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On the notion of Jacobi fields in constrained calculus of variations

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Abstract. In variational calculus, the minimality of a given functional under arbitrary deformations with fixed end-points is established through an analysis of the so called *second variation*. In this paper, the argument is examined in the context of constrained variational calculus, assuming piecewise differentiable extremals, commonly referred to as *extremaloids*. The approach relies on the existence of a fully covariant representation of the second variation of the action functional, based on a family of *local* gauge transformations of the original Lagrangian and on a set of scalar attributes of the extremaloid, called the corners' *strengths* [16]. In discussing the positivity of the second variation, a relevant role is played by the *Jacobi fields*, defined as infinitesimal generators of 1-parameter groups of diffeomorphisms preserving the extremaloids. Along a piecewise differentiable extremal, these fields are generally discontinuous across the corners. A thorough analysis of this point is presented. An alternative characterization of the Jacobi fields as solutions of a suitable *accessory variational problem* is established.

Introduction

The study of extremals with corners dates back to the works of Weierstrass and Erdmann [8]. Since then, many Authors made their own contribution to the subject. Among others, we cite Caratheodory [2], [3], [4], Dresden [5], [6], [7], Bolza [1], Hadamard [12], Rider [22], Graves [10], [11], Reid [21] and, more recently, Milyutin, Osmolovskii and Lempio [17], [18], [19].

In three recent papers [14], [15], [16], a geometrical setup for the study of the subject is thoroughly worked out. In [14], the role of the Pontryagin equations [20], [9] and of the Erdmann-Weierstrass corner conditions [8], [13], [23] as necessary

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and sufficient conditions for a continuous, piecewise differentiable curve to be an extremal of the action functional is analysed.

The resulting framework is employed in [15] in order to discuss the problem of minimality for differentiable extremals. A tensorial algorithm is set up through an adaptation technique, consisting in a systematic replacement of the original Lagrangian by a gauge-equivalent one, fulfilling the property of being *critical* along the extremal in study.

In [16], the analysis is extended to piecewise differentiable extremals, here called *extremaloids*. It is shown that a global adaptation procedure is generally unavailable, due to the presence of a set of scalars, called the corners' *strengths*, whose non vanishing precludes the existence of a differentiable gauge transformation yielding a Lagrangian critical along the whole extremaloid. The construction of a tensorial algorithm is nonetheless possible, resorting to a family of *local* gauge transformations, one for each differentiable arc. In this way, all significant ideas involved in the study of the second variation (matrix Riccati equation, Jacobi fields, focal points) find their place in the piecewise differentiable context.

In this paper we resume and complete the theory of Jacobi fields along an extremaloid developed in [16], henceforth called *broken* Jacobi fields. We show that, in addition to the standard definition as infinitesimal deformations preserving extremaloids, they are equally well characterized as solutions of an accessory variational problem, with action functional given by the second variation of the original functional, viewed as a quadratic form over the vector space of admissible infinitesimal deformations. The result, entirely straightforward in the case of differentiable extremals [15], presents here some non-trivial aspects, mainly in connection with the management of the *jumps* of the Jacobi fields at the corners.

The presentation is organized as follows: Sections 1 and 2 summarize the results of [14], [16] more closely related to the present developments.

In Section 3, the definition of the Jacobi fields along an extremaloid is reviewed. The accessory variational problem, meant as the study of the extremals of the second-variation within the class of admissible deformations, is then formulated. The resulting Pontryagin equations are compared with the Jacobi ones. A discussion of the behaviour of the solutions at the corners is presented.

1 Brief overview of foregoing results

All the definitions, conventions and results described in [14], [15], [16] will be freely used throughout. For convenience of the reader, we review here some basic ideas, laying particular emphasis on those related to piecewise differentiability.

1.1 Preliminaries

(i) Let $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$ denote a $(n+1)$-dimensional fiber bundle over the real line, referred to local fibred coordinates $t, q^1, \ldots, q^n$ and called the *event space*. The notation q^0 will be occasionally adopted in place of t, with Greek indices running from 0 to n and Latin ones running from 1 to n.

The first jet-space $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$, referred to local coordinates $t, q^i, \dot{q}^i$, is called the *velocity space*. Given any section $\gamma\colon \mathbb{R} \to \mathcal{V}_{n+1}$, the corresponding jet-extension is denoted by $j_1(\gamma)\colon \mathbb{R} \to j_1(\mathcal{V}_{n+1})$.

The presence of non-holonomic constraints is accounted for by an embedded submanifold of the first jet-bundle $j_1(\mathcal{V}_{n+1})$, fibred over $\mathcal{V}_{n+1}$ and referred to local fibred coordinates $t, q^1, \ldots, q^n, z^1, \ldots, z^r$. The imbedding $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$ is locally expressed as

$$\dot{q}^i = \psi^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r)\,, \qquad i = 1, \ldots, n. \tag{1}$$

A section $\gamma\colon \mathbb{R} \to \mathcal{V}_{n+1}$ is called *admissible* if and only there exists a section $\hat{\gamma}\colon \mathbb{R} \to \mathcal{A}$, called the lift of γ, locally described as $q^i = q^i(t), z^A = z^A(t)$ and satisfying $i \cdot \hat{\gamma} = j_1(\gamma)$. Under the stated assumptions, the section $\hat{\gamma}$ is itself called admissible. In coordinates, the admissibility condition reads

$$\frac{dq^i}{dt} = \psi^i\big(t, q^1(t), \ldots, q^n(t), z^1(t), \ldots, z^r(t)\big). \tag{2}$$

Given an admissible γ, we denote by $\dot{\gamma}$ its tangent vector, locally represented as

$$\dot{\gamma} := \frac{dq^\mu}{dt}\left(\frac{\partial}{\partial q^\mu}\right)_\gamma = \left(\frac{\partial}{\partial t}\right)_\gamma + \psi^i{}_{|\hat{\gamma}}\left(\frac{\partial}{\partial q^i}\right)_\gamma. \tag{3}$$

(ii) A noteworthy differential operator mapping the Grassmann algebra over $\mathcal{V}_{n+1}$ into the Grassmann algebra over $\mathcal{A}$ is the *symbolic time derivative* $\frac{d}{dt}$, uniquely defined by the properties (see [14] and references therein)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \psi^k \frac{\partial f}{\partial q^k} := \dot{f}\,, \qquad \frac{d}{dt}(df) = d\dot{f} \qquad \forall f \in \mathscr{F}(\mathcal{V}_{n+1}), \tag{4a}$$

$$\frac{d}{dt}(\omega \wedge \eta) = \frac{d\omega}{dt} \wedge \eta + \omega \wedge \frac{d\eta}{dt} \qquad \forall \omega, \eta \in \mathcal{G}(\mathcal{V}_{n+1}). \tag{4b}$$

Given any admissible section γ, the value of the restriction $\frac{d\omega}{dt}\big|_{\hat{\gamma}}$ is easily seen to depend only on the value of the restriction $\omega_{|\gamma}$. It makes therefore perfectly good sense to define the symbolic time derivative of an exterior r-form *along* γ, meant as an r-form along $\hat{\gamma}$. In particular, for $r = 1$, we have the expression

$$\frac{d}{dt}\left(\nu_0(t)dt_{|\gamma} + \nu_i(t)dq^i{}_{|\gamma}\right) = \frac{d\nu_0}{dt}\, dt_{|\hat{\gamma}} + \frac{d\nu_i}{dt}\, dq^i{}_{|\hat{\gamma}} + \nu_i d\psi^i{}_{|\hat{\gamma}}\,. \tag{5}$$

(iii) Let $V(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ and $V^*(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ respectively denote the *vertical bundle* relative to the fibration $\mathcal{V}_{n+1} \to \mathbb{R}$ and the associated dual bundle, isomorphic to the quotient of the cotangent bundle $T^*(\mathcal{V}_{n+1})$ by the equivalence relation

$$\sigma \sim \sigma' \iff \begin{cases} \pi(\sigma) = \pi(\sigma') \\ \sigma - \sigma' \propto dt_{|\pi(\sigma)}\,. \end{cases} \tag{6}$$

The elements of $V^*(\mathcal{V}_{n+1})$ are called the *virtual 1-forms* over $\mathcal{V}_{n+1}$.

Every local coordinate system t, q^i in $\mathcal{V}_{n+1}$ induces fibred coordinates t, q^i, p_i in $V^*(\mathcal{V}_{n+1})$, with $p_i(\hat{\lambda}) := \big\langle \hat{\lambda}, \big(\frac{\partial}{\partial q^i}\big)_{\pi(\lambda)} \big\rangle\ \forall \hat{\lambda} \in V^*(\mathcal{V}_{n+1})$.

For any $g \in \mathscr{F}(\mathcal{V}_{n+1})$, the linear functional on $V(\mathcal{V}_{n+1})$ determined by the differential dg is denoted by δg and is called the *virtual differential* of g.

The vector bundles $V(\mathcal{V}_{n+1})$ and $V^*(\mathcal{V}_{n+1})$ generate a tensor algebra, whose elements are called the *virtual tensors* over $\mathcal{V}_{n+1}$.

The fibred product $\mathcal{C}(\mathcal{A}) := \mathcal{A} \times_{\mathcal{V}_{n+1}} V^*(\mathcal{V}_{n+1})$, referred to local coordinates t, q^i, z^A, p_i, is called the *contact bundle*. By construction, $\mathcal{C}(\mathcal{A})$ is a vector bundle over $\mathcal{A}$, canonically isomorphic to the subbundle of the cotangent space $T^*(\mathcal{A})$ locally spanned by the *contact 1-forms* $\omega^i = dq^i - \psi^i dt$.

(iv) Given any admissible section γ, we denote by $V(\gamma) \xrightarrow{t} \mathbb{R}$ the bundle of vertical vectors along γ, by $A(\hat\gamma) \xrightarrow{t} \mathbb{R}$ the totality of vectors along $\hat\gamma$ annihilating the 1-form dt and by $V(\hat\gamma) \subset A(\hat\gamma)$ the totality of vertical vectors relative to the fibration $A(\hat\gamma) \xrightarrow{\pi} V(\gamma)$. All spaces are referred to fibred coordinates, respectively denoted by t, u^i, t, u^i, v^A and t, v^A, symbolically defined as $u^i = \langle dq^i, \cdot\rangle$, $v^A = \langle dz^A, \cdot\rangle$.

The restriction of $V^*(\mathcal{V}_{n+1})$ to the curve γ determines a vector bundle $V^*(\gamma) \xrightarrow{t} \mathbb{R}$, dual to the vertical bundle $V(\gamma)$. The elements of $V^*(\mathcal{V}_{n+1})$ are called the *virtual 1-forms* along γ. The elements of the tensor algebra generated by $V(\gamma)$ and $V^*(\gamma)$ are called the *virtual tensors* along γ.

