goal. "More next".

Acknowledgement

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References

[1] J. Adámek: *Theory of Mathematical Structures*. Reidel, Dordrecht (1983).

[2] S. Bugajski: Statistical maps I. Basic properties. *Math. Slovaca* 51 (3) (2001) 321–342.

[3] S. Bugajski: Statistical maps II. Operational random variables. *Math. Slovaca* 51 (3) (2001) 343–361.

[4] F. Chovanec, R. Frič: States as morphisms. *Internat. J. Theoret. Phys.* 49 (12) (2010) 3050–3060.

[5] F. Chovanec, F. Kôpka: D-posets. *Handbook of Quantum Logic and Quantum Structures: Quantum Structures*(2007) 367–428. Edited by K. Engesser, D. M. Gabbay and D. Lehmann

[6] A. Dvurečenskij, S. Pulmannová: *New Trends in Quantum Structures.* Kluwer Academic Publ. and Ister Science, Dordrecht and Bratislava (2000).

[7] R. Frič: Łukasiewicz tribes are absolutely sequentially closed bold algebras. *Czechoslovak Math. J.* 52 (2002) 861–874.

[8] R. Frič: Remarks on statistical maps and fuzzy (operational) random variables. *Tatra Mt. Math. Publ* 30 (2005) 21–34.

[9] R. Frič: Extension of domains of states. *Soft Comput.* 13 (2009) 63–70.

[10] R. Frič: On D-posets of fuzzy sets. *Math. Slovaca* 64 (2014) 545–554.

[11] R- Frič, M. Papčo: A categorical approach to probability. *Studia Logica* 94 (2010) 215–230.

[12] R. Frič, M. Papčo: Fuzzification of crisp domains. *Kybernetika* 46 (2010) 1009–1024.

[13] R. Frič, M. Papčo: On probability domains. *Internat. J. Theoret. Phys.* 49 (2010) 3092–3100.

[14] R. Frič, M. Papčo: On probability domains II. *Internat. J. Theoret. Phys.* 50 (2011) 3778–3786.

[15] R. Frič, M. Papčo: On probability domains III. *Internat. J. Theoret. Phys.* 54 (2015) 4237–4246.

[16] J. A. Goguen: A categorical manifesto. *Math. Struct. Comp. Sci.* 1 (1991) 49–67.

[17] S. Gudder: Fuzzy probability theory. *Demonstratio Math.* 31 (1998) 235–254.

[18] A. N. Kolmogorov: *Grundbegriffe der wahrscheinlichkeitsrechnung.* Springer, Berlin (1933).

[19] F. Kôpka, F. Chovanec: D-posets. *Math. Slovaca* 44 (1994) 21–34.

[20] M. Kuková, M. Navara: What observables can be. In: R.K. Gubaidullina (ed.): *Theory of Functions, Its Applications, and Related Questions, Transactions of the Mathematical Institute of N.I. Lobachevsky 46.* Kazan Federal University (2013) 62–70.

[21] M. Loève: *Probability theory.* D. Van Nostrand, Inc., Princeton, New Jersey (1963).

[22] R. Mesiar: Fuzzy sets and probability theory. *Tatra Mt. Math. Publ.* 1 (1992) 105–123.

[23] M. Navara: Triangular norms and measures of fuzzy sets. In: E.P. Klement, R. Mesiar (eds.): *Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms.* Elsevier (2005) 345–390.

[24] M. Navara: Probability theory of fuzzy events. In: E. Montseny, P. Sobrevilla (eds.): *Fourth Conference of the European Society for Fuzzy Logic and Technology and 11 Rencontres Francophones sur la Logique Floue et ses Applications.* Universitat Polit ecnica de Catalunya, Barcelona, Spain (2005) 325–329.

[25] M. Navara: Tribes revisited. In: U. Bodenhofer, B. De Baets, E.P. Klement, S. Saminger-Platz (eds.): *30th Linz Seminar on Fuzzy Set Theory: The Legacy of 30 Seminars, Where Do We Stand and Where Do We Go?.* Johannes Kepler University, Linz, Austria (2009) 81–84.

[26] M. Papčo: On measurable spaces and measurable maps. *Tatra Mt. Math. Publ.* 28 (2004) 125–140.

[27] M. Papčo: On fuzzy random variables: examples and generalizations. *Tatra Mt. Math. Publ.* 30 (2005) 175–185.

[28] M. Papčo: On effect algebras. *Soft Comput.* 12 (2008) 373–379.

[29] M. Papčo: Fuzzification of probabilistic objects. In: *8th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2013), doi:10.2991/eusat.2013.10.* (2013) 67–71.

[30] B. Riečan, D. Mundici: Probability on MV-algebras. In: E.Pap (ed.): *Handbook of Measure Theory, Vol. II.* North-Holland, Amsterdam (2002) 869–910.

[31] B. Riečan, T. Neubrunn: *Integral, Measure, and Ordering.* Kluwer Acad. Publ., Dordrecht-Boston-London (1997).

[32] L. A. Zadeh: Probability measures of fuzzy events. *J. Math. Anal. Appl.* 23 (1968) 421–427.

[33] L. A. Zadeh: Fuzzy probabilities. *Inform. Process. Manag.* 19 (1984) 148–153.

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