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The calculus of variations on jet bundles as a universal approach for a variational formulation of fundamental physical theories

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Abstract. As widely accepted, justified by the historical developments of physics, the background for standard formulation of postulates of physical theories leading to equations of motion, or even the form of equations of motion themselves, come from empirical experience. Equations of motion are then a starting point for obtaining specific conservation laws, as, for example, the well-known conservation laws of momenta and mechanical energy in mechanics. On the other hand, there are numerous examples of physical laws or equations of motion which can be obtained from a certain variational principle as Euler-Lagrange equations and their solutions, meaning that the "true trajectories" of the physical systems represent stationary points of the corresponding functionals.

It turns out that equations of motion in most of the fundamental theories of physics (as e.g. classical mechanics, mechanics of continuous media or fluids, electrodynamics, quantum mechanics, string theory, etc.), are Euler-Lagrange equations of an appropriately formulated variational principle. There are several well established geometrical theories providing a general description of variational problems of different kinds. One of the most universal and comprehensive is the calculus of variations on fibred manifolds and their jet prolongations. Among others, it includes a complete general solution of the so-called strong inverse variational problem allowing one not only to decide whether a concrete equation of motion can be obtained from a variational principle, but also to construct a corresponding variational functional. Moreover, conservation laws can be derived from symmetries of the Lagrangian defining this functional, or directly from symmetries of the equations.

In this paper we apply the variational theory on jet bundles to tackle some fundamental problems of physics, namely the questions on existence of a Lagrangian and the problem of conservation laws. The aim is to demonstrate that the methods are universal, and easily applicable to distinct physical disciplines: from classical mechanics, through special relativity, waves, classical electrodynamics, to quantum mechanics.

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This paper is dedicated to the memory of prof. Gennadi Sardanashvily

1 Introduction

It is obvious and widely accepted in physics that experiments serve as the starting point for formulation of hypotheses or postulates lying in the foundations of physical theories. The postulates either lead to equations of motion of corresponding physical systems, or they represent the equations of motion themselves.

The former of the two mentioned situations can be demonstrated e.g. on electrodynamics: the empirical physical laws lead to Maxwell equations, i.e. equations of motion in electrodynamics representing the differential form of the empirical integral laws:

- **Coulomb law** for the electrostatic interaction of particles with charges q_1 and q_2 separated by a vector $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$:

  $$-\vec{F}_{12} = \vec{F}_{21} = \frac{q_1 q_2}{4\pi\varepsilon}\frac{\vec{r}_{12}}{r_{12}^3} \;\Rightarrow\; \int\limits_S \vec{E}\,\mathrm{d}\vec{S} = \frac{Q}{\varepsilon} \;\Rightarrow\; \operatorname{div}\vec{E} = \frac{\varrho}{\varepsilon},$$

  where $\vec{E}(\vec{r})$ or $\vec{E}(t, \vec{r})$ is the intensity of the electrostatic, or, electric field in general, at the point $\vec{r}$, S is a closed surface surrounding the charge Q, ε is the electric permeability.

- The **law of continuity of the magnetic flow** which states that here exist no magnetic monopoles:

  $$\int\limits_S \vec{B}\,\mathrm{d}\vec{S} = 0 \;\Rightarrow\; \operatorname{div}\vec{B} = 0,$$

  where $\vec{B}$ is the magnetic induction.

- **Faraday law** of electromagnetic induction stating that the variable magnetic flow induces between the ends of a closed loop the voltage

  $$U_{\mathrm{ind}} = -\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_S \vec{B}\,\mathrm{d}\vec{S} \;\Rightarrow\; \operatorname{rot}\vec{E} = -\frac{\partial\vec{B}}{\partial t}.$$

- **Ampère law** concerning the interaction of static magnetic fields with electric currents generalized for variable magnetic fields and combined with the continuity equation representing the conservation law for the electric charge:

  $$\mathrm{d}\vec{F} = I\,\mathrm{d}\vec{\ell}\times\vec{B} \;\Rightarrow\; \int\limits_C \vec{B}\,\mathrm{d}\vec{\ell} = \mu I, \quad \operatorname{rot}\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{j},$$

  where $\mathrm{d}\vec{F}$ is the elementary force acting on the length element $\mathrm{d}\vec{\ell}$ of an electric conductor with total current I placed in a magnetic field, $\vec{H}$ is the magnetic intensity, $\vec{j}$ is the current density and μ the magnetic permeability. (In a general situation I represents the total current, i.e. the free current and the magnetization current together.)

Examples when equations of motion themselves are formulated as postulates, are e.g. the following:

- In **Newtonian mechanics** the second Newton law for a mass particle (one of the postulates of the theory) represents the equation of motion of a particle with mass m,
$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F},$$
where $\vec{p} = m\vec{v}$ is the momentum of the particle, $\vec{v}$ is its velocity and $\vec{F}$ is the total force acting on the particle.

- Some approaches of **quantum mechanics** present the Schrödinger equation as a postulate,
$$\mathrm{i}\hbar\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle.$$
(This equation describes the time development of the state vector $|\psi\rangle$ and thus it is the equation of motion; recall that $\hat{H}$ is the Hamiltonian operator — operator of the total energy of the quantum mechanical system).

On the other hand, it is known that in numerous cases important physical laws and equations of motion are derived from an appropriately formulated variational principle. In fact, this was a fascinating discovery which initiated rapid developments of the calculus of variations and its extremely fruitful interference with physics. As a well-known example we can mention the "geometrical" rules for light reflection (law of reflection) and transmission (Snell law) of light on the boundary of two optical media. These laws result from the Fermat principle. Another typical example is Lagrangian mechanics concerning mechanical systems with defined potential energy (recall that this does not concern systems of particles moving in non-conservative fields).

In physics there are various, more or less empirical approaches to find out whether concrete equations of motion are variational, i.e. if they can be obtained from a certain variational principle, as well as various approaches how to obtain conservation laws important from the point of view of physics. However, remarkably, there are mathematical tools which allow one to treat these questions in a unified and systematic way, at any degree of generality (mechanics or field theory, first order or higher order problems). Among them, one of the most powerful is the calculus of variations on fibred manifolds. The aim of this paper is to stress that (and how) it provides a unified, universal and general setting for solving problems concerning variational properties of different physical theories. In particular, we shall show how

- using results on the solution of the so-called *inverse problem of the calculus of variations* one can examine whether a given equation of motion or a set of equations of motion comes from a variational principle, and determine a Lagrangian;

- one can find *Noether symmetries* of a Lagrangian defining a variational principle, i.e. transformations leaving the Lagrangian invariant, and obtain the corresponding quantities conserved along solutions of equations of motion, so called *Noetherian currents*.

We want to demonstrate that the method is universal and easily applicable to fundamental equations of distinct physical disciplines: from classical mechanics, through special relativity, waves, classical electrodynamics, to quantum mechanics.

The paper is organized as follows: First, we briefly summarize basic results of the calculus of variations in fibred manifolds required for solving the questions above (sections 2 and 3), and then we apply them to equations of motion of selected fundamental physical theories (section 4) to study variationality and determine the corresponding conservation laws. We asume that the reader is familiar with basic definitions and theorems concerning the geometry of fibred manifolds as well as with fundamentals of the calculus of variations. For more information and systematic study we refer to the books [2], [3], [4], [6], [8], [13], [14].

2 Basic concepts and notations

As underlying structures we consider fibred manifolds and their jet prolongations. Geometrical concepts and notation we use are standard. We briefly summarize only those of them we will use in the following text (without proofs of their properties). Einstein summation is used throughout the paper.

2.1 Fibred manifolds and fibred charts

Let $\pi : Y \to X$, $\dim Y = m + n$, $\dim X = n$, be a fibred manifold, $\pi_r : J^rY \to X$ its r-th jet prolongation, $r = 0, 1, 2, \ldots$, where $J^0Y = Y$. For $n = 1$ the corresponding theory is called *mechanics*, for $n > 1$ *field theory*. Let $V \subset Y$ be an open set. Then (V, ψ), where $\psi = (t, q^\sigma)$, respectively, $\psi = (x^i, y^\sigma)$, are coordinate functions, $1 \le i \le n$, $1 \le \sigma \le m$, is a *fibred chart on* Y and (U, φ), $U = \pi(V)$, $\varphi = (t)$, respectively, $\varphi = (x^i)$, $1 \le i \le n$, is the *associated chart on* X. (V_1, ψ_1), $\psi_1 = (t, q^\sigma, \dot{q}^\sigma)$, respectively, $\psi_1 = (x^i, y^\sigma, y^\sigma_i)$, $V_1 = \pi^{-1}_{1,0}(V)$, is the *associated fibred chart on* J^1Y. Similarly, (V_2, ψ_2), $\psi_2 = (t, q^\sigma, q^\sigma_1, q^\sigma_2) = (t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$, or $\psi_2 = (x^i, y^\sigma, y^\sigma_i, y^\sigma_{ij})$, $V_2 = \pi^{-1}_{2,0}(V)$, is the *associated fibred chart on* J^2Y. Charts for higher order prolongations of the underlying fibred manifold are defined quite analogously.