Preserving the notation δ for the virtual differential, every virtual tensor field w along γ is locally represented as $w = w^i{}_{j\cdots}(t)\big(\frac{\partial}{\partial q^i}\big)_\gamma \otimes \delta q^j{}_{|\gamma} \otimes \cdots$.

(v) In the forthcoming discussion we shall consider not only sections in the ordinary sense but also *piecewise differentiable* evolutions, defined on *closed* intervals. In this connection, we recall the following definitions [14]:

- an admissible closed arc $\big(\gamma, [m,n]\big)$ in $\mathcal{V}_{n+1}$ is the restriction to a closed interval $[m,n]$ of an admissible section $\gamma\colon (m',n') \to \mathcal{V}_{n+1}$ defined on some open interval $(m',n') \supset [m,n]$;
- a piecewise differentiable evolution of the system in the interval $[t_0,t_1]$ is a finite collection

$$\big(\gamma, [t_0,t_1]\big) := \big\{ \big(\gamma^{(s)}, [a_{s-1},a_s]\big),\ s = 1,\dots,N,\ t_0 = a_0 < \cdots < a_N = t_1 \big\}$$

of admissible closed arcs satisfying the matching conditions

$$\gamma^{(s)}(a_s) = \gamma^{(s+1)}(a_s)\,, \qquad \forall s = 1,\dots,N-1\,. \tag{7}$$

On account of Eq. (7), the image $\gamma(t)$ is well-defined and continuous for all $t_0 \leqslant t \leqslant t_1$. The points $x_s := \gamma(a_s)$, $s = 1,\dots,N-1$, are called the *corners* of γ. The tangent vector to the arc $\gamma^{(s)}$ is denoted by $\dot\gamma^{(s)}$.

In like manner, the lift of an admissible closed arc $\big(\gamma,[m,n]\big)$ is the restriction to $[m,n]$ of the lift $\hat\gamma\colon (m',n') \to \mathcal{A}$, while the lift $\hat\gamma$ of a piecewise differentiable evolution $\big\{\big(\gamma^{(s)}, [a_{s-1},a_s]\big)\big\}$ is the family of lifts $\hat\gamma^{(s)}$, each restricted to the corresponding closed interval $[a_{s-1},a_s]$. The image $\hat\gamma(t)$ is well-defined for $t \neq a_1,\dots,a_{N-1}$, thus allowing to regard $\hat\gamma\colon [t_0,t_1] \to \mathcal{A}$ as a (generally discontinuous) section of the velocity space.

1.2 Deformations

The geometric setup associated with the representation of deformations in the presence of constraints is regarded as known [14], [15]. For the purposes of the present work, a few technical aspects are briefly reviewed.

(i) An admissible deformation of an admissible closed arc $\big(\gamma, [m,n]\big)$ is a 1-parameter family $\big(\gamma_\xi, [m(\xi), n(\xi)]\big)$, $|\xi| < \varepsilon$, of admissible closed arcs depending differentiably on ξ and satisfying the condition

$$\big(\gamma_0, [m(0), n(0)]\big) = \big(\gamma, [m,n]\big)\,.$$

An admissible deformation of a piecewise differentiable evolution $\big(\gamma, [t_0, t_1]\big)$ is likewise a collection $\big\{\big(\gamma^{(s)}_\xi, [a_{s-1}(\xi), a_s(\xi)]\big)\big\}$ of admissible deformations of each single arc, satisfying the matching conditions

$$\gamma^{(s)}_\xi(a_s(\xi)) = \gamma^{(s+1)}_\xi(a_s(\xi)) \qquad \forall |\xi| < \varepsilon, \quad s = 1, \dots, N-1\,. \tag{8}$$

Each lift $\hat\gamma^{(s)}_\xi$, restricted to the interval $[a_{s-1}(\xi), a_s(\xi)]$, is easily recognized to provide a deformation for the lift $\hat\gamma^{(s)}\colon [a_{s-1}, a_s] \to \mathcal{A}$.

In what follows, we shall only consider deformations leaving the interval $[t_0, t_1]$ fixed, namely those satisfying the conditions $a_0(\xi) = t_0, a_N(\xi) = t_1$; no restriction is posed on the functions $a_s(\xi), s = 1, \dots, N-1$.

For each s, the curve $c_s(\xi) := \gamma^{(s)}_\xi(a_s(\xi)) = \gamma^{(s+1)}_\xi(a_s(\xi))$ is called the *orbit* of the corner x_s under the given deformation.

In local coordinates, setting $q^i(\gamma^{(s)}_\xi(t)) = \varphi^i_{(s)}(\xi, t)$, the matching conditions (8) read

$$\varphi^i_{(s)}(\xi, a_s(\xi)) = \varphi^i_{(s+1)}(\xi, a_s(\xi)), \tag{9}$$

while the representation of the orbit $c_s(\xi)$ takes the form

$$c_s(\xi)\colon \qquad t = a_s(\xi)\,, \qquad q^i = \varphi^i_{(s)}(\xi, a_s(\xi)) \tag{10}$$

with $c_s(0) = x_s$.

(ii) An admissible infinitesimal deformation of a closed arc $\big(\gamma, [m,n]\big)$ is a triple (α, X, β), where X is the restriction to $[m,n]$ of an admissible infinitesimal deformation of the arc $\gamma\colon (m', n') \to \mathcal{V}_{n+1}$, while α, β are the derivatives

$$\alpha = \left.\frac{dm}{d\xi}\right|_{\xi=0}, \qquad \beta = \left.\frac{dn}{d\xi}\right|_{\xi=0}.$$

Likewise, an admissible infinitesimal deformation of a piecewise differentiable evolution $\big(\gamma, [t_0, t_1]\big)$ is a collection $\big\{\cdots \alpha_{s-1}, X_{(s)}, \alpha_s \cdots\big\}$ of admissible infinitesimal deformations of each single closed arc, with $\alpha_s = \frac{da_s}{d\xi}\big|_{\xi=0}$ (in particular, with $\alpha_0 = \alpha_N = 0$ whenever the interval $[t_0, t_1]$ is held fixed).

The admissibility of each $X_{(s)}$ requires the existence of a corresponding lift $\hat{X}_{(s)} = X^i_{(s)}\big(\frac{\partial}{\partial q^i}\big)_{\hat{\gamma}^{(s)}} + X^A_{(s)}\big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}^{(s)}}$ satisfying the variational equation

$$\frac{dX^i_{(s)}}{dt} = \left(\frac{\partial \psi^i}{\partial q^k}\right)_{\hat{\gamma}^{(s)}} X^k_{(s)} + \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} X^A_{(s)} \,. \tag{11}$$

Eqs. (9) imply the jump relations

$$\big[X^i\big]_{x_s} =: X^i_{(s+1)}(a_s) - X^i_{(s)}(a_s) = -\alpha_s\big[\psi^i\big]_{x_s} \tag{12}$$

as well as the representation

$$W_{(s)} := c_{s*}\left(\frac{d}{d\xi}\right)_{\xi=0} = \alpha_s\left(\frac{\partial}{\partial t}\right)_{x_s} + \big(\alpha_s\psi^i + X^i\big)_{x_s}\left(\frac{\partial}{\partial q^i}\right)_{x_s} \tag{13}$$

for the tangent vector to the orbit of the corner x_s at $\xi = 0$.

1.3 Infinitesimal controls

(i) Given an admissible differentiable section $\gamma\colon \mathbb{R} \to \mathcal{V}_{n+1}$, an *infinitesimal control* along γ is a linear section $h\colon V(\gamma) \to A(\hat{\gamma})$, described in fibred coordinates as $v^A = h_i{}^A(t)u^i$. The image $h(V(\gamma))$ defines a distribution along $\hat{\gamma}$, locally spanned by the vector fields

$$\tilde{\partial}_i := h\left[\left(\frac{\partial}{\partial q^i}\right)_{\gamma}\right] = \left(\frac{\partial}{\partial q^i}\right)_{\hat{\gamma}} + h_i{}^A\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} \,, \tag{14}$$

called the *horizontal distribution* associated with h.

Every $\hat{X} = X^i(t)\big(\frac{\partial}{\partial q^i}\big)_{\hat{\gamma}} + X^A(t)\big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}} \in A(\hat{\gamma})$ may be uniquely decomposed into the sum of a horizontal vector $\mathcal{P}_H(\hat{X})$ and a vertical vector $\mathcal{P}_V(\hat{X})$, respectively defined by the equations

$$\begin{aligned}
&\mathcal{P}_H(\hat{X}) := h\big(\pi_*(\hat{X})\big) = X^i\tilde{\partial}_i \,,\\
&\mathcal{P}_V(\hat{X}) := \hat{X} - \mathcal{P}_H(\hat{X}) = \big(X^A - X^i h_i{}^A\big)\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} =: U^A\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} \,.
\end{aligned}$$

A section $X\colon \mathbb{R} \to V(\gamma)$ is said to be h-transported along γ if the composite map $h \cdot X\colon \mathbb{R} \to A(\hat{\gamma})$ is an admissible infinitesimal deformation of $\hat{\gamma}$. In view of Eqs. (11), (14), the h-transported sections form an n-dimensional vector space V_h, isomorphic to the standard fibre of $V(\gamma)$. Every infinitesimal control provides therefore a *trivialization* of the vector bundle $V(\gamma) \to \mathbb{R}$, summarized into the identification $V(\gamma) \simeq \mathbb{R} \times V_h$. By duality, this entails the analogous identification $V^*(\gamma) \simeq \mathbb{R} \times V_h{}^*$.

(ii) The notion of h-transport induces a derivation $\frac{D}{Dt}$ of the virtual tensor algebra along γ, called the *absolute time derivative*. Introducing the *temporal connection coefficients* $\tau_k{}^i := -\tilde{\partial}_k\big(\psi^i\big)$, we have the representation

$$\frac{D}{Dt}\left[Z^i{}_{j\ldots}(t)\left(\frac{\partial}{\partial q^i}\right)_\gamma \otimes \delta q^j{}_{|\gamma} \otimes \cdots\right] := \frac{DZ^i{}_{j\ldots}}{Dt}\left(\frac{\partial}{\partial q^i}\right)_\gamma \otimes \delta q^j{}_{|\gamma} \otimes \cdots\,,$$

with

$$\frac{DZ^i{}_{j\ldots}}{Dt} = \frac{dZ^i{}_{j\ldots}}{dt} + \tau_k{}^i Z^k{}_{j\ldots} - \tau_j{}^k Z^i{}_{k\ldots} + \cdots\,. \tag{15a}$$

Matters get simplified referring both bundles $V(\gamma), V^*(\gamma)$ to h-transported dual bases $e_{(a)} = e_{(a)}{}^i\big(\frac{\partial}{\partial q^i}\big)_\gamma$, $e^{(a)} = e^{(a)}{}_i\delta q^i{}_{|\gamma}$.

Setting $Z = \widetilde{Z}^a{}_{b\ldots}e_{(a)} \otimes e^{(b)} \otimes \cdots$, Eq. (15a) takes then the form

$$\frac{DZ}{Dt} = \frac{d\widetilde{Z}^a{}_{b\ldots}}{dt}\, e_{(a)} \otimes e^{(b)} \otimes \cdots\,, \tag{15b}$$

i.e. it reduces to the ordinary derivative of the components.

Given any admissible infinitesimal deformation $X = X^i\big(\frac{\partial}{\partial x^i}\big)_\gamma$ of γ lifting to a deformation $\hat{X}$ of $\hat{\gamma}$ and denoting by $U = U^A\big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}}$ the vertical projection of $\hat{X}$, the variational equation (11) and the lift process may be cast into the form

$$\frac{DX^i}{Dt} = U^A\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat{\gamma}}\,, \tag{16a}$$

$$\hat{X} = h(X) + U. \tag{16b}$$

(iii) Assigning an infinitesimal control $h^{(s)}$ along each arc $\gamma^{(s)}$ of a piecewise differentiable admissible evolution $\big(\gamma, [t_0, t_1]\big)$ and arguing as above, we conclude that every admissible infinitesimal deformation $\big\{\cdots\alpha_{s-1}, X_{(s)}, \alpha_s\cdots\big\}$ of γ is determined, up to initial data, by the coefficients $\alpha_1, \ldots, \alpha_{N-1}$ and by N vertical vector fields $U_{(s)} = U^A_{(s)}\big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}^{(s)}}$ through the covariant variational equation

$$\frac{DX^i_{(s)}}{Dt} = U^A_{(s)}\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat{\gamma}^{(s)}}\,, \qquad s = 1, \ldots, N\,, \tag{17a}$$

completed with the jump relations (12). Each lift $\hat{X}_{(s)}$ is then given by

$$\hat{X}_{(s)} = h^{(s)}\big(X_{(s)}\big) + U_{(s)}\,. \tag{17b}$$

(iv) Given a piecewise differentiable admissible evolution $\big(\gamma, [t_0, t_1]\big)$ and a family $h = \{h^{(s)}\}$ of infinitesimal controls, we glue $h^{(s)}$-transport along each arc $\big(\gamma^{(s)}, [a_{s-1}, a_s]\big)$ and continuity at the corners into a global *h-transport law* along γ.

Once again, the resulting algorithm provides a trivialization of the vector bundle $V(\gamma)$ into the cartesian product $[t_0, t_1] \times V_h$, with $V_h \simeq V(\gamma)_{|t}$ $\forall t \in [t_0, t_1]$.

Given any infinitesimal deformation $\{\cdots \alpha_{s-1}, X_{(s)}, \alpha_s \cdots\}$, we merge all sections $X_{(s)}$ into a piecewise differentiable map $X\colon [t_0, t_1] \to V_h$, with jump discontinuities at $t = a_s$ expressed by Eq. (12).

The vertical projections $U_{(s)} = \mathcal{P}_V(\hat{X}_{(s)})$ are similarly merged into a single object U, henceforth (improperly) called a vertical vector field along $\hat{\gamma}$.

In this way Eqs. (17) becomes formally identical to Eqs. (16). In particular, in h-transported bases, the determination of the components $\tilde{X}^a$ in terms of U^A and of the scalars α_s relies on the equations

$$\frac{d\tilde{X}^a}{dt} = U^A e^{(a)}{}_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} \qquad \forall t \neq a_s\,, \tag{18a}$$

completed with the jump conditions

$$[\tilde{X}^a]_{x_s} = -\alpha_s e^{(a)}{}_i(a_s)[\psi^i]_{x_s} \qquad s = 1, \ldots, N-1\,. \tag{18b}$$

1.4 Extremaloids

The first step in the solution of the fixed-endpoints variational problem based on the functional

$$\mathcal{I}[\gamma] := \int_{\hat{\gamma}} \mathscr{L}(t, q^1, \ldots, q^n, z^1, \ldots, z^r)\, dt\,, \qquad \mathscr{L} \in \mathscr{F}(\mathcal{A}) \tag{19}$$

is the analysis of its *first variation*. The argument has been worked out in [14]. The main results are reported below.

(i) Given an admissible piecewise differentiable section $\gamma\colon [t_0, t_1] \to \mathcal{V}_{n+1}$, let $\mathfrak{V}$ and $\mathfrak{W}$ respectively denote the infinite-dimensional vector space formed by the totality of vertical vector fields $U = \{U_{(s)}, s = 1, \ldots, N\}$ along $\hat{\gamma}$ and the direct sum $\mathfrak{V} \oplus \mathbb{R}^{N-1}$. Also, let $h = (h^{(1)}, \ldots, h^{(N)})$ denote a collection of (arbitrarily chosen) infinitesimal controls along the arcs of γ.

By Eqs. (18), every admissible infinitesimal deformation X of γ is determined, up to initial data, by an element $(U, \underset{\sim}{\alpha}) := (U, \alpha_1, \ldots, \alpha_{N-1}) \in \mathfrak{W}$.

In h-transported bases, for any $t \in (a_{r-1}, a_r]$, $r = 1, \ldots, N$, the resulting expression reads

$$X(t) = \left(\tilde{X}^a(t_0) + \int_{t_0}^{t} U^A e^{(a)}{}_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} dt - \sum_{s=1}^{r-1} \alpha_s e^{(a)}{}_i(a_s)[\psi^i(\hat{\gamma})]_{x_s}\right) e_{(a)}(t)\,.$$

In particular, denoting by $\Upsilon\colon \mathfrak{W} \to V_h$ linear map defined by the equation

$$\Upsilon(U, \underset{\sim}{\alpha}) := \left(\int_{t_0}^{t_1} U^A e^{(a)}{}_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} dt - \sum_{s=1}^{N-1} \alpha_s e^{(a)}{}_i(a_s)[\psi^i(\hat{\gamma})]_{x_s}\right) e_{(a)}\,, \tag{20}$$

the admissible infinitesimal deformations vanishing at the endpoints of γ are in 1-1 correspondence with the elements of the subspace $\ker(\Upsilon) \subset \mathfrak{W}$.

Several important properties of the evolution γ (ordinariness, normality, local normality) are related to the nature of the map (20). A detailed account may be found in [14] and references therein. For the purposes of the present analysis we report here two basic results.

Proposition 1. *An evolution* $\gamma\colon [t_0, t_1] \to \mathcal{V}_{n+1}$ *is* normal *if and only if there exists no non-zero* continuous *virtual* 1*-form* $\hat{\lambda}\colon [t_0, t_1] \to V^*(\gamma)\mathcal{V}_{n+1}$ *satisfying the homogeneous system*

$$\lambda_i \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}} = 0\,,$$

$$\lambda_i(a_s)[\psi^i(\hat{\gamma})]_{x_s} = 0\,, \qquad s = 1, \ldots, N-1.$$

It is locally normal *if the same property holds on any subinterval* $I \subset [t_0, t_1]$.

Proposition 2. *Given an admissible evolution* γ*, let* $\wp(\gamma)$ *denote the totality of piecewise differentiable virtual* 1*-forms* $\hat{\rho} = p_i(t)\delta q^i{}_{|\gamma}$ *along* γ *satisfying the equations*

$$\frac{dp_i}{dt} + \frac{\partial \psi^k}{\partial q^i} p_k = \frac{\partial \mathscr{L}}{\partial q^i}\,, \tag{21a}$$

$$p_i \frac{\partial \psi^i}{\partial z^A} = \frac{\partial \mathscr{L}}{\partial z^A} \tag{21b}$$

as well as the matching conditions

$$\big[p_i\big]_{a_s} = \big[p_i \psi^i{}_{|\hat{\gamma}} - \mathscr{L}_{|\hat{\gamma}}\big]_{a_s} = 0\,, \qquad s = 1, \ldots, N-1\,. \tag{21c}$$

Then:

a) *the condition* $\wp(\gamma) \neq \emptyset$ *is* sufficient *for* γ *to be an extremaloid for the functional* (19)*;*

b) *if* γ *is a* normal *evolution, the condition* $\wp(\gamma) \neq \emptyset$ *is also* necessary *for* γ *to be an extremaloid.*

Eqs. (21a), (21b), completed with the admissibility conditions (2), are the well known *Pontryagin equations*; Eqs. (21c) reproduce the content of the *Erdmann--Weierstrass corner conditions.*

For future reference we observe that the whole set of equations is invariant under arbitrary transformations of the form

$$\mathscr{L} \to \mathscr{L} + \dot{S}\,, \qquad p_i(t) \to p_i(t) + \left(\frac{\partial S}{\partial q^i} \right)_{\gamma(t)}\,, \tag{22}$$

$S(t, q^1, \ldots, q^n)$ being any differentiable function over $\mathcal{V}_{n+1}$, and $\dot{S} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i}\psi^i$ denoting its symbolic time derivative [14].

An alternative derivation of Eqs. (21) is obtained working in the environment $\mathcal{C}(\mathcal{A})$ and introducing the *Pontryagin Hamiltonian*

$$\mathscr{H}(t, q^i, p_i, z^A) := p_i \psi^i(t, q^i, z^A) - \mathscr{L}(t, q^i, z^A)\,. \tag{23}$$

Every ordinary extremaloid is then the projection of a (generally discontinuous) curve $\tilde{\gamma}$: $q^i = q^i(t), p_i = p_i(t), z^A = z^A(t)$ satisfying the equations

$$\frac{dq^i}{dt} = \frac{\partial \mathscr{H}}{\partial p_i}\,, \qquad \frac{dp_i}{dt} = -\frac{\partial \mathscr{H}}{\partial q^i}\,, \qquad \frac{\partial \mathscr{H}}{\partial z^A} = 0 \tag{24a}$$

as well as the matching conditions

$$\left[q^i{}_{|\tilde{\gamma}}\right]_{a_s} = \left[p_{i|\tilde{\gamma}}\right]_{a_s} = \left[\mathscr{H}_{|\tilde{\gamma}}\right]_{a_s} = 0\,. \tag{24b}$$

In particular, when γ is normal, there exists a unique such $\tilde{\gamma}$.