2.2 Sections and vector fields

A mapping $\gamma : U \to Y$, $\pi \circ \gamma = \mathrm{id}_U$, denotes a section of the projection π defined on an open subset $U \subset X$, $\gamma(t) = (t, q^\sigma\gamma(t))$, or $\gamma(x^i) = (x^i, y^\sigma\gamma(x^i))$; $J^r\gamma$, $r = 1, 2, \ldots$, are prolongations of γ. The set of all sections defined on U is denoted $\Gamma_U(\pi)$; $\Gamma_\Omega(\pi)$, where $\Omega \subset X$ is a compact connected n-dimensional submanifold of X with boundary, denotes a set of all sections defined on open sets containing Ω.

The standard concepts of projectable and vertical vector fields (which are adapted to the fibred structure) are used: A vector field ξ on J^rY, $r = 0, 1, 2, \ldots$, is *π_r-projectable* if there exists a vector field ξ_0 on X (its *π_r-projection*) such that $T\pi_r(\xi) = \xi_0 \circ \pi$. A vector field ξ on J^rY is *$\pi_{r,s}$-projectable*, $0 \le s < r$, if there exists a vector field $\xi_{r,s}$ on J^sY (its $\pi_{r,s}$-*projection*) such that $T\pi_{r,s}(\xi) = \xi_{r,s} \circ \pi_{r,s}$. A vector field ξ on J^rY is π_r-*vertical*, respectively, $\pi_{r,s}$-*vertical* if its π_r-projection, respectively, $\pi_{r,s}$-projection is the zero vector field.

For π-projectable vector fields on Y we can define their jet prolongations for $r = 1, 2, \ldots$. In fibred coordinates it holds

$$\xi = \xi^0(t)\frac{\partial}{\partial t} + \Xi^\sigma(t, q^\nu)\frac{\partial}{\partial q^\sigma}, \tag{1}$$

$$J^r\xi = \xi^0(t)\frac{\partial}{\partial t} + \sum_{j=0}^{r} \Xi^\sigma_j(t, q^\nu, \ldots, q^\nu_j)\frac{\partial}{\partial q^\sigma_j}, \quad \Xi^\sigma_j = \frac{\mathrm{d}\Xi^\sigma_{j-1}}{\mathrm{d}t} - q^\sigma_j \frac{\mathrm{d}\xi^0}{\mathrm{d}t},$$

$1 \leq j \leq r$, for mechanics and

$$\xi = \xi^j(x^i)\frac{\partial}{\partial x^j} + \Xi^\sigma(x^i, y^\nu)\frac{\partial}{\partial y^\sigma},$$

$$J^r\xi = \xi^j(x^i)\frac{\partial}{\partial x^j} + \sum_{s=0}^{r} \Xi^\sigma_{j_1\ldots j_s}(x^i, y^\sigma, \ldots, y^\sigma_{i_1\ldots i_s})\frac{\partial}{\partial y^\sigma_{j_1\ldots j_s}}, \tag{2}$$

$$\Xi^\sigma_{j_1\ldots j_s} = \mathrm{d}_{j_s}\Xi^\sigma_{j_1\ldots j_{s-1}} - y^\sigma_{j_1\ldots j_{s-1}k}\,\mathrm{d}_{j_s}\xi^k, \ \ 0 \leq j_1, \ldots, j_s \leq n, \ \ 1 \leq s \leq r,$$

for field theory. The vector field $\xi_0 = \xi^0(t)\frac{\partial}{\partial t}$ or $\xi_0 = \xi^j(x^i)\frac{\partial}{\partial x^j}$ is the π-*projection* of ξ and simultaneously π_r-*projection* of $J^r\xi$, $r = 1, 2, \ldots$, and $\frac{\mathrm{d}}{\mathrm{d}t}$ and $\mathrm{d}_i = \frac{\mathrm{d}}{\mathrm{d}x^i}$ are total derivative operators with respect to coordinates on the base X.

2.3 Horizontal and contact forms

Differential forms on fibred manifolds and their jet prolongations, adapted to the fibred structure, are well-known. π-*horizontal forms* on Y, or π_r-*horizontal forms* on J^rY, $r = 1, 2, \ldots$, are defined by the condition $i_\xi\eta = 0$, where i_ξ is the contraction by an arbitrary vertical vector field. The chart expressions of horizontal forms on J^rY are

$$\eta = A(t, q^\sigma, \ldots, q^\sigma_r)\,\mathrm{d}t, \quad \text{or} \quad \eta = A_{i_1\ldots i_q}(x^i, y^\sigma, y^\sigma_{j_1\ldots j_r})\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_q},$$

$1 \leq i_1, \ldots i_r \leq n$, $1 \leq q \leq n$, for mechanics or field theory, respectively. We denote

$$\omega_0 = \mathrm{d}t, \quad \text{respectively,} \quad \omega_0 = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$$

and

$$\omega_i = i_{\frac{\partial}{\partial x^i}}\omega_0, \quad \omega_{ij} = i_{\frac{\partial}{\partial x^j}} i_{\frac{\partial}{\partial x^i}}\omega_0, \quad \ldots$$

A form η on J^rY is $\pi_{r,s}$-*horizontal*, $0 \leq s < r$, if $i_\xi\eta = 0$ for every $\pi_{r,s}$-vertical vector field on J^rY.

Contact forms on J^rY are defined by the condition $J^r\gamma^*\omega = 0$ for every section γ of π. Contact 1-forms on prolongations of (Y, π, X) are used to define bases *adapted to the contact structure*:

$$(\mathrm{d}t, \omega^\sigma, \ldots, \omega^\sigma_{r-1}, \mathrm{d}q^\sigma_r), \quad \omega^\sigma = \mathrm{d}q^\sigma - q^\sigma_1\,\mathrm{d}t, \quad \omega^\sigma_j = \mathrm{d}q^\sigma_j - q^\sigma_{j+1}\,\mathrm{d}t,$$

$1 \le j \le r-1$, $1 \le \sigma \le m$, for mechanics, and

$$(\mathrm{d}x^i,\, \omega^\sigma,\, \ldots,\, \omega^\sigma_{j_1\ldots j_{r-1}},\, \mathrm{d}y^\sigma_{j_1\ldots j_r}),$$
$$\omega^\sigma = \mathrm{d}y^\sigma - y^\sigma_j\,\mathrm{d}x^j, \quad \omega^\sigma_{j_1\ldots j_s} = \mathrm{d}y^\sigma_{j_1\ldots j_s} - y^\sigma_{j_1\ldots j_s i}\,\mathrm{d}x^i,$$

$1 \le j_1, \ldots, j_r \le n$, $1 \le s \le r-1$, $1 \le \sigma \le m$, for field theory.

Every q-form is locally generated by forms of the adapted basis by means of the exterior product. A q-form on J^rY is called k-contact, $1 \le k \le q$, if its contraction by every π_r-vertical vector field is a $(k-1)$-contact $(q-1)$-form. A horizontal form is considered as 0-contact.

Let ω be a q-form on J^rY. Then there exists a unique decomposition

$$\pi^*_{r+1,r}\omega = \sum_{k=0}^{q} p_k\omega$$

where $h\omega = p_0\omega$ is the *horizontal component* of ω, $p_k\omega$, $1 \le k \le q$, is the *k-contact component* of ω. Mappings $h : \omega \to h\omega$ and $p : \omega \to p_1\omega + \cdots + p_q\omega$ are the *horizontalization* and *contactization*, respectively. These mappings are linear. Moreover it holds $h(\omega \wedge \varrho) = h\omega \wedge h\varrho$.

2.4 Lepage forms and equivalents

Lepage forms are very important objects in the geometry of the calculus of variations. A general definition of Lepage form and its properties are introduced and discussed in [11] in terms of the finite order variational sequence with the use of the interior Euler operator (see also [1] and [5]). In this section we recall Lepage forms only to the extent needed in this paper.

A $(n+k)$-form ϱ on J^sY, $k \ge 0$, is called *Lepage form* if

$$p_{k+1}\,\mathrm{d}\varrho = \mathcal{I}(\mathrm{d}\varrho), \quad \text{where} \quad \mathcal{I}(\eta) = \frac{1}{k}\omega^\sigma \wedge \sum_{|J|=0}^{s} (-1)^{|J|} d_J \left(i_{\frac{\partial}{\partial y^\sigma_J}} p_k\eta \right) \tag{3}$$

defines the action of the *interior Euler operator* on a $(n+k)$-form η on J^sY, $d_J = (\mathrm{d}_{j_1} \ldots \mathrm{d}_{j_l})$, $|J| = l$. For our purposes Lepage n-forms and Lepage $(n+1)$-forms will be relevant.

For Lepage n-forms we can use an equivalent definition as follows: A n-form ϱ on J^sY is Lepage if $p_1\,\mathrm{d}\varrho$ is $\pi_{s+1,0}$-horizontal, i.e. $p_1\,\mathrm{d}\varrho = E_\sigma\,\omega^\sigma \wedge \omega_0$ where E_σ are functions on $J^{s+1}Y$. Or equivalently $h(i_\xi\,\mathrm{d}\varrho) = 0$ for every $\pi_{s,0}$-vertical vector field ξ. The chart expression of a Lepage n-form on J^sY for mechanics is

$$\pi^*_{2s,s}\varrho = f_0\,\mathrm{d}t + \sum_{j=0}^{s} \left[\sum_{l=0}^{s-j} (-1)^l \frac{\mathrm{d}^l}{\mathrm{d}t^l} \left(\frac{\partial f_0}{\partial q^\sigma_{l+j+1}} \right) \right] \omega^\sigma_j, \tag{4}$$

where $\omega^\sigma_0 = \omega^\sigma$ and f_0 is a function on $J^{s+1}Y$ affine in variables q^ν_{s+1}, $1 \le \sigma \le m$. For field theory it holds

$$\begin{aligned} \pi^*_{2s,s}\varrho &= f_0\,\omega_0 \\ &+ \sum_{q=0}^{s} \left[\sum_{p=0}^{s-q} (-1)^p\, \mathrm{d}_{l_1}\,\mathrm{d}_{l_2}\, \ldots\, \mathrm{d}_{l_p} \left(\frac{\partial f_0}{\partial y^\sigma_{j_1\ldots j_q l_1\ldots l_p i}} \right) \right] \omega^\sigma_{j_1\ldots j_q} \wedge \omega_i + p_1\,\mathrm{d}\nu + \mu, \end{aligned} \tag{5}$$

where f_0 is again a function on $J^{s+1}Y$ affine in variables $y^\sigma_{j_1\dots j_{s+1}}$, ν is a $(n-1)$-form with zero horizontal component, i.e. *at least 1-contact form* and μ is a n-form with zero horizontal and 1-contact components, i.e. *at least 2-contact form*. An analogous decomposition as for Lepage n-forms holds true for general Lepage $(n+k)$-forms, $k \geq 0$, i.e. $\pi^*_{s+1,s}\varrho = \Theta_\varrho + p_{k+1}\,\mathrm{d}\nu + \mu$ where ν is at least $(k+1)$-contact form and μ is at least $(k+2)$-contact form. The chart expression of the form Θ_ϱ can be obtained by a direct application of the interior Euler operator.

For applications, especially in physics, so-called *Lepage equivalents* of π_r-horizontal n-forms and of 1-contact $\pi_{r,0}$-horizontal $(n+1)$-forms on J^rY, are of importance. For a π_r-horizontal n-form $\lambda = L\,\omega_0$ on J^rY (*r-th order Lagrangian*) we define its *Lepage equivalent* as a Lepage n-form ϱ_λ such that $h\varrho_\lambda = \lambda$ up to a possible projection. Such a form is defined on $J^{2r-1}Y$ in general, because $f_0 = L$, i.e. f_0 is a function on J^rY. We obtain its chart expression from (4) for mechanics and from (5) for field theory writing L instead of f_0 and setting $s = r-1$. In mechanics the Lepage equivalent of λ (called the *Cartan form*) is unique and global, while in field theory the decomposition $\varrho_\lambda = (\Theta_\lambda + p_1\,\mathrm{d}\nu) + \mu$ is neither unique nor global in general, because the decomposition of the term $(\Theta_\lambda + p_1\,\mathrm{d}\nu)$ to a sum of Θ_λ and $p_1\,\mathrm{d}\nu$ depends on the choice of coordinates. (The form Θ_λ is called the *Poincaré-Cartan* form.)

For a 1-contact and $\pi_{r,0}$-horizontal $(n+1)$-form $E = E_\sigma\,\omega^\sigma \wedge \omega_0$ on J^rY (*r-th order dynamical form*) we define its *Lepage equivalent* as a Lepage form α such that $p_1\alpha = E$. From the point of view of physics dynamical forms affine in "highest" variables q^σ_r (in mechanics) and $y^\sigma_{j_1\dots j_r}$ (in field theory) are highly relevant. They are directly related to equations of motion (see the following sections).

3 Calculus of variations on fibred manifolds

In this section we give a very short summary of some basic concepts and results of the calculus of variations on fibred manifolds and their prolongations, again only to the extent needed for our considerations concerning variational physical theories.

3.1 Basic structures

A variational problem of order r on a fibred manifold (Y, π, X) is defined by a *variational integral*

$$S : \Gamma_\Omega(\pi) \ni \gamma \longrightarrow S[\gamma] = \int_\Omega J^1\gamma^*\lambda \in \mathbb{R},$$

where the *Lagrangian* $\lambda = L\omega_0$ is a horizontal n-form on J^rY. The pair (π, λ) is called *Lagrange structure*. Let ξ be a π-projectable vector field (*a variation*) on Y defined on an open set V, such that $\Omega \subset \pi(V)$. The mapping

$$\delta S : \Gamma_\Omega(\pi) \ni \gamma \longrightarrow \delta S[\gamma] = \int_\Omega J^r\gamma^*\partial_\xi\lambda \in \mathbb{R}$$

is the *variational derivative of S*, ∂_ξ denotes the Lie derivative. The definition of the variational derivative leads to one of the most important formulas in the calculus of variations, the well-known *first variation formula*, in its integral form (two equivalent formulations) and the infinitesimal form, as follows:

$$\int_{\Omega} J^r\gamma^* \partial_{J^r\xi}\lambda = \int_{\Omega} J^{2r-1}\gamma^*\, i_{J^{2r-1}\xi}\, \mathrm{d}\varrho + \int_{\partial\Omega} J^{2r-1}\gamma^*\, i_{J^{2r-1}\xi}\varrho, \tag{6}$$

$$\int_{\Omega} J^r\gamma^* \partial_{J^r\xi}\lambda = \int_{\Omega} J^{2r}\gamma^*\, i_{J^{2r}\xi}\, E_\lambda + \int_{\partial\Omega} J^{2r-1}\gamma^*\, i_{J^{2r-1}\xi}\varrho, \tag{7}$$

$$\pi^*_{2r,r}\partial_{J^r\xi}\lambda = h\, i_{J^{2r-1}\xi}\, \mathrm{d}\varrho + h\, \mathrm{d}i_{J^{2r-1}\xi}\varrho, \tag{8}$$

where ϱ is an arbitrary Lepage equivalent of the Lagrangian and

$$E_\lambda = \left[\sum_{l=1}^{r}(-1)^l\, \mathrm{d}_{j_1}\ldots \mathrm{d}_{j_l}\left(\frac{\partial L}{\partial y^\sigma_{j_1\ldots j_l}}\right)\right]\omega^\sigma\wedge\omega_0, \quad 1\le j_1,\, \ldots,\, j_l\le n, \tag{9}$$

is the *Euler-Lagrange form* of the r-th order Lagrangian λ. In contrast to the Lepage equivalent, the Euler–Lagrange form is always unique, even in field theory. Notice that it is a dynamical form defined in general on $J^{2r}Y$, affine in variables $y^\sigma_{j_1\ldots j_{2r}}$. It is worth noting that the first variation formula plays a key role not only for obtaining equations of motion of a given physical system with a Lagrangian λ but also for finding conservation laws.

A section $\gamma\in\Gamma_\Omega(\pi)$ defined on an open set $U\subset X$ is an *extremal*, i.e. a *stationary point* of the r-th order Lagrange structure $(\pi,\,\lambda)$, if $\delta S[\gamma]=0$ for every π-vertical vector field ξ on Y and every $\Omega\subset U$. Equations for extremals (for a physical system equations of motion) result from this definition and from the first variation formula and take the following form:

$$J^{2r-1}\gamma^*\, i_{J^{2r-1}\xi}\, \mathrm{d}\varrho = 0 \quad \text{or equivalently} \quad J^{2r}\gamma^*\, i_{J^{2r}\xi}\, E_\lambda = 0, \tag{10}$$

for every π-projectable vector field ξ on Y (see e.g. [8]).

Recall that the Euler-Lagrange form of a r-th order Lagrangian λ is identically zero iff $\lambda_0 = h\,\mathrm{d}\eta$ where η is an arbitrary $(n-1)$-form on $J^{r-1}Y$ (see [7] or [11]). Such Lagrangians are called *trivial* or *null Lagrangians*. Thus, coordinate expressions of trivial Lagrangians are $(\mathrm{d}_j f^j)\omega_0$ in field theory, resp. $\frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{d}t$ in mechanics, where f^j, resp. f are local functions on $J^{r-1}Y$.