A further enhancement of Proposition 2 comes from the use of the symbolic time derivative (5). Resuming the notation $\dot{\gamma}$ for the tangent vector to γ, we have the following

Proposition 3. *The class $\wp(\gamma)$ defined in Proposition 2 is in 1-1 correspondence with the family of continuous 1-forms $\rho = p_0 dt_{|\gamma} + p_i dq^i{}_{|\gamma}$ along γ satisfying the conditions*

$$\langle \rho, \dot{\gamma} \rangle = \mathscr{L}_{|\hat{\gamma}}\,, \qquad \frac{d\rho}{dt} = d\mathscr{L}_{|\hat{\gamma}}\,. \tag{25}$$

The correspondence $\rho \to \hat{\rho}$ is given by the quotient map $\hat{\rho} = [\rho]$ associated with the equivalence relation (6).

Remark 1. On account of Eqs. (3), (23), the first equation (25) may be cast into the form

$$p_0(t) = \mathscr{L}_{|\hat{\gamma}(t)} - p_i(t)\psi^i{}_{|\hat{\gamma}(t)} = -\mathscr{H}_{|\tilde{\gamma}(t)}\,, \tag{26}$$

relating the component $p_0(t)$ to the value of the Pontryagin Hamiltonian along the curve $\tilde{\gamma}$. This fact is reflected into the second relation (25): in addition to restoring the content of Eqs. (21a), (21b), the latter provides in fact the evolution equation

$$\frac{dp_0}{dt} = \left(\frac{\partial \mathscr{L}}{\partial t}\right)_{\hat{\gamma}} - p_i \left(\frac{\partial \psi^i}{\partial t}\right)_{\hat{\gamma}} = -\left(\frac{\partial \mathscr{H}}{\partial t}\right)_{\hat{\gamma}}\,, \tag{27}$$

expressing the non-holonomic counterpart of the transport law for the Hamiltonian.

Remark 2. According to Propositions 1, 2, along any normal extremaloid, the 1-form ρ is uniquely determined by the knowledge of the Lagrangian. Any gauge transformation (22) modifies ρ according to the prescription

$$\rho \to \rho - dS_{|\gamma} = \left[p_0(t) - \left(\frac{\partial S}{\partial t}\right)_{\gamma}\right] dt_{|\gamma} + \left[p_i(t) - \left(\frac{\partial S}{\partial q^i}\right)_{\gamma}\right] dq^i{}_{|\gamma}\,. \tag{28}$$

2 The second variation

2.1 Adapted Lagrangians

(i) In order to create a tensorial setup for the study of the second variation of the action functional along a normal extremaloid $\gamma = \{(\gamma^{(s)}, [a_{s-1}, a_s])\}$, the gauge structure of the theory has to be exploited, so as to make each point $\hat{\gamma}^{(s)}(t)$, $t \in (a_{s-1}, a_s)$ into a *critical* point of the Lagrangian. Referring to [16] for the details, we report here the main ideas involved in the procedure.

Given the extremaloid γ, let $\rho = \{\rho^{(s)}\}$ denote the unique 1-form along γ satisfying the requests of Proposition 3. The correspondence $t \to (\gamma(t), \rho(t))$ defines then a continuous, piecewise differentiable curve $\varsigma\colon [t_0, t_i] \to T^*(\mathcal{V}_{n+1})$, consisting of N contiguous arcs $\varsigma^{(s)}\colon q^i = q^i_{(s)}(t)$, $p_0 = p^{(s)}_0(t)$, $p_i = p^{(s)}_i(t)$.

Discontinuities of the tangent vector $\dot{\varsigma}(t)$, with jumps $\dot{\varsigma}^{(s+1)}(a_s) - \dot{\varsigma}^{(s)}(a_s)$, may occur at $t = a_s$, $s = 1, \ldots, N-1$.

By means of the symplectic structure of $T^*(\mathcal{V}_{n+1})$, to each pair $\dot{\varsigma}^{(s)}(a_s)$, $\dot{\varsigma}^{(s+1)}(a_s)$ we associate the symplectic product $A_{(s)} := (\dot{\varsigma}^{(s)}(a_s), \dot{\varsigma}^{(s+1)}(a_s))$. We call $A_{(s)}$ it the *strength* of the corner x_s. A straightforward calculation provides the evaluation

$$A_{(s)} = \dot{\gamma}^{\mu}_{(s+1)}(a_s)\left(\frac{dp^{(s)}_{\mu}}{dt}\right)_{a_s} - \dot{\gamma}^{\mu}_{(s)}(a_s)\left(\frac{dp^{(s+1)}_{\mu}}{dt}\right)_{a_s} . \tag{29}$$

On account of Eq. (28), it is easily seen that the representation (29) is invariant under arbitrary gauge transformations of the form (22). For a normal extremaloid, the strengths $A_{(1)}, \ldots, A_{(N-1)}$ express therefore a set of intrinsic attributes, uniquely determined by the action functional. When at least one strength $A_{(s)}$ is nonzero, no differentiable gauge transformation can therefore exist, yielding a Lagrangian $\mathscr{L}' = \mathscr{L} - \dot{S}$ critical along the whole of γ: one can, of course, resort to N distinct transformations

$$\mathscr{L}(t, q^i, z^A) \to \mathscr{L}'_{(s)}(t, q^i, z^A) := \mathscr{L}(t, q^i, z^A) - \dot{S}^{(s)} , \tag{30}$$

each one yielding a Lagrangian $\mathscr{L}'_{(s)}$ critical along the arc $\hat{\gamma}^{(s)}$.

A "smart" choice of the transformations (30) relies on the following result, established in [16] and reported here without proof

Theorem 1. *There exists (at least) one family of differentiable functions $S^{(s)} \in \mathscr{F}(U_s)$, $s = 1, \ldots, N$ such that:*

(i) the relation

$$dS^{(s)}{}_{|\gamma^{(s)}} = \rho^{(s)} \tag{31}$$

holds along each arc $\gamma^{(s)}$;

(ii) at each corner x_s, the difference $\mathfrak{S}^{(s)} := S^{(s+1)} - S^{(s)} \in \mathscr{F}(U_s) \cap \mathscr{F}(U_{s+1})$ satisfies the conditions

$$\mathfrak{S}^{(s)}(x_s) = \left(\frac{\partial \mathfrak{S}^{(s)}}{\partial q^\mu}\right)_{x_s} = 0\,, \qquad \mu = 0, \ldots, n\,, \tag{32a}$$

$$\left(\frac{\partial^2 \mathfrak{S}^{(s)}}{\partial q^\mu \partial q^\nu}\right)_{x_s} = -A_{(s)}\delta^0_\mu \delta^0_\nu\,. \tag{32b}$$

Any family $\{S^{(s)}\}$ of functions satisfying Eqs. (31), (32a), (32b) is said to be *adapted* to the curve γ. The same terminology applies to the *local Lagrangians* $\mathscr{L}'_{(s)} \in \mathscr{F}(U_s)$ obtained from the original one through the local gauge transformations $\mathscr{L}'_{(s)} = \mathscr{L} - \dot{S}^{(s)}$. In view of Eqs. (4a), (25), (31), each adapted Lagrangian satisfies the relations

$$\big(\mathscr{L}'_{(s)}\big)_{\hat{\gamma}^{(s)}} = \big(\mathscr{L} - \dot{S}^{(s)}\big)_{\hat{\gamma}^{(s)}} = \big(\mathscr{L}\big)_{\hat{\gamma}^{(s)}} - \big\langle \rho, \dot{\gamma} \big\rangle = 0\,, \tag{33a}$$

$$\big(d\mathscr{L}'_{(s)}\big)_{\hat{\gamma}^{(s)}} = \big(d\mathscr{L}\big)_{\hat{\gamma}^{(s)}} - \frac{d}{dt}\big(dS^{(s)}\big)_{\gamma^{(s)}} = \big(d\mathscr{L}\big)_{\hat{\gamma}^{(s)}} - \frac{d\rho^{(s)}}{dt} = 0\,, \tag{33b}$$

ensuring the tensorial behavior of the Hessian of $\mathscr{L}'_{(s)}$ at each point $\hat{\gamma}^{(s)}(t)$.

Every family $\mathscr{L}' := \{\mathscr{L}'_{(s)}, s = 1, \ldots, N\}$ of adapted Lagrangians determines therefore a (generally discontinuous) symmetric covariant tensor field along $\hat{\gamma}$, henceforth denoted by $\big(d^2\mathscr{L}'\big)_{\hat{\gamma}}$ and called the Hessian of $\mathscr{L}'$.

In particular, if $\hat{X} = X^i\big(\frac{\partial}{\partial q^i}\big)_{\hat{\gamma}} + X^A\big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}}$, $\hat{Y} = Y^i\big(\frac{\partial}{\partial q^i}\big)_{\hat{\gamma}} + Y^A\big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}}$ is any pair of vectors in $A(\hat{\gamma})$, we write

$$\Big\langle \big(d^2\mathscr{L}'\big)_{\hat{\gamma}}, \hat{X} \otimes \hat{Y} \Big\rangle := \left(\frac{\partial^2 \mathscr{L}'}{\partial q^i \partial q^j}\right)_{\hat{\gamma}(t)} X^i Y^j + \\ + \left(\frac{\partial^2 \mathscr{L}'}{\partial q^i \partial z^A}\right)_{\hat{\gamma}(t)} \big(X^i Y^A + Y^i X^A\big) + \left(\frac{\partial^2 \mathscr{L}'}{\partial z^A \partial z^B}\right)_{\hat{\gamma}(t)} X^A Y^B\,, \tag{34}$$

with the understanding that, along any arc $\hat{\gamma}^{(s)}$, the symbol $\mathscr{L}'$ is meant as a shorthand for $\mathscr{L}'_{(s)}$.

Following the standard usage, the matrix $\Big(\frac{\partial^2 \mathscr{L}'}{\partial z^A \partial z^B}\Big)_{\hat{\gamma}}$ is denoted by G_{AB}. As pointed out in [15], if γ is a locally normal extremaloid lifting to a (generally discontinuous) section $\tilde{\gamma}\colon [t_0, t_1] \to \mathcal{C}(\mathcal{A})$, we have the identification

$$G_{AB}(t) = \left(\frac{\partial^2 \mathscr{H}}{\partial z^A \partial z^B}\right)_{\tilde{\gamma}(t)}\,,$$

$\mathscr{H}$ denoting the Pontryagin Hamiltonian (23). The matrix G_{AB} is therefore an intrinsic object, independent of the adaptation process. A normal extremaloid is called *regular* if and only if G_{AB} is everywhere non singular.