3.2 The Inverse Problem of the Calculus of Variations

Equations of motion of a physical system with m degrees of freedom usually are a system of m second order differential equations $E_\sigma\circ J^2\gamma = 0$ (ordinary in mechanics, partial in field theories), where E_σ, $1\le\sigma\le m$, are functions on J^2Y affine in "accelerations", i.e. in the second order coordinates. The inverse problem of calculus of variations is then based on the question whether these equations can be derived from an appropriate variational functional S, i.e. whether there exists a lagrangian λ (usually of the first order) such that the dynamical form $E = E_\sigma\,\omega^\sigma\wedge\omega_0$ is its Euler-Lagrange form. This question can be answered on a completely general level, for dynamical forms on an arbitrary prolongation of the underlying fibred manifold. A universal and elegant approach to this problem is the use of properties of the finite order variational sequence see e.g. [6], [5], in details see [11]. Let us summarize basic definitions and results.

A dynamical form E on J^sY is *locally variational* if to every point of its domain there exists an open neighborhood W and a Lagrangian λ such that $E = E_\lambda$ on W. (Meantime let us leave ahead the question of the order of such a Lagrangian.) Let us formulate the main theorem solving the inverse problem. It is the immediate consequence of properties of variational sequences.

Theorem 1. *A dynamical form $E = E_\sigma\, \omega^\sigma \wedge \omega_0$, $1 \le \sigma \le m$, on J^sY is locally variational if and only if there exists its closed Lepage equivalent.*

The property formulated in Theorem 1 is the most practical one of the three equivalent conditions for a general inverse variational problem proved in [11]. Note that if a globally defined dynamical form E in mechanics is (at least locally) variational then its closed Lepage equivalent is unique and global.

Expressing Theorem 1 in coordinates we obtain "practical" formulas for verifying variationality of a dynamical form, so called *Helmholtz conditions*. They are given in the next theorem which simultaneously presents a formula for a corresponding local Lagrangian (of the same order as the equations).

Theorem 2. *Let $E = E_\sigma\, \omega^\sigma \wedge \omega_0$ be a dynamical form on J^sY.*

- *E is (locally) variational if and only if to every point z of its domain there exists a neighborhood and a chart (V, ψ) on Y such that this neighborhood is a subset of $\pi^{-1}_{s,0}(V)$ and the following relations hold on W:*

$$\frac{\partial E_\sigma}{\partial q^\nu_l} - (-1)^l \frac{\partial E_\nu}{\partial q^\sigma_l} - \sum_{j=l+1}^{s} (-1)^j \binom{j}{l} \frac{\mathrm{d}^{j-l}}{\mathrm{d}t^{j-l}} \left(\frac{\partial E_\nu}{\partial q^\sigma_j} \right) = 0, \tag{11}$$

  for mechanics and

$$\frac{\partial E_\sigma}{\partial y^\nu_{p_1\ldots p_l}} - (-1)^l \frac{\partial E_\nu}{\partial y^\sigma_{p_1\ldots p_l}} - \sum_{j=l+1}^{s} (-1)^j \binom{j}{l} \mathrm{d}_{p_{l+1}} \ldots \mathrm{d}_{p_j} \frac{\partial E_\nu}{\partial y^\sigma_{p_1\ldots p_l p_{l+1}\ldots p_j}} = 0, \tag{12}$$

  for field theory, where $0 \le l \le s$ in both cases.

- *Let E be a locally variational dynamical form defined on an open subset W of J^sY and let a mapping $\chi : [0, 1] \times W \to W$ be defined as follows:*

$$\chi\big(u, (x^i, y^\mu, y^\mu_{p_1}, \ldots y^\mu_{p_1\ldots p_s})\big) = \big(x^i, uy^\mu, uy^\mu_{p_1}, \ldots uy^\mu_{p_1\ldots p_s}\big). \tag{13}$$

  Then E is the Euler-Lagrange form of the s-th order Lagrangian

$$\Lambda = y^\sigma \left(\int_0^1 (E_\sigma \circ \chi)\, \mathrm{d}u \right) \omega_0. \tag{14}$$

  For mechanics the formula is analogous.

- *Let E be a locally variational dynamical form on J^sY in mechanics. Then locally there exists a Lagrangian of the (minimal possible) order $\frac{s}{2}$ for even s, or of the order $\frac{s+1}{2}$ for odd s (see [8]).*

3.3 Symmetries and conservation laws

By a *symmetry of a Lagrangian* λ we mean a π-projectable vector field ξ on Y such that the local isomorphisms of the one-parameter group of $J^1\xi$ are invariance transformations of λ. This leads to the Noether equation and corresponding consequences resulting from the first variation formula (6)

$$\partial_{J^1\xi}\lambda = 0, \tag{15}$$

$$0 = \int_\Omega J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\,\mathrm{d}\varrho + \int_\Omega \mathrm{d}J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\varrho. \tag{16}$$

Because of the fact that the first term is zero along extremals, we obtain a conservation law corresponding to Noether symmetry ξ,

$$\int_\Omega J^{2r-1}\gamma^*\,\mathrm{d}i_{J^{2r-1}\xi}\varrho = \int_\Omega dJ^{2r-1}\gamma^* i_{J^{2r-1}\xi}\varrho = 0 \implies$$

$$\Phi(\xi) = hi_{J^{2r-1}\xi}\varrho \quad \text{is closed along extremals.} \tag{17}$$

The quantity $\Phi(\xi)$ is the *Noether current* corresponding to the symmetry ξ. Chart expressions of Noether equation (15) and Noether current (17) for an r-th order Lagrangian $\lambda = L\,\mathrm{d}t$ are then

$$i_{J^1\xi}\,\mathrm{d}L + L\frac{\mathrm{d}\xi^i}{\mathrm{d}x^i} = 0, \tag{18}$$

$$\Phi(\xi) = \left[L\xi^i + \sum_{q=0}^{r-1}\left(\sum_{p=0}^{r-q-1}(-1)^p \mathrm{d}_{l_1}\ldots\mathrm{d}_{l_p}\left(\frac{\partial L}{\partial y^\sigma_{j_1\ldots j_q l_1\ldots l_p i}}\right)\right.\right.$$
$$\left.\left.\times\left(\Xi^\sigma_{j_1\ldots j_q} - y^\sigma_{j_1\ldots j_q j}\xi^j\right)\right)\right]\omega_i \tag{19}$$

for field theory and

$$i_{J^r\xi}\,\mathrm{d}L + L\frac{\mathrm{d}\xi^0}{\mathrm{d}t} = 0, \tag{20}$$

$$\Phi(\xi) = L\xi^0 + \sum_{j=0}^{r-1}\sum_{l=0}^{r-j-1}(-1)^l\frac{\mathrm{d}^l}{\mathrm{d}t^l}\left(\frac{\partial L}{\partial q^\sigma_{j+l+1}}\right)\left(\Xi^\sigma_j - q^\sigma_{j+1}\xi^0\right) \tag{21}$$

for mechanics. Note that the conservation law (17) now reads: (the function) $\Phi(\xi)$ is constant along extremals.

4 Variational theories in physics

In this section we apply the variational theory on jet bundles to tackle some fundamental problems of physics, namely the questions on existence of a Lagrangian and the problem of conservation laws. The aim is to demonstrate that the methods are universal, and easily applicable to distinct physical disciplines. Here we shall deal with classical mechanics and special relativity, waves in continuous media, quantum mechanics and classical electrodynamics. In classical mechanics we shall provide the form of the so-called *variational forces* leading to variational equations of motion. We shall also study the Noether symmetries and corresponding conservation laws for variational equations of motion in the individual cases.

4.1 Mechanics and special relativity

Equations of motion of a mechanical system with m degrees of freedom in classical mechanics are second order ordinary differential equations $E_\sigma \circ J^2\gamma = 0$, $1 \leq \sigma \leq m$, where E_σ are functions on J^2Y affine in accelerations q_2^σ. From now on we denote $q_1^\sigma = \dot{q}^\sigma$ and $q_2^\sigma = \ddot{q}^\sigma$, as usual in physics. We have $E_\sigma = f_\sigma - g_{\sigma\nu}\ddot{q}^\nu$ where the functions f_σ, $g_{\sigma\nu}$ are defined on an open subset of J^1Y, and we can interprete the E_σ as components of a dynamical form $E = E_\sigma \omega^\sigma \wedge \mathrm{d}t$. In general E is not variational. Helmholtz conditions (11) for a second order variational dynamical form are

$$\begin{aligned}
\frac{\partial E_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial E_\nu}{\partial \ddot{q}^\sigma} &= 0, \\
\frac{\partial E_\sigma}{\partial \dot{q}^\nu} + \frac{\partial E_\nu}{\partial \dot{q}^\sigma} - \frac{d}{dt}\left(\frac{\partial E_\sigma}{\partial \ddot{q}^\nu} + \frac{\partial E_\nu}{\partial \ddot{q}^\sigma}\right) &= 0, \\
\frac{\partial E_\sigma}{\partial q^\nu} - \frac{\partial E_\nu}{\partial q^\sigma} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial E_\sigma}{\partial \dot{q}^\nu} - \frac{\partial E_\nu}{\partial \dot{q}^\sigma}\right) &= 0.
\end{aligned} \tag{22}$$

In variational mechanics, usual equations of motion take the form

$$-g_{\sigma\nu}\ddot{q}^\nu + \Gamma_{\sigma\nu\lambda}\dot{q}^\nu\dot{q}^\lambda = 0, \quad \text{and} \quad - m g_{\sigma\nu}\ddot{q}^\nu - \frac{\partial U}{\partial q^\sigma} = 0,$$

where $g = (g_{\sigma\nu})$, is a Riemannian metric (a regular symmetric matrix of functions of coordinates q^ρ), $\Gamma = (\Gamma_{\sigma\nu\lambda})$, $1 \leq \sigma$, ν, $\lambda \leq m$, are Christoffel symbols of g and U is a function of time and coordinates (a potential). The first of the equations, as well as the second ones with $U = 0$, are equations of *geodesics*; the second ones, as the consequence of constant metric ($\Gamma = 0$), have the form of Newton equations.

There is a question under what conditions a general dynamical form

$$E = (-g_{\sigma\nu}\ddot{q}^\nu + f_\sigma)\,\omega^\sigma \wedge \mathrm{d}t$$

is variational. In view of the above examples, assume that the functions $g_{\sigma\nu}$ are locally defined on Y and f_σ are local functions on J^1Y. The first set of Helmholtz conditions (22) leads immediately to symmetry relations $g_{\sigma\nu} = g_{\nu\sigma}$. Applying the second and the third set of the conditions we obtain

$$\frac{\partial f_\sigma}{\partial \dot{q}^\nu} + \frac{\partial f_\nu}{\partial \dot{q}^\sigma} + 2\left(\frac{\partial g_{\sigma\nu}}{\partial t} + \frac{\partial g_{\sigma\nu}}{\partial q^\lambda}\dot{q}^\lambda\right) = 0,$$

$$\begin{aligned}
-\left(\frac{\partial g_{\sigma\lambda}}{\partial q^\nu} - \frac{\partial g_{\nu\lambda}}{\partial q^\sigma}\right)\ddot{q}^\lambda &+ \left(\frac{\partial f_\sigma}{\partial q^\nu} - \frac{\partial f_\nu}{\partial q^\sigma}\right) \\
&- \frac{1}{2}\frac{\mathrm{d}'}{\mathrm{d}t}\left(\frac{\partial f_\sigma}{\partial \dot{q}^\nu} - \frac{\partial f_\nu}{\partial \dot{q}^\sigma}\right) - \frac{1}{2}\left(\frac{\partial^2 f_\sigma}{\partial \dot{q}^\lambda \partial \dot{q}^\nu} - \frac{\partial^2 f_\nu}{\partial \dot{q}^\lambda \partial \dot{q}^\sigma}\right)\ddot{q}^\lambda = 0,
\end{aligned}$$

where $\frac{\mathrm{d}'}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}^\mu \frac{\partial}{\partial q^\mu}$. Thus, the functions f_σ are of the form $\Gamma_{\sigma\nu\lambda}\dot{q}^\nu\dot{q}^\lambda + a_{\sigma\nu}\dot{q}^\nu + b_\sigma$, where $\Gamma_{\sigma\nu\lambda}$, $a_{\sigma\nu}$, b_σ are functions on Y and $\Gamma_{\sigma\nu\lambda} = \Gamma_{\sigma\lambda\nu}$. Rewriting Helmholtz conditions for this case we obtain

$$\Gamma_{\sigma\nu\lambda} = \frac{1}{2}\left(\frac{\partial g_{\nu\lambda}}{\partial q^\sigma} - \frac{\partial g_{\sigma\lambda}}{\partial q^\nu} - \frac{\partial g_{\sigma\nu}}{\partial q^\lambda}\right), \quad a_{\sigma\nu} + a_{\nu\sigma} = 2\frac{\partial g_{\sigma\nu}}{\partial t}.$$

If in addition, $(g_{\sigma\nu})$ is independent of time, as usual, we can see that functions $a_{\sigma\nu}$ are antisymmetric. The remaining conditions

$$\frac{\partial f_\sigma}{\partial q^\nu} - \frac{\partial f_\nu}{\partial q^\sigma} = \frac{1}{2}\frac{\mathrm{d}'}{\mathrm{d}t}\left(\frac{\partial f_\sigma}{\partial \dot{q}^\nu} - \frac{\partial f_\nu}{\partial \dot{q}^\sigma}\right)$$

give

$$a_{\sigma\nu} = \left(\frac{\partial \phi_\nu}{\partial q^\sigma} - \frac{\partial \phi_\sigma}{\partial q^\nu}\right), \quad b_\sigma = -\left(\frac{\partial \phi_\sigma}{\partial t} + \frac{\partial \varphi}{\partial q^\sigma}\right),$$

where ϕ_σ, $1 \le \sigma \le m$, and φ are arbitrary functions on Y. Notice that addition of a trivial Lagrangian $\frac{\mathrm{d}f(t,\,q^\mu)}{\mathrm{d}t}\,\mathrm{d}t$ can affect only the functions Φ_σ, where $\Phi_\sigma = a_{\sigma\nu}\dot{q}^\nu + b_\sigma$:

$$\Phi_\sigma \longrightarrow \Phi_\sigma + \frac{\partial f}{\partial q^\nu}, \quad \varphi \longrightarrow \varphi - \frac{\partial f}{\partial t}, \tag{23}$$

This represents a *gauge transformation.* We conclude that variational forces in mechanics are of the form

$$F_\sigma = \Gamma_{\sigma\nu\lambda}\dot{q}^\nu\dot{q}^\lambda + \dot{q}^\nu\left(\frac{\partial \phi_\nu}{\partial q^\sigma} - \frac{\partial \phi_\sigma}{\partial q^\nu}\right) - \left(\frac{\partial \phi_\sigma}{\partial t} + \frac{\partial \varphi}{\partial q^\sigma}\right). \tag{24}$$

Because of the fact that metric $(g_{\sigma\nu})$ depends only on coordinates we can construct directly first order Lagrangians corresponding to E. Denote $p_\sigma = \frac{\partial L}{\partial \dot{q}^\sigma}$. Then

$$\frac{\partial p_\sigma}{\partial \dot{q}^\nu} = \frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu} = g_{\sigma\nu}.$$

The forms $\eta_\sigma = g_{\sigma\nu}\,\mathrm{d}\dot{q}^\nu$ and $\omega = g_{\sigma\nu}\dot{q}^\nu\,\mathrm{d}\dot{q}^\sigma$ are closed along every fiber over a fixed point in Y, $\eta_\sigma = \mathrm{d}(p_\sigma) = \mathrm{d}\left(\frac{\partial L}{\partial \dot{q}^\sigma}\right)$ and $\omega = \mathrm{d}\left(-L + p_\sigma\dot{q}^\sigma\right)$. Consider the parametrization of the line connecting points $(t,\,q^\mu,\,0)$ and $(t,\,q^\mu,\,\dot{q}^\mu)$ in the fibre $\pi_{1,0}^{-1}(\{(t,\,q^\mu)\}) \subset J^1Y$ over a point $(t,\,q^\mu)$,

$$\chi:\,[0,\,1] \times J^1Y \ni (u,\,(t,\,q^\mu,\,\dot{q}^\mu)) \to \chi(u,\,(t,\,q^\mu,\,\dot{q}^\mu)) = (t,\,q^\mu,\,u\dot{q}^\mu) \in J^1Y.$$

We can easily obtain functions p_σ and $H = -L + p_\sigma\dot{q}^\sigma$ (*momenta* and *Hamilton function*) integrating the forms η_σ and ω along this line:

$$p_\sigma = \int_0^1 (g_{\sigma\nu} \circ \chi)\,\chi^*\,\mathrm{d}\dot{q}^\nu = \dot{q}^\nu \int_0^1 (g_{\sigma\nu} \circ \chi)\,\mathrm{d}u + \phi_\sigma(t,\,q^\mu) \;=\; g_{\sigma\nu}\dot{q}^\nu + \phi_\sigma(t,\,q^\mu),$$

$$\begin{aligned} H &= \int_0^1 [(g_{\sigma\nu}\dot{q}^\nu) \circ \chi]\,\chi^*\,\mathrm{d}\dot{q}^\sigma \;=\; \dot{q}^\sigma\dot{q}^\nu \int_0^1 (g_{\sigma\nu} \circ \chi)u\,\mathrm{d}u + \varphi(t,\,q^\mu) \\ &= \frac{1}{2}g_{\sigma\nu}\dot{q}^\sigma\dot{q}^\nu + \varphi(t,\,q^\mu), \end{aligned}$$

where $\phi_\sigma(t, q^\mu)$ and $\varphi(t, q^\mu)$, $1 \leq \sigma \leq m$, are arbitrary functions of time and coordinates (integration "constants"). Finally $L = p_\sigma \dot{q}^\sigma - H$,

$$L = \frac{1}{2} g_{\sigma\nu} \dot{q}^\sigma \dot{q}^\nu + \dot{q}^\sigma \phi_\sigma - \varphi. \tag{25}$$

For $m = 3$, Euclidean metrics and cartesian coordinates we can write the variational forces Φ_σ above in a vector form (denoting the vector by $\vec{F}$)

$$\vec{F} = \vec{v} \times \operatorname{rot} \vec{\phi} - \frac{\partial \vec{\phi}}{\partial t} - \operatorname{grad} \varphi.$$

Denoting $\vec{\phi} = e\vec{A}$, $\operatorname{rot} \vec{A} = \vec{B}$, $\varphi = eU$ and $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \operatorname{grad} U$ we can write

$$\vec{F} = e\vec{v} \times \vec{B} + e\vec{E}, \quad \operatorname{div} \vec{B} = 0, \quad \operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{26}$$

We have obtained a Lorentz-type force and two of the well-known three-dimensional Maxwell equations. This is purely a result of variationality!