Another important consequence of Theorem 1 is the fact that, on account of Eqs. (32a), at each corner x_s the Hessian $\big(\frac{\partial^2 \mathfrak{S}^{(s)}}{\partial q^\mu \partial q^\nu}\big)_{x_s}$ is a tensor. We denote it by $\big[d^2 S\big]_{x_s}$. In coordinates, Eq. (32b) provides the representation

$$\big[d^2 S\big]_{x_s} = -A_{(s)} dt \otimes dt\,. \tag{35}$$

(iii) Under the regularity assumption $\det G^{(s)}_{AB} \neq 0$, the Hessian $\big(d^2 \mathscr{L}'_{(s)}\big)_{\hat\gamma}$ determines an infinitesimal control along each arc $\gamma^{(s)}$, uniquely defined by the condition

$$\Big\langle \big(d^2 \mathscr{L}'_{(s)}\big)_{\hat\gamma^{(s)}}, h^{(s)}(X) \otimes Y \Big\rangle = 0 \qquad \forall X \in V(\gamma^{(s)}), Y \in V(\hat\gamma^{(s)})\,. \tag{36a}$$

In view of Eqs. (17b), (34), the requirement (36a) is locally expressed by the relations

$$\left\langle \big(d^2 \mathscr{L}'_{(s)}\big)_{\hat\gamma^{(s)}}, \tilde\partial_i \otimes \left(\frac{\partial}{\partial z^A}\right)_{\hat\gamma^{(s)}} \right\rangle = \left(\frac{\partial^2 \mathscr{L}'_{(s)}}{\partial q^i \partial z^A}\right)_{\hat\gamma^{(s)}} + G^{(s)}_{AB}\, h_i^{(s)B} = 0\,. \tag{36b}$$

Under the assumption $\det G^{(s)}_{AB} \neq 0$, these may be solved for the unknowns $h_i^{(s)B}$, yielding the expressions

$$h_i^{(s)A} = -G^{AB}_{(s)} \left(\frac{\partial^2 \mathscr{L}'_{(s)}}{\partial q^i \partial z^B}\right)_{\hat\gamma^{(s)}},$$

whence also

$$\tilde\partial_i := h^{(s)}\left(\frac{\partial}{\partial q^i}\right)_{\gamma^{(s)}} = \left(\frac{\partial}{\partial q^i}\right)_{\hat\gamma^{(s)}} - G^{AB}_{(s)} \left(\frac{\partial^2 \mathscr{L}'_{(s)}}{\partial q^i \partial z^B}\right)_{\hat\gamma^{(s)}} \left(\frac{\partial}{\partial z^A}\right)_{\hat\gamma^{(s)}}, \tag{37}$$

with $G^{(s)}_{AB} G^{BC}_{(s)} = \delta^C_A$.

The absolute time derivative along $\gamma^{(s)}$ induced by $h^{(s)}$ is denoted by $\big(\frac{D}{Dt}\big)_{\gamma^{(s)}}$. On account of Eq. (37), the corresponding temporal connection coefficients read

$$\tau_i^{(s)j} = -\big(\tilde\partial_i \psi^j\big)_{\hat\gamma^{(s)}} = -\left(\frac{\partial \psi^j}{\partial q^i}\right)_{\hat\gamma^{(s)}} + G^{AB}_{(s)} \left(\frac{\partial \psi^j}{\partial z^A}\right)_{\hat\gamma^{(s)}} \left(\frac{\partial^2 \mathscr{L}'_{(s)}}{\partial q^i \partial z^B}\right)_{\hat\gamma^{(s)}}. \tag{38}$$

2.2 The second variation of the action functional

Let γ be a locally normal extremaloid, $\gamma_\xi = \big\{\big(\gamma_\xi^{(s)}, [a_{s-1}(\xi), a_s(\xi)]\big)\big\}$ a deformation of γ with fixed endpoints, and X the resulting infinitesimal deformation, with jumps $[X^i]_{x_s}$ related to the scalars $\alpha_s = \frac{da_s}{d\xi}\big|_{\xi=0}$ by Eq. (12).

Given a family of functions $S^{(s)}$ adapted to γ, for any $s = 1, \ldots, N-1$ we denote by $\mathfrak{S}^{(s)}(c_s(\xi))$ the restriction of the difference $\mathfrak{S}^{(s)} := S^{(s+1)} - S^{(s)}$ to the orbit of the corner x_s. On account of Eqs. (33) we have then the equality

$$\begin{aligned}\int_{\hat\gamma_\xi} \mathscr{L}\, dt - \int_{\hat\gamma} \mathscr{L}\, dt &= \sum_{s=1}^{N} \left\{ \int_{\hat\gamma_\xi^{(s)}} \Big(\mathscr{L}'_{(s)} + \dot S^{(s)} \Big) dt - \int_{\hat\gamma^{(s)}} \big(\dot S^{(s)} \big) dt \right\} = \\ &= \sum_{s=1}^{N} \int_{\hat\gamma_\xi^{(s)}} \mathscr{L}'_{(s)}\, dt - \sum_{s=1}^{N-1} \big(S^{(s+1)} - S^{(s)} \big)_{c_s(\xi)} = \\ &= \sum_{s=1}^{N} \int_{a_{s-1}(\xi)}^{a_s(\xi)} \mathscr{L}'_{(s)} \big(\hat\gamma_\xi^{(s)}(t) \big) dt - \sum_{s=1}^{N-1} \mathfrak{S}^{(s)}(c_s(\xi)) \,. \end{aligned} \tag{39}$$

Studying the second variation of the action functional is therefore equivalent to analysing the behaviour of the second derivative of the right-hand term of Eq. (39) in a neighborhood of $\xi = 0$. To this end, we restore the notation $\hat X$ for the lift of the field X and $W_{(s)}$ for the tangent vector to the orbit $c_s(\xi)$ at $\xi = 0$. A straightforward calculation then yields the evaluation

$$\begin{aligned}\frac{d^2}{d\xi^2} \int_{a_{s-1}(\xi)}^{a_s(\xi)} \mathscr{L}'_{(s)} \big(\hat\gamma_\xi^{(s)}(t) \big) dt \bigg|_{\xi=0} &= \int_{a_{s-1}}^{a_s} \left[\frac{d^2}{d\xi^2} \mathscr{L}'_{(s)} \big(\hat\gamma_\xi^{(s)}(t) \big) \right]_{\xi=0} dt \\ &= \int_{a_{s-1}}^{a_s} \Big\langle \big(d^2 \mathscr{L}'_{(s)} \big)_{\hat\gamma^{(s)}}, \hat X_{(s)} \otimes \hat X_{(s)} \Big\rangle\, dt \,.\end{aligned}$$

In a similar way, by the very definition of the Hessian, Eqs. (32b), (33a), (33b), (35) and the definition of $\mathfrak{S}^{(s)}(c_s(\xi))$ entail the relation

$$\frac{d^2 \mathfrak{S}^{(s)}(c_s(\xi))}{d\xi^2} \bigg|_{\xi=0} = \Big\langle \big[d^2 S \big]_{x_s}, W_{(s)} \otimes W_{(s)} \Big\rangle = -A_{(s)} \alpha_s^2 \,.$$

Summing up, we reach the final, plainly covariant expression

$$\frac{d^2 \mathcal{I}[\gamma_\xi]}{d\xi^2} \bigg|_{\xi=0} = \sum_{s=1}^{N} \int_{a_{s-1}}^{a_s} \Big\langle \big(d^2 \mathscr{L}'_{(s)} \big)_{\hat\gamma^{(s)}}, \hat X_{(s)} \otimes \hat X_{(s)} \Big\rangle\, dt + \sum_{s=1}^{N-1} A_{(s)} \alpha_s^2 \,. \tag{40}$$

In particular, whenever $\gamma^{(s)}$ is *regular*, introducing the horizontal basis (37) associated with the Hessian $\big(d^2 \mathscr{L}'_{(s)} \big)_{\hat\gamma^{(s)}}$ and expressing $\hat X_{(s)}$ in components as $\hat X_{(s)} = X^i_{(s)} \tilde\partial_i + U^A_{(s)} \big(\frac{\partial}{\partial z^A} \big)_{\hat\gamma^{(s)}}$, Eq. (36b) provides the identification

$$\Big\langle \big(d^2 \mathscr{L}'_{(s)} \big)_{\hat\gamma^{(s)}}, \hat X_{(s)} \otimes \hat X_{(s)} \Big\rangle = N^{(s)}_{kr} X^k_{(s)} X^r_{(s)} + G^{(s)}_{AB} U^A_{(s)} U^B_{(s)} \,, \tag{41}$$

with

$$N^{(s)}_{ij} := \Big\langle \big(d^2 \mathscr{L}'_{(s)} \big)_{\hat\gamma^{(s)}}, \tilde\partial_i \otimes \tilde\partial_j \Big\rangle = \left[\frac{\partial^2 \mathscr{L}'_{(s)}}{\partial q^i \partial q^j} - G^{AB}_{(s)} \frac{\partial^2 \mathscr{L}'_{(s)}}{\partial q^i \partial z^A} \frac{\partial^2 \mathscr{L}'_{(s)}}{\partial q^j \partial z^B} \right]_{\hat\gamma^{(s)}} . \tag{42}$$

3 Jacobi fields

3.1 Broken Jacobi fields

(i) Given a regular, locally normal extremaloid γ of the action functional $\mathcal{I}[\gamma]$, we define a *broken Jacobi field* X along γ as the infinitesimal deformation tangent to the orbits of a finite deformation γ_ξ consisting of a 1-parameter family of *extremaloids* of $\mathcal{I}[\gamma]$. No condition is placed on the behavior of X at the end-points of γ. For each ξ, we denote by $\rho_\xi = p_0(\xi,t)dt_{|\gamma_\xi} + p_i(\xi,t)dq^i{}_{|\gamma_\xi}$ the (unique) continuous, piecewise differentiable 1-form along γ_ξ satisfying the requirements of Proposition 3.