A special example of variational forces are *fictive forces* connected with dynamics in non-inertial reference frames:

$$\vec{F}^* = -m\vec{a} + (-2m\vec{\omega} \times \vec{v}) + (-m\vec{\varepsilon} \times \vec{r}) + (-m\vec{\omega} \times (\vec{\omega} \times \vec{r})).$$

where the vectors $\vec{r}$ and $\vec{v}$ are a position and velocity of a particle with respect to a non-inertial reference frame moving (with respect to an inertial reference frame) with translational acceleration $\vec{a}$ and rotating with angular velocity $\vec{\omega}$, angular acceleration is $\varepsilon = \dot{\vec{\omega}}$. Individual terms are $\vec{F}^*_T = -m\vec{a}$, $\vec{F}^*_C = -2m\vec{\omega} \times \vec{v}$ (Coriolis force), $\vec{F}^*_E = -m\vec{\varepsilon} \times \vec{r}$ (Euler force), and $\vec{F}^*_o = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ (centrifugal force). Quantities $\vec{a}$, $\vec{\omega}$ and $\vec{\varepsilon}$ are only functions of time. We have

$$\vec{\phi} = m\vec{\omega} \times \vec{r}, \quad m\vec{\varepsilon} \times \vec{r} = \frac{\partial \vec{\phi}}{\partial t}, \quad \varphi = m\vec{r}\vec{a} + \frac{m}{2}\left[(\vec{r}\vec{\omega})^2 - r^2\omega^2\right].$$

In special relativity a Minkowski metric on $\mathbb{R}^4$ is considered, and we have a fibred manifold $(\mathbb{R} \times \mathbb{R}^4, \pi, \mathbb{R})$, where a coordinate on the base $X = \mathbb{R}$ is a parameter of the curves, s, without a physical meaning. As we have seen above, a Lagrangian "corresponding to a four-dimensional observer" must be of the form (25), hence, explicitly, it reads

$$\tilde{\lambda} = \mathcal{L}\, \mathrm{d}s = \left(-\frac{1}{2} \left[(\dot{q}^4)^2 - \sum_{p=1}^{3} (\dot{q}^p)^2 \right] + \dot{q}^\sigma \phi_\sigma - \psi \right) \mathrm{d}s,$$

where $\phi(q^\mu)$, $\psi(q^\mu)$, $1 \leq \mu \leq 4$, are functions on the space-time. Moreover, a non-holonomic constraint condition holds for the four-velocity, $(\dot{q}^4)^2 - \sum_{p=1}^{3} (\dot{q}^p)^2 = m^2c^2$, $q^4 = ct$, where t represents the time variable, c is the speed of light in vacuum. The thee-dimensional velocity of a particle is $\vec{v} = (v^l)$, $v^l = \frac{\mathrm{d}q^l}{\mathrm{d}t} = \frac{c\dot{q}^l}{\dot{q}^4}$, $1 \leq l \leq 3$, and the constraint condition in the latter coordinates reads $\dot{q}^4 \sqrt{1 - \frac{v^2}{c^2}} = mc$. Then $\dot{q}^4 = \frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\mathrm{d}s = \frac{c\,\mathrm{d}t}{\dot{q}^4}$. Putting $\psi = 0$ we obtain

$$\tilde{\lambda} = \left[-\frac{1}{2} \left(1 - \frac{v^2}{c^2} \right) (\dot{q}^4)^2 + \left(\vec{v}\vec{\phi} + c\phi_4 \right) \frac{\dot{q}^4}{c} \right] \mathrm{d}s$$

This 1-form gives rise to the *constrained Lagrangian* $\tilde{\lambda}_C = L\,\mathrm{d}t = \frac{1}{2} m^2 c^2\,\mathrm{d}s$, the first part $L\,\mathrm{d}t$ of which is relevant as an unconstrained Lagrangian for a "three-dimensional observer",

$$\lambda = L\,\mathrm{d}t = \left(-mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \vec{v}\vec{\phi} - \varphi \right) \mathrm{d}t, \tag{27}$$

where $\phi_4 = -\varphi$. (See [12] for getting a deeper understanding of the problem of a relativistic particle as a non-holonomic constrained system.) The Lagrangian (27) leads, indeed, to well-known equations of motion of a relativistic particle,

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{v} \times \mathrm{rot}\,\vec{\phi} - \frac{\partial \vec{\phi}}{\partial t} - \mathrm{grad}\,\varphi, \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}},$$

$\vec{\phi} = \frac{e}{c}\vec{A}$ and $\varphi = eU$ are proportional to vector potential $\vec{A}$ and scalar potential U, respectively, e is the charge of the particle.

4.2 Waves

Waves occur as solutions of equations of motion in various physical disciplines, especially mechanics of continuous media (acoustic waves) and electrodynamics (light waves). As a basic equation of motion we can consider the wave equation. It can be obtained from a variational principle as well. Let us consider only one-dimensional case of wave propagation with a constant phase speed v. The wave equation for a function $y(t, x)$ of time and x-coordinate reads

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = 0.$$

The concrete expression of the phase speed depends on a concrete physical situation. For example, for mechanical waves $v = \sqrt{\frac{E}{s}}$, where E is the Young module and s is the density of a medium, for light waves in vacuum $v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, etc. In what follows we consider units where the phase speed is 1.

The left-hand side of the wave equation defines a dynamical form $E = E_1\,\omega^1 \wedge \omega_0$ on J^2Y where $(Y, \pi, X) = (\mathbb{R}^2 \times \mathbb{R}, \pi, \mathbb{R}^2)$, i.e. $n = 2$, $m = 1$. In a chart (V, ψ), $\psi = (t, x, y)$ on Y, the associated chart $(\pi(V), \varphi)$, $\varphi = (t, x)$ on X and the associated fibred charts $(\pi^{-1}_{1,0}(V), \psi_1)$, $\psi_1 = (t, x, y, y_t, y_x)$, and $(\pi^{-1}_{2,0}(V), \psi_2)$, $\psi_2 = (t, x, y, y_t, y_x, y_{tt}, y_{tx}, y_{xx})$, on J^1Y and J^2Y, respectively, we have

$$E = E_1 \omega^1 \wedge \mathrm{d}t \wedge \mathrm{d}x = (y_{xx} - y_{tt})\omega^1 \wedge \mathrm{d}t \wedge \mathrm{d}x, \quad \omega^1 = \mathrm{d}y - y_t\,\mathrm{d}t - y_x\,\mathrm{d}x.$$

Helmholtz variationality conditions (12) for a second order dynamical form are in general

$$\begin{aligned} \frac{\partial E_\sigma}{\partial y^\nu} - \frac{\partial E_\nu}{\partial y^\sigma} + \mathrm{d}_i \frac{\partial E_\nu}{\partial y^\sigma_i} - \mathrm{d}_i\,\mathrm{d}_j \frac{\partial E_\nu}{\partial y^\sigma_{ij}} &= 0, \\ \frac{\partial E_\sigma}{\partial y^\nu_i} + \frac{\partial E_\nu}{\partial y^\sigma_i} - 2\mathrm{d}_j \frac{\partial E_\nu}{\partial y^\sigma_{ij}} &= 0, \\ \frac{\partial E_\sigma}{\partial y^\nu_{ij}} - \frac{\partial E_\nu}{\partial y^\sigma_{ij}} &= 0. \end{aligned} \tag{28}$$

For our form E ($1 \le i,\, j \le 2$, $\sigma = \nu = 1$) they are fulfilled trivially. The Vainberg-Tonti Lagrangian is $\Lambda = \mathcal{L}\,\mathrm{d}t \wedge \mathrm{d}x$, where

$$\Lambda = \left[y \int_0^1 (u y_{xx} - u y_{tt})\,\mathrm{d}u \right] \mathrm{d}t \wedge \mathrm{d}x = \frac{y}{2}(y_{xx} - y_{tt})\mathrm{d}t \wedge \mathrm{d}x.$$

Subtracting the form $\frac{1}{2}\left[\mathrm{d}_x\,(y y_x) - \mathrm{d}_t\,(y y_t)\right] \mathrm{d}t \wedge \mathrm{d}x$ representing a trivial Lagrangian we obtain the first order Lagrangian

$$\lambda = \frac{1}{2}\left(y_t^2 - y_x^2\right) \mathrm{d}t \wedge \mathrm{d}x.$$

Let us now discuss the problem of Noether symmetries and currents of this Lagrangian. Let $\xi = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \Xi \frac{\partial}{\partial y}$. Noether equation (18) then reads

$$\begin{aligned} y_t \left(\mathrm{d}_t \Xi - y_t\,\mathrm{d}_t \xi^t - y_x\,\mathrm{d}_t \xi^x\right) - y_x \left(\mathrm{d}_x \Xi - y_t\,\mathrm{d}_x \xi^t - y_x\,\mathrm{d}_x \xi^x\right) \\ + \frac{1}{2}\left(y_t^2 - y_x^2\right)\left(\mathrm{d}_t \xi^t + \mathrm{d}_x \xi^x\right) = 0. \end{aligned}$$

We can immediately see that basic symmetries are generators of time and space translations, $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ (the Lagrangian λ does not depend on t and x explicitly). Other Noether symmetries are the vector fields $\frac{\partial}{\partial y}$ and $x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}$. The corresponding Noether currents are (see (19))

$$\begin{aligned} \Phi\left(\frac{\partial}{\partial t}\right) &= -y_t y_x\,\mathrm{d}t - \frac{1}{2}(y_t^2 + y_x^2)\,\mathrm{d}x, \\ \Phi\left(\frac{\partial}{\partial x}\right) &= -\frac{1}{2}(y_t^2 + y_x^2)\,\mathrm{d}t - y_t y_x\,\mathrm{d}x, \\ \Phi\left(\frac{\partial}{\partial y}\right) &= y_x\,\mathrm{d}t + y_t\,\mathrm{d}x, \\ \Phi\left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right) &= -\left[\frac{1}{2}t\left(y_t^2 + y_x^2\right) + x y_t y_x\right]\mathrm{d}t - \left[\frac{1}{2}x\left(y_t^2 + y_x^2\right) + t y_t y_x\right]\mathrm{d}x. \end{aligned}$$

These currents are consistent with the fact that the general solution of the wave equation is of the form $y(t,\, x) = f_1(x - vt) + f_2(x + vt)$, i.e. $y(t,\, x) = f_1(x - t) + f_2(x + t)$ for the phase speed $v = 1$ as assumed for simplicity. There f_1 and f_2 are arbitrary functions of phases $\phi_1 = x - t$ and $\phi_2 = x + t$, respectively.

4.3 Quantum mechanics

Equation of motion for a quantum mechanical system is the Schrödinger equation $\mathrm{i}\hbar\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle$ (see sec. 1). In the coordinate representation a quantum mechanical state of a microscopic particle (mass m) is described by a complex *wave function* of time and position $\psi(t,\,\vec{r})$. (Recall that $|\psi(t,\,\vec{r})|^2$ is the probability density for finding the particle at the moment t at the point $\vec{r}$.) For simplicity let us consider only one-dimensional motion of a free particle, i.e. $\psi = \psi(t,\,x)$. Then the Schrodinger equation reads

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}, \qquad \mathrm{i}\hbar\psi_t + \frac{\hbar^2}{2m}\psi_{xx} = 0,$$

using again the standard notation for partial derivatives. The left-hand side of this equation represents two expressions, for the real and the imaginary part of the wave function, $\psi(t,\,x) = v(t,\,x) + \mathrm{i}w(t,\,x)$. So, the corresponding fibred space is $(\mathbb{R}^2 \times \mathbb{R}^2,\,\pi,\,\mathbb{R}^2)$, coordinates on the base $X = \mathbb{R}^2$ are $(x^1,\,x^2) = (t,\,x)$ and coordinates on fibres $\mathbb{R}^2$ are $(y^1,\,y^2) = (v,\,w)$. The left-hand sides of the equations of motion correspond to two components of the dynamical form

$$\begin{aligned} E = E_1\,\omega^1 \wedge \omega_0 + E_2\,\omega^2 \wedge \omega_0 = {} & \left(\frac{\hbar^2}{2m}v_{xx} - \hbar w_t\right)\mathrm{d}v \wedge \mathrm{d}t \wedge \mathrm{d}x \\ & + \left(\frac{\hbar^2}{2m}w_{xx} + \hbar v_t\right)\mathrm{d}w \wedge \mathrm{d}t \wedge \mathrm{d}x. \end{aligned}$$

Verification of Helmholtz conditions (12) for $n = 2$, $m = 2$ and $r = 2$ (or (28)) is trivial, these conditions are fulfilled.

Let us construct the corresponding Vainberg-Tonti Lagrangian using (14). We have

$$\begin{aligned} \Lambda &= \left[v\int_0^1 u\left(\frac{\hbar^2}{2m}v_{xx} - \hbar w_t\right)\mathrm{d}u + w\int_0^1 u\left(\frac{\hbar^2}{2m}w_{xx} + \hbar v_t\right)\mathrm{d}u\right]\mathrm{d}t \wedge \mathrm{d}x \\ &= \left[\frac{\hbar^2}{4m}(vv_{xx} + ww_{xx}) - \frac{\hbar}{2}(vw_t - wv_t)\right]\mathrm{d}t \wedge \mathrm{d}x. \end{aligned}$$

This is not, of course, a minimal-order Lagrangian. Subtracting the trivial Lagrangian $\frac{\hbar^2}{4m}\mathrm{d}_x(vv_x + ww_x)$ from Λ we obtain the first-order Lagrangian

$$\lambda = \left[-\frac{\hbar^2}{4m}\left(v_x^2 + w_x^2\right) - \frac{\hbar}{2}\left(vw_t - wv_t\right)\right]\mathrm{d}t \wedge \mathrm{d}x.$$

Expressing real functions v and w with help of ψ, i.e.

$$v = \frac{1}{2}(\psi + \psi^*)\,, \quad w = -\frac{\mathrm{i}}{2}(\psi - \psi^*)\,,$$

star denoting the complex conjugate, we finally obtain

$$\lambda = \left[-\frac{\hbar^2}{4m}(\psi\psi^*) + \frac{\mathrm{i}\hbar}{4}(\psi^*\psi_t - \psi\psi_t^*)\right]\mathrm{d}t \wedge \mathrm{d}x.$$

For a particle moving in a potential field $V(t,\, x)$ the Schrodinger equation is variational as well and the term $-\frac{1}{2}V\psi\psi^* = -\frac{1}{2}V(v^2+w^2)$ is added to the Lagrangian.

Let us find basic symmetries of the free particle Lagrangian. Noether equation reads

$$-\frac{\hbar^2}{2m}\left(v_x\Xi^v_x + w_x\Xi^w_x\right) - \frac{\hbar}{2}\left(v\Xi^w_t + w_t\Xi^v - w\Xi^v_t - v_t\Xi^w\right) \\ - \left[\frac{\hbar^2}{4m}\left(v_x^2+w_x^2\right) + \frac{\hbar}{2}\left(vw_t - wv_t\right)\right]\left(\mathrm{d}_t\xi^t + \mathrm{d}_x\xi^x\right) = 0.$$

Its basic solutions are, indeed, translation generators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$. (Note that these vector fields remain symmetries even for a particle in a potential field $V(t,\, x)$ if $\frac{\partial V}{\partial t} = 0$, or $\frac{\partial V}{\partial x} = 0$, respectively.) Noether currents corresponding to these symmetries are obtained from the general first order expression

$$\Phi\left(\xi\right) = \left[L\xi^i + \frac{\partial L}{\partial y^\sigma_i}\left(\Xi^\sigma - y^\sigma_j\xi^j\right)\right]\omega_i, \quad i = t,\, x, \quad y^1 = v,\, y^2 = w,$$

and take the form

$$\Phi\left(\frac{\partial}{\partial t}\right) = \left[-\frac{\hbar^2}{4m}(v_x^2+w_x^2)\right]\mathrm{d}x - \left[\frac{\hbar^2}{2m}(v_xv_t + w_xw_t)\right]\mathrm{d}t\,,$$
$$\Phi\left(\frac{\partial}{\partial x}\right) = \left[\frac{\hbar}{2}(vw_x - wv_x)\right]\mathrm{d}x - \left[\frac{\hbar^2}{4m}(v_x^2+w_x^2) - \frac{\hbar}{2}(vw_t - wv_t)\right]\mathrm{d}t\,.$$

Physical interpretation of these currents can be as follows: Taking into account that $\psi = v + \mathrm{i}w$ we obtain

$$vw_t - wv_t = \frac{\mathrm{i}}{2}(\psi\psi^*_t - \psi^*\psi_t), \;\; vw_x - wv_x = \frac{\mathrm{i}}{2}(\psi\psi^*_x - \psi^*\psi_x) = \frac{m}{\hbar}j,$$
$$v_tw_x + w_tv_x = \frac{1}{2}(\psi_t\psi^*_x - \psi^*_t\psi_tx), \;\; v_x^2 + w_x^2 = \psi_x\psi^*_x.$$

The quantity j (vector $\vec{j}$ in three-dimensional space) is the density of the probability flow. The solution of the Schrödinger equation for a free particle is $\psi(t,\, x) = K\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(Et-kx)}$ where E and k are constants (energy and wave number, respectively). We can easily make sure that the obtained Noether currents are constant along these solutions.