In order to set up a tensorial algorithm, we replace the original Lagrangian $\mathscr{L}$ by an adapted family $\mathscr{L}' := \{\mathscr{L}'_{(s)}, s = 1,\ldots,N\}$ of gauge equivalent ones, and denote by $\rho'_\xi = p'_0(\xi,t)dt_{|\gamma_\xi} + p'_i(\xi,t)dq^i{}_{|\gamma_\xi}$ the (generally discontinuous) 1-form along γ_ξ defined by the condition

$$\rho'_\xi(t) := \rho_\xi(t) - dS^{(s)}{}_{|\gamma^{(s)}_\xi(t)} \qquad \forall t \in [a_{s-1}(\xi), a_s(\xi)]\,.$$

Whenever necessary, we denote by $\rho'^{(s)}_\xi = p'^{(s)}_0 dt_{|\gamma^{(s)}_\xi} + p'^{(s)}_i dq^i{}_{|\gamma^{(s)}_\xi}$ the restriction of the 1-form ρ'_ξ to the arc $\gamma^{(s)}_\xi$. In this way, on account of Eq. (4a), Eqs. (25) read

$$\langle \rho'^{(s)}_\xi, \dot\gamma_\xi \rangle = \mathscr{L}'_{(s)|\hat\gamma_\xi}\,, \qquad \frac{d}{dt}\rho'^{(s)}_\xi = \big(d\mathscr{L}'_{(s)}\big)_{\hat\gamma_\xi}\,, \qquad s = 1,\ldots,N\,,$$

while the continuity of ρ_ξ along γ_ξ is expressed by the conditions

$$\big(\rho'^{(s+1)}_\xi - \rho'^{(s)}_\xi\big)_{a_s(\xi)} + d\mathfrak{S}^{(s)}{}_{|c_s(\xi)} = 0 \qquad s = 1,\ldots,N-1\,.$$

In the overall, adopting the representation $q^i = \varphi^i(\xi,t)$, $z^A = \zeta^A(\xi,t)$ for the lift $\hat\gamma_\xi$, we have thus a family of $2n+r+1$ functions satisfying the inner relation

$$p'_0(\xi,t) = \mathscr{L}'{}_{|\hat\gamma_\xi} - p'_k(\xi,t)\psi^k{}_{|\hat\gamma_\xi}\,, \tag{43}$$

the Pontryagin equations

$$\frac{\partial\varphi^i}{\partial t} = \psi^i(t,\varphi^i,\zeta^A)\,, \tag{44a}$$

$$\frac{\partial p'_i}{\partial t} + \left(\frac{\partial\psi^k}{\partial q^i}\right)_{\hat\gamma_\xi} p'_k = \left(\frac{\partial\mathscr{L}'}{\partial q^i}\right)_{\hat\gamma_\xi}\,, \tag{44b}$$

$$p'_i\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma_\xi} = \left(\frac{\partial\mathscr{L}'}{\partial z^A}\right)_{\hat\gamma_\xi}\,, \tag{44c}$$

the Jacobi equation

$$\frac{\partial p'_0}{\partial t} + \left(\frac{\partial\psi^k}{\partial t}\right)_{\hat\gamma_\xi} p'_k = \left(\frac{\partial\mathscr{L}'}{\partial t}\right)_{\hat\gamma_\xi} \tag{45}$$

and the matching conditions

$$\big(\varphi^i_{(s+1)} - \varphi^i_{(s)}\big)_{(\xi,a_s(\xi))} = 0\,, \qquad \big(p'^{(s+1)}_\mu - p'^{(s)}_\mu\big)_{(\xi,a_s(\xi))} = -\left(\frac{\partial\mathfrak{S}^{(s)}}{\partial q^\mu}\right)_{c_s(\xi)}\,. \tag{46}$$

Due to the normality of γ, Eqs. (44b), (44c) entail the relation $p'_i(0,t)=0$. On account of Eqs. (33a), (43), the latter implies also $p'_0(0,t)=0$, thus ensuring the tensorial behaviour of the derivatives $\frac{\partial p'_\mu}{\partial\xi}\big|_{\xi=0}$.

(ii) After these preliminaries, the analysis of the Jacobi fields splits into two parts. To start with, we deal with the differentiable aspects of the problem, along the same lines outlined in [15]. To this end we focus on a single arc $\gamma^{(s)}$, derive Eqs. (43), (44), (45) with respect to ξ and evaluate everything at $\xi=0$.

In this way, setting $X^i=\frac{\partial\varphi^i}{\partial\xi}\big|_{\xi=0}$, $X^A=\frac{\partial\zeta^A}{\partial\xi}\big|_{\xi=0}$, $\lambda_\alpha=\frac{\partial p'_\alpha}{\partial\xi}\big|_{\xi=0}$ and recalling Eqs. (33a), (33b), we obtain the system

$$\lambda_0+\lambda_i\psi^i{}_{|\hat\gamma}=0\,, \tag{47}$$

$$\frac{dX^i}{dt}=\left(\frac{\partial\psi^i}{\partial q^k}\right)_{\hat\gamma}X^k+\left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma}X^A\,, \tag{48a}$$

$$\frac{d\lambda_i}{dt}+\lambda_k\left(\frac{\partial\psi^k}{\partial q^i}\right)_{\hat\gamma}=\left(\frac{\partial^2\mathscr{L}'}{\partial q^i\partial q^k}\right)_{\hat\gamma}X^k+\left(\frac{\partial^2\mathscr{L}'}{\partial q^i\partial z^A}\right)_{\hat\gamma}X^A\,, \tag{48b}$$

$$\lambda_k\left(\frac{\partial\psi^k}{\partial z^A}\right)_{\hat\gamma}=\left(\frac{\partial^2\mathscr{L}'}{\partial z^A\partial q^k}\right)_{\hat\gamma}X^k+G_{AB}X^B\,, \tag{48c}$$

$$\frac{d\lambda_0}{dt}+\lambda_k\left(\frac{\partial\psi^k}{\partial t}\right)_{\hat\gamma}=\left(\frac{\partial^2\mathscr{L}'}{\partial t\partial q^k}\right)_{\hat\gamma}X^k+\left(\frac{\partial^2\mathscr{L}'}{\partial t\partial z^A}\right)_{\hat\gamma}X^A\,, \tag{49}$$

with $G_{AB}=\big(\frac{\partial^2\mathscr{L}'}{\partial z^A\partial z^B}\big)_{\hat\gamma}$.

As already pointed out, the functions $\lambda_\mu(t)$ form the components of a 1-form λ along γ. The functions $\lambda_i(t)$ are therefore the components of a *virtual* 1-form $\hat\lambda$ along γ, identical to the image of λ under the quotient map associated with the equivalence relation (6).

Because of the regularity assumption $\det G_{AB}\neq 0$, Eqs. (48) may be cast into Hamiltonian form. This is easily accomplished by means of the infinitesimal control $h^{(s)}\colon V(\gamma^{(s)})\to A(\hat\gamma^{(s)})$ induced by the Lagrangian $\mathscr{L}'_{(s)}$ along the arc $\hat\gamma^{(s)}$, as described at the end of Sec. 2.1.

To this end, we split each field $\hat X_{(s)}=X^i\big(\frac{\partial}{\partial q^i}\big)_{\hat\gamma^{(s)}}+X^A\big(\frac{\partial}{\partial z^A}\big)_{\hat\gamma^{(s)}}$ into the sum $X^i\tilde\partial_i+U^A\big(\frac{\partial}{\partial z^A}\big)_{\hat\gamma^{(s)}}$ of a horizontal and a vertical part, with

$$U^A_{(s)}=X^A_{(s)}-X^i_{(s)}h_i^{(s)A}=X^A_{(s)}+G^{AB}_{(s)}\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial z^B}\right)_{\hat\gamma^{(s)}}X^i_{(s)}\,. \tag{50}$$

Eq. (48c) takes then the form

$$\lambda_k^{(s)} \left(\frac{\partial \psi^k}{\partial z^A} \right)_{\hat{\gamma}^{(s)}} = G_{AB}^{(s)} U_{(s)}^B \quad \Longrightarrow \quad U_{(s)}^A = G_{(s)}^{AB} \lambda_k^{(s)} \left(\frac{\partial \psi^k}{\partial z^B} \right)_{\hat{\gamma}^{(s)}} , \tag{51}$$

showing that each field $\hat{X}$ is fully determined by the knowledge of $X_{(s)}^i$, $\lambda_i^{(s)}(t)$. Setting

$$M_{(s)}^{ij} = G_{(s)}^{AB} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}^{(s)}} \left(\frac{\partial \psi^j}{\partial z^B} \right)_{\hat{\gamma}^{(s)}} \tag{52}$$

and recalling the expression for the absolute time derivative induced by the control $h^{(s)}$, from Eqs. (42), (48a), (48b), (51) we derive the following system of differential equations for the unknowns $X_{(s)}^i(t)$, $\lambda_i^{(s)}(t)$:

$$\left(\frac{D X_{(s)}^i}{Dt} \right)_{\gamma^{(s)}} = G_{(s)}^{AB} \left(\frac{\partial \psi^i}{\partial z^A} \right)_{\hat{\gamma}^{(s)}} \left(\frac{\partial \psi^j}{\partial z^B} \right)_{\hat{\gamma}^{(s)}} \lambda_j^{(s)} = M_{(s)}^{ij} \lambda_j^{(s)} , \tag{53a}$$

$$\begin{aligned} \left(\frac{D \lambda_i^{(s)}}{Dt} \right)_{\gamma^{(s)}} &= \left[\left(\frac{\partial^2 \mathscr{L}'_{(s)}}{\partial q^i \partial q^j} \right)_{\hat{\gamma}^{(s)}} - G_{(s)}^{AB} \left(\frac{\partial^2 \mathscr{L}'_{(s)}}{\partial q^i \partial z^A} \right)_{\hat{\gamma}^{(s)}} \left(\frac{\partial^2 \mathscr{L}'_{(s)}}{\partial q^j \partial z^B} \right)_{\hat{\gamma}^{(s)}} \right] X_{(s)}^j \\ &= N_{ij}^{(s)} X_{(s)}^j . \end{aligned} \tag{53b}$$

The remaining equations fit naturally into the scheme: Eq. (47) determines λ_0 in terms of λ_i, while Eq. (49) follows identically from Eqs. (47), (48) and from the vanishing of $d\mathscr{L}'_{(s)}$ along $\hat{\gamma}$.

For each $s = 1, \dots, N$, the whole information is therefore carried by a vector field $X_{(s)} = X_{(s)}^i \big(\frac{\partial}{\partial q^i} \big)_{\gamma^{(s)}}$ and a virtual 1-form $\hat{\lambda}^{(s)} = \lambda_i^{(s)} \delta q^i{}_{|\gamma^{(s)}}$ satisfying Eqs. (53). We call them a *Jacobi pair* along $\gamma^{(s)}$.

Straightforward consequences of Eqs. (51), (53) are the assertions:

- the simultaneous vanishing of $X_{(s)}$ and $\hat{\lambda}^{(s)}$ at a point $t^* \in [a_{s-1}, a_s]$ entails the vanishing of $X_{(s)}(t)$ and $\hat{\lambda}^{(s)}(t)$ all over $\gamma^{(s)}$;

- the vanishing of $\hat{X}_{(s)}$ along the whole of $\hat{\gamma}^{(s)}$ yields the relations

$$\left(\frac{D \lambda_i^{(s)}}{Dt} \right)_{\gamma^{(s)}} = 0 , \qquad \lambda_i^{(s)}(t) \frac{\partial \psi^i}{\partial z^A} \bigg|_{\hat{\gamma}^{(s)}} = 0 ,$$

  which, together with the local normality of $\gamma^{(s)}$, ensure the vanishing of $\lambda^{(s)}$ all over $\gamma^{(s)}$.

(iii) To complete the picture, we have still to consider the implications of the matching conditions (46). From these, deriving with respect to ξ, evaluating everything at $\xi = 0$ and taking Eqs. (13), (32b) as well as the identities $p_\mu'^{(s)}(0,t) = p_\mu'^{(s+1)}(0,t) = 0$ into account, we get the *jump relations*

$$\left[X^i \right]_{x_s} = -\alpha_s \left[\psi^i \right]_{x_s} , \tag{54}$$

identical to Eq. (12), and

$$\left[\lambda_\mu\right]_{x_s} + \left(\cancel{\frac{\partial p'^{(s+1)}_\mu}{\partial t}}\bigg|_{\xi=0} - \cancel{\frac{\partial p'^{(s)}_\mu}{\partial t}}\bigg|_{\xi=0} \right)\alpha_s = -W_s^{(J)}\left(\frac{\partial \mathfrak{S}^{(s)}}{\partial q^\mu}\right) = A_{(s)}\alpha_s\delta^0_\mu \,.$$

In view of Eq. (47), the latter breaks up into the pair of conditions

$$\left[\lambda_i\right]_{x_s} = 0\,, \qquad \lambda_i(a_s)\left[\psi^i\right]_{x_s} = -A_{(s)}\alpha_s \tag{55}$$

which, in turn, can be merged into the single expression

$$\left[\lambda_i(dq^i - \psi^i dt)\right]_{x_s} = A_{(s)}\alpha_s\, dt_{|x_s}\,.$$

Whenever $A_{(s)} \neq 0$, Eqs. (54), (55) can be uniquely solved for the unknowns $[X^i]_{x_s}$, $[\lambda_i]_{x_s}$, α_s, thereby determining the jumps of the fields $X, \hat{\lambda}$ as well as the tangent vector to the orbit of the corner x_s.

The circumstance $A_{(s)} = 0$ constitutes, on the contrary, a criticality of the algorithm: in this case, the second equation (55) does no longer determine the value of α_s, but becomes a *constraint* for the components $\lambda_i(a_s)$. The fulfilment of the latter at the first corner x_s in which the condition $A_{(s)} = 0$ occurs entails a restriction on the values $X^i(t_0)$, $\lambda_i(t_0)$: there exist, therefore, Jacobi fields in the interval $[t_0, a_s]$ which cannot be prolonged beyond x_s.

On the other hand, when the initial values are consistent with the constraint, the coefficient α_s may be arbitrarily chosen, thus giving rise to infinite possible prolongations of the fields from $\gamma^{(s)}$ to $\gamma^{(s+1)}$. For example, nothing can prevent a Jacobi pair null within the interval $[t_0, a_s]$ from "waking up" at the corner x_s, jumping to $\lambda_i^{(s+1)}(a_s) = 0$, $X^i_{(s+1)}(a_s) = -\alpha_s\left[\psi^i\right]_{x_s}$, α_s being any real number.

In any case, independently of the values of $A_{(s)}$, any pair of fields $(X(t), \hat{\lambda}(t))$ satisfying Eqs. (53) and the jump conditions (54), (55) are said to form a *broken Jacobi pair* along γ.

3.2 The accessory variational problem

A further insight into the content of the Jacobi equations is gained regarding the second variation (40) as an action functional over the space of admissible infinitesimal deformations. This gives rise to a new variational problem, henceforth called the *accessory* problem, whose solutions are precisely the Jacobi fields along $\hat{\gamma}$. This statement, well known in the differentiable context (see e.g. [15] and references therein), will now be extended to an arbitrary, locally normal extremaloid $\gamma = \left\{\left(\gamma^{(s)}, [a_{s-1}, a_s]\right)\right\}$.

To this end, resuming the notation of Section 1, we consider the vector bundles $V(\gamma) \to \mathbb{R}$, with coordinates t, u^i, and $A(\hat{\gamma}) \to V(\gamma)$, with coordinates t, u^i, v^A. To these we add the dual bundle $V^*(\gamma)$, referred to dual coordinates t, π_i, the jet-bundle $j_1(V(\gamma)) \to V(\gamma)$, referred to jet coordinates $t, u^i, \dot{u}^i$ and the vertical subbundle $V(\hat{\gamma}) \subset A(\hat{\gamma})$, meant as the kernel of the projection $A(\hat{\gamma}) \to V(\gamma)$, and referred to coordinates t, v^A.

We adopt $V(\gamma)$ as our "accessory" event space, and regard $A(\hat{\gamma})$ as a piecewise continuous linear subbundle of $j_1(V(\gamma))$, consisting of N disjoint open subsets $\Sigma_s := \mathcal{A}(\hat{\gamma}^{(s)})$, described in coordinates as

$$\dot{u}^i = \left(\frac{\partial \psi^i}{\partial q^k}\right)_{\hat{\gamma}^{(s)}} u^k + \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} v^A := \tilde{\psi}^i_{(s)}(t, u^i, v^A)\,, \qquad a_{s-1} < t < a_s\,. \tag{56a}$$

For simplicity, we summarize all previous expressions into a single symbolic equation

$$\dot{u}^i = \tilde{\psi}^i(t, u^i, v^A)\,. \tag{56b}$$

Notice that, as it stands, Eq. (56b) is not defined for $t = a_1, \ldots, a_{N-1}$. For each such value of t, we assign instead the vector

$$[\psi]_{x_s} := \dot{\gamma}^{(s+1)}(a_s) - \dot{\gamma}^{(s)}(a_s) = [\psi^i]_{x_s} \big(\tfrac{\partial}{\partial q^i}\big)_{x_s} \in V(\mathcal{V}_{n+1})\,,$$

expressing the *jump* of the tangent vector to γ at the corner $x_s = \gamma(a_s)$.

A piecewise continuous section $X \colon [t_0, t_1] \to V(\gamma)$ is called *admissible* if and only if

- the restriction of X to each open interval (a_{s-1}, a_s), $s = 1, \ldots, N$ may be prolonged to a differentiable section $X_{(s)} \colon (b_{s-1}, b_s) \to V(\gamma)$, defined on an open interval $(b_{s-1}, b_s) \supset [a_{s-1}, a_s]$;
- each section $X_{(s)} \colon (b_{s-1}, b_s) \to V(\gamma)$ admits a lift $\hat{X}_{(s)} \colon (b_{s-1}, b_s) \to A(\hat{\gamma})$ satisfying the condition $i \cdot \hat{X}_{(s)} = j_1(X_{(s)})$;
- at each $t = a_1, \ldots, a_{N-1}$, the jump $[X]_{a_s} := X_{(s+1)}(a_s) - X_{(s)}(a_s)$ is proportional to the vector $[\psi]_{x_s}$, i.e. it admits a representation of the form

$$[X]_{a_s} = -\alpha_s [\psi]_{x_s}, \qquad \alpha_s \in \mathbb{R}\,. \tag{57}$$

With the stated conventions, the admissible sections $X \colon [t_0, t_1] \to V(\gamma)$ are readily seen to coincide with the admissible deformations of the extremaloid γ.

In order to set up a dynamical scheme, we now introduce a "Lagrangian" on $A(\hat{\gamma})$, meant as a family of differentiable functions $\hat{L}_{(s)}$, each one defined on the open domain $\Sigma_s \colon a_{s-1} < t < a_s$, and merged into a single function $\hat{L}(t, u^i, v^A)$. Once again, the function $\hat{L}$ is undefined for $t = a_1, \ldots, a_{N-1}$. For each such value of t, we assign instead a function $f_s(\alpha_s)$, depending on the entity of the jump $[X]_{a_s}$.

By means of $\hat{L}$ and $f_s(a_s)$ we eventually construct the *action functional*

$$\mathcal{J}[X] = \sum_{s=1}^{N} \int_{a_{s-1}}^{a_s} \hat{L}_{(s)}\big(t, X^i_{(s)}, X^A_{(s)}\big)\, dt + \sum_{s=1}^{N-1} f_s(\alpha_s)\,, \tag{58}$$

assigning to each admissible section $X \colon [t_0, t_1] \to V(\gamma)$ an overall "cost", embodying possible contributions from the arcs of X *and* from the jumps.

To study the extremals of the functional (58), given an admissible section $X \colon [t_0, t_1] \to V(\gamma)$, we consider an arbitrary infinitesimal deformation X_ξ with fixed end points, and lift both X and X_ξ to sections $\hat{X}$, $\hat{X}_\xi$ of $A(\hat{\gamma})$. In coordinates, on each open interval (a_{s-1}, a_s) we have the representations

$$\hat{X}\colon u^i = X^i_{(s)}(t), v^A = X^A_{(s)}(t)\,, \qquad \hat{X}_\xi\colon u^i = X^i_{(s)}(t,\xi), v^A = X^A_{(s)}(t,\xi)$$

with the functions $X^i_{(s)}$, $X^A_{(s)}$ satisfying the admissibility requirement

$$\frac{\partial X^i_{(s)}}{\partial t} = \tilde{\psi}^i_{(s)}\big(t, X^i_{(s)}(t,\xi), X^A_{(s)}(t,\xi)\big) \tag{59}$$

as well as jump conditions of the form $[X_\xi]_{a_s} = -\alpha_s(\xi)[\psi]_{x_s}$.

Setting $U^i_{(s)} := \left(\frac{\partial X^i_{(s)}}{\partial \xi}\right)_{\xi=0}$, $V^A_{(s)} := \left(\frac{\partial X^A_{(s)}}{\partial \xi}\right)_{\xi=0}$, $\beta_s := \left(\frac{d\alpha_s}{d\xi}\right)_{\xi=0}$ and taking the vectorial character of the fibres of $V(\gamma)$ and the linearity of Eqs. (59) into account, it is now an easy matter to verify that the infinitesimal deformation tangent to X_ξ is itself a section $U\colon [t_0, t_1] \to V(\gamma)$, vanishing at the endpoints and consisting of N arcs $U_{(s)}$ satisfying the admissibility condition

$$\frac{dU^i_{(s)}}{dt} = \frac{\partial \tilde{\psi}^i_{(s)}}{\partial u^k} U^k_{(s)} + \frac{\partial \tilde{\psi}^i_{(s)}}{\partial v^A} V^A_{(s)} = \tilde{\psi}^i_{(s)}\big(t, U^i_{(s)}, V^A_{(s)}\big) \tag{60}$$

as well as the jump relations $[U]_{a_s} = -\beta_s[\psi]_{x_s}$.

From this, arguing as in Section 1, we conclude that every infinitesimal deformation of the section X is uniquely determined by the knowledge of the functions $V^A_{(s)}(t)$, $s = 1, \dots, N$ and of the scalars β_s, $s = 1, \dots, N-1$.

After these preliminaries, given the section X and the deformation X_ξ, let us now consider the *first variation* of the action functional (58)

$$\left.\frac{d\mathcal{J}}{d\xi}\right|_{\xi=0} = \sum_{s=1}^{N} \int_{a_{s-1}}^{a_s} \left(\frac{\partial \hat{L}_{(s)}}{\partial u^i} U^i_{(s)} + \frac{\partial \hat{L}_{(s)}}{\partial v^A} V^A_{(s)}\right) dt + \sum_{s=1}^{N-1} f'_s(\alpha_s)\beta_s\,. \tag{61}$$

To evaluate the latter, we introduce an auxiliary virtual 1-form λ, namely a continuous, piecewise differentiable section of the dual bundle $V^*(\gamma)$, described in coordinates on each interval $a_{s-1} \le t \le a_s$ as $\pi_i = \lambda^{(s)}_i(t)$, and related to X by the evolution equation

$$\frac{d\lambda^{(s)}_i}{dt} + \lambda^{(s)}_k \frac{\partial \tilde{\psi}^k_{(s)}}{\partial u^i} = \frac{\partial \hat{L}_{(s)}}{\partial u^i}\big(t, X^i_{(s)}, X^A_{(s)}\big), \qquad s = 1, \dots, N, \tag{62}$$

completed by the matching conditions $\lambda^{(s+1)}(a_s) = \lambda^{(s)}(a_s) := \lambda(a_s)$.

In view of Eq. (62), taking Eqs. (57), (60) and the continuity of λ into account, the right-hand side of Eq. (61) may be written in the form

$$\begin{aligned}\left.\frac{d\mathcal{J}}{d\xi}\right|_{\xi=0} &= \sum_{s=1}^{N}\int_{a_{s-1}}^{a_s}\left[\frac{d}{dt}\lambda_i^{(s)}U^i_{(s)} - \left(\lambda_i^{(s)}\frac{\partial\tilde{\psi}^i_{(s)}}{\partial v^A} - \frac{\partial\hat{L}_{(s)}}{\partial v^A}\right)V^A_{(s)}\right]dt \\ &\quad + \sum_{s=1}^{N-1} f_s'(\alpha_s)\beta_s \\ &= -\sum_{s=1}^{N}\int_{a_{s-1}}^{a_s}\left(\lambda_i^{(s)}\frac{\partial\tilde{\psi}^i_{(s)}}{\partial v^A} - \frac{\partial\hat{L}_{(s)}}{\partial v^A}\right)V^A_{(s)}dt \\ &\quad + \sum_{s=1}^{N-1}\Big(f_s'(\alpha_s) + \lambda_i(a_s)[\psi]_{x_s}\Big)\beta_s\,. \end{aligned} \tag{63}$$

Due to the independence of the quantities $V^A_{(s)}, \beta_s$, the vanishing of the first variation (61) is equivalent to the validity of the equations

$$\lambda_i^{(s)}\frac{\partial\tilde{\psi}^i_{(s)}}{\partial v^A} = \frac{\partial\hat{L}_{(s)}}{\partial v^A}\,, \qquad s = 1,\ldots,N\,, \tag{64a}$$

$$f_s'(\alpha_s) = -\lambda_i(a_s)[\psi]_{x_s}\,, \qquad s = 1,\ldots,N-1\,. \tag{64b}$$

These, together with Eqs. (56a), (62), provide a set of $2n+r$ equations for the unknowns $X^i_{(s)}, X^A_{(s)}, \lambda_i^{(s)}$, formally analogous to (a linearized counterpart of) the Pontryagin equations (2), (21a), (21b), completed with the jump conditions (64b) and with the continuity requirement $[\lambda]_{a_s} = 0$.

Returning to the variational characterization of the broken Jacobi fields along a locally normal extremaloid γ, we now observe that the expression (40) for the second variation of the original action functional $\mathcal{I}[\gamma]$ is precisely of the form (58). More specifically, if we consider the accessory functional

$$\mathcal{J}[X] = \frac{1}{2}\left[\sum_{s=1}^{N}\int_{a_{s-1}}^{a_s}\Big\langle\big(d^2\mathscr{L}'_{(s)}\big)_{\hat{\gamma}^{(s)}}, \hat{X}_{(s)}\otimes\hat{X}_{(s)}\Big\rangle dt + \sum_{s=1}^{N-1}A_{(s)}\alpha_s^2\right], \tag{65}$$

corresponding to the ansatz

$$\begin{aligned}\hat{L}_{(s)} &= \frac{1}{2}\left[\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial q^j}\right)_{\hat{\gamma}^{(s)}}u^iu^j + 2\left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial z^A}\right)_{\hat{\gamma}^{(s)}}u^iv^A + \left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial z^A\partial z^B}\right)_{\hat{\gamma}^{(s)}}v^Av^B\right], \\ f_s(\alpha_s) &:= \frac{1}{2}A_{(s)}\alpha_s^2\,.\end{aligned}$$

Eqs. (56a), (62), (64) yield the system

$$\frac{dX^i_{(s)}}{dt} = \left(\frac{\partial\psi^i}{\partial q^k}\right)_{\hat\gamma^{(s)}} X^k_{(s)} + \left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} X^A_{(s)}\,, \tag{66a}$$

$$\frac{d\lambda^{(s)}_i}{dt} + \lambda^{(s)}_k \left(\frac{\partial\psi^k}{\partial q^i}\right)_{\hat\gamma^{(s)}} = \left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial q^k}\right)_{\hat\gamma^{(s)}} X^k_{(s)} + \left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial z^A}\right)_{\hat\gamma^{(s)}} X^A_{(s)}\,, \tag{66b}$$

$$\lambda^{(s)}_i \left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} = \left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial q^i\partial z^A}\right)_{\hat\gamma^{(s)}} X^i_{(s)} + \left(\frac{\partial^2\mathscr{L}'_{(s)}}{\partial z^A\partial z^B}\right)_{\hat\gamma^{(s)}} X^B_{(s)}\,, \tag{66c}$$

$$A_{(s)}\alpha_s = -\lambda_i(a_s)[\psi^i]_{x_s}\,. \tag{66d}$$

Comparison with Eqs. (48a), (48b), (48c), (55) shows that the extremals of the accessory variational problem based on the functional (65) are indeed the broken Jacobi fields along γ.

References

[1] O. Bolza: The Determination of the Conjugate Points for Discontinuous Solutions in the Calculus of Variations. *Amer. J. Math.* 30 (1908) 209–221.

[2] C. Caratheodory: *Über die diskontinuierlichen Lösungen in der Variationsrechnung, Doctor-Dissertation, Universität Göttingen 1904.* C. H. Beck'sche Verlagsbuchhandlulng (1954).

[3] C. Caratheodory: Über die starken Maxima und Minima bei einfachen Integralen, Habilitationsschrift, Universität Gottingen 1905. *Mathematische Annalen* 62 (1906) 449–503.

[4] C. Caratheodory: Sur les points singuliers du problème du Calcul des Variations dans le plan. *Annali di Matematica pura e applicata* 21 (1913) 153–171.

[5] A. Dresden: An Example of the Indicatrix in the Calculus of Variations. *Am. Math. Mon.* 14 (1907) 119–126.

[6] A. Dresden: An Example of the Indicatrix in the Calculus of Variations (continued). *Am. Math. Mon.* 14 (1907) 143–150.

[7] A. Dresden: The Second Derivatives of the Extremal-Integral. *Trans. Amer. Math. Soc.* 9 (1908) 467–486.

[8] G. Erdmann: Über die unstetige Lösungen in der Variationsrechnung. *J. Reine Angew. Math.* 82 (1877) 21–30.

[9] M. Giaquinta, S. Hildebrandt: *Calculus of variations I, II.* Springer-Verlag, Berlin, Heidelberg, New York (1996).

[10] L.M. Graves: Discontinuous Solutions in the Calculus of Variations. *Bull. Amer. Math. Soc.* 36 (1930) 831–846.

[11] L.M. Graves: Discontinuous Solutions in Space Problems of the Calculus of Variations. *Amer. J. Math.* 52 (1930) 1–28.

[12] J. Hadamard: Leçons sur le calcul des variations. *Hermann et fils, Paris*(1910) 3–88.

[13] M. R. Hestenes: *Calculus of variations and optimal control theory.* Wiley, New York, London, Sydney (1966).

[14] E. Massa, D. Bruno, G. Luria, E. Pagani: Geometric constrained variational calculus. I: Piecewise smooth extremals. *Int. J. Geom. Methods Mod. Phys.* 12 (2015). 1550061

[15] E. Massa, D. Bruno, G. Luria, E. Pagani: Geometric constrained variational calculus. II: The second variation (Part I). *Int. J. Geom. Methods Mod. Phys.* 13 (2016). 1550132

[16] E. Massa, G. Luria, E. Pagani,: Geometric constrained variational calculus. III: The second variation (Part II). *Int. J. Geom. Methods Mod. Phys.* 13 (2016). 1650038

[17] A. A. Milyutin, N. P. Osmolovskii: *Calculus of Variations and Optimal Control (Translations of Mathematical Monographs).* American Mathematical Society (1998).

[18] N.P. Osmolovskii, F. Lempio: Jacobi conditions and the Riccati equation for a broken extremal. *J. Math. Sci* 100 (2000) 2572–2592.

[19] N.P. Osmolovskii, F. Lempio: Transformation of Quadratic Forms to Perfect Squares for Broken Extremals. *Set-Valued Var. Anal.* 10 (2002) 209–232.

[20] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko: *The mathematical theory of optimal processes.* Interscience Publishers John Wiley & Sons Inc, New York-London (1962).

[21] W.T. Reid: Discontinuous Solutions in the Non-Parametric Problem of Mayer in the Calculus of Variations. *Amer. J. Math.* 57 (1935) 69–93.

[22] P.R. Rider: The figuratix in the Calculus of Variations. *Trans. Amer. Math. Soc.* 28 (1926) 640–653.

[23] H. Sagan: *Introduction to the calculus of variations.* McGraw–Hill Book Company, New York (1969).

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