4.4 Classical electrodynamics

Equations of motion in electrodynamics are the Maxwell equations

$$\varepsilon_0\,\mathrm{div}\,\vec{E} = \varrho, \quad \mathrm{div}\,\vec{B} = 0, \quad \mathrm{rot}\,\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad \mathrm{rot}\,\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{j}$$

with usual notation of physical quantities. Second and third of these equations are of type (26), which was a consequence of the requirement of variationality of forces in mechanics. We also have relations (notation above equations (26))

$$\vec{B} = \mathrm{rot}\,\vec{A}, \quad \vec{E} = -\mathrm{grad}\,U - \frac{\partial\vec{A}}{\partial t}, \tag{29}$$

where $\vec{A}$ and U in electrodynamics have the meaning of vector and scalar potential, respectively.

Now we shall show that equations (29) result from a variational principle. For this purpose we rewrite them into a standard four-dimensional notation. As an underlying fibred manifold consider $(Y, \pi, X) = (\mathbb{R}^4 \times \mathbb{R}^4, \pi, \mathbb{R}^4)$, where the base X is the time-space with the Minkowski metric (signature $(1, -1, -1, -1)$). In coordinates, with standard notation $t = c^{-1}x^0$ for time, and cartesian coordinates, we have $(x^0, x^1, x^2, x^3) \in X$, i.e. $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$. Coordinates in fibres are co-vectors $(A) = (A_0, A_1, A_2, A_3)$, $A_0 = c^{-1}U$. Corresponding vector (A^0, A^1, A^2, A^3) has components $A^i = g^{ij}A_j$, i.e. $A_i = g_{ij}A^j$, where (g^{ij}) is the inverse matrix to (g_{ij}). The first and last equation in four-dimensional notation read

$$-\frac{1}{c}\left(j^i + \frac{1}{\mu_0}\frac{\partial F^{ij}}{\partial x^j}\right) = 0, \quad 0 \le i, j \le 3, \tag{30}$$

where $(j) = (c\varrho, \vec{j})$, (F^{ij}) and (F_{ij}), $F_{ij} = g_{ik}g_{jl}F^{kl}$ (and vice versa) are contravariant and covariant components of the *electromagnetic field tensor*,

$$F_{ij} = A_{j,i} - A_{i,j} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}.$$

Coordinates on (J^1Y, π, X) then are $(x^i, A_i, A_{i,j})$, $0 \le i, j \le 3$. Taking into account relations between the four-potential (A) and quantities $\vec{E}$, $\vec{B}$ and U we can write the covariant tensor (F_{ij}) as follows:

$$(F_{ij}) = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & -B^3 & B^2 \\ -E^2/c & B^3 & 0 & -B^1 \\ -E^3/c & -B^2 & B^1 & 0 \end{pmatrix}.$$

The contravariant tensor (F^{ij}) differs from (F_{ij}) only by changing the first row to the first column and vice versa. Equation (30) for $i = 0$ gives $\varepsilon_0 \operatorname{div}\vec{E} = \varrho$, while for $i = 1, 2, 3$ leads to a vector equation $\operatorname{rot}\vec{B} = \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{j}$, where we put $c^{-2} = \varepsilon_0\mu_0$.

We rewrite the set of equations of motion (30) in agreement with our choice of the underlying fibred space (Y, π, X) and its first and second prolongations. Components of the corresponding dynamical form are then

$$E^i = -\frac{1}{c}\left[g^{ik}j_k + \frac{1}{\mu_0}g^{ij}g^{kl}\left(A_{l,jk} - A_{j,lk}\right)\right].$$

We obtain Helmholtz conditions of variationality from (28) changing $\sigma \to i$, $0 \le i \le 3$, and $y^\sigma \to A_i$, $y^\sigma_k \to A_{i,k}$ and $y^\sigma_{kl} \to A_{i,kl}$. Let us verify the last of them, verification of the remaining ones is trivial.

$$\frac{\partial E^i}{\partial A_{p,rs}} = -\frac{1}{c\mu_0}g^{ij}g^{kl}(\delta_{lp}\delta_{rj}\delta_{sk} - \delta_{jp}\delta_{lr}\delta_{ks}) = -\frac{1}{c\mu_0}\left(g^{ir}g^{sp} - g^{ip}g^{sr}\right),$$

analogously $\frac{\partial E^p}{\partial A_{i,rs}} = g^{pr}g^{si} - g^{pi}g^{sr}$, which equals $\frac{\partial E^i}{\partial A_{p,rs}}$ because of the symmetry of starting expressions in the indices r and s.

The mapping (13) is

$$\chi : \big(u,\, (x^i,\, A_j,\, A_{j,k},\, A_{j,kl})\big) \longrightarrow \big(x^i,\, uA_j,\, uA_{j,k},\, uA_{j,kl}\big)$$

and the Vainberg-Tonti Lagrangian corresponding to our dynamical form is

$$\begin{aligned}\Lambda = A_i \int_0^1 (E^i \circ \chi)\,\mathrm{d}u &= -\frac{1}{c} A_i \left[g^{il} j_l + \frac{1}{\mu_0} g^{ij} g^{kl} \left(A_{l,jk} - A_{j,lk}\right) u\,\mathrm{d}u \right] \\ &= -\frac{1}{c}\left[A_i j^i + \frac{1}{2\mu_0} g^{ij} g^{kl} A_i \left(A_{l,jk} - A_{j,lk}\right)\right].\end{aligned}$$

Taking into account that

$$\mathrm{d}_k \left[A_i \left(A_{l,j} - A_{j,l}\right)\right] = A_{i,k}(A_{l,j} - A_{j,l}) + A_i \left(A_{l,jk} - A_{j,lk}\right)$$

and $A_{i,k}F^{ik} = -A_{k,i}F^{ik}$ (due to the antisymmetry of the electromagnetic field tensor) we finally obtain the well-known first order Lagrangian (see e.g. [10]).

$$\lambda = L\,\omega_0, \quad L = -\frac{1}{c}\left(j^i A_i + \frac{1}{4\mu_0} F_{ik}F^{ik}\right) \tag{31}$$

Finally, let us find basic Noether symmetries and the corresponding Noether currents for the electromagnetic field without charges and currents, i.e. for $(j^i) = (c\varrho,\, \vec{j}) = (0)$. Noether equation (18) has the form

$$i_{J^1\xi}\left(\frac{\partial L}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial L}{\partial A_i}\,\mathrm{d}A_i + \frac{\partial L}{\partial A_{i,l}}\,\mathrm{d}A_{i,l}\right) + L\,\mathrm{d}_i \xi^i = 0.$$

At first sight it is evident that generators of translations $\xi_i = \frac{\partial}{\partial x^i}$ are Noether symmetries, as well as $\frac{\partial}{\partial A_s}$. The Noether current corresponding to a symmetry $\xi = \xi^i \frac{\partial}{\partial x^i}$ on the base X (i.e. a linear combination of generators ξ_i) is

$$\Phi(\xi) = L\xi^i + \frac{\partial L}{\partial A_{r,i}}(-A_{r,j}\xi^j),$$

where

$$\begin{aligned}\frac{\partial L}{\partial A_{r,i}} &= -\frac{1}{4c\mu_0}\frac{\partial}{\partial A_{r,i}}(F_{kl}F^{kl}) = -\frac{1}{4c\mu_0}\frac{\partial}{\partial A_{r,i}} g^{kp}g^{lq}(A_{l,k} - A_{k,l})(A_{q,p} - A_{p,q}) \\ &= -\frac{1}{2c\mu_0}(g^{ip}g^{rq} - g^{rp}g^{iq})(A_{q,p} - A_{q,p}) = -\frac{1}{2c\mu_0}(F^{ir} - F^{ri}) = -\frac{1}{c\mu_0}F^{ir}.\end{aligned}$$

Hence, the Noether current is

$$\Phi(\xi) = -\frac{1}{c\mu_0}\left[\frac{1}{4}F_{kl}F^{kl}\xi^i - A_{r,j}F^{ir}\xi^j\right]\omega_i.$$

Components of the generators $\xi_s = \frac{\partial}{\partial x^s}$ are $\xi_s^i = \delta_s^i$ and we have

$$\Phi\left(\frac{\partial}{\partial x^s}\right) = -\frac{1}{c\mu_0}\left(\frac{1}{4}F_{kl}F^{kl}\delta_s^i - A_{r,s}F^{ir}\right)\omega_i$$

Components of these forms are components of the well-known *energy-momentum tensor*. Their contravariant form (standardly used) is

$$T^{ji} = -\frac{1}{c\mu_0}\left(\frac{1}{4}g^{ji}F_{kl}F^{kl} - g^{js}g_{rl}A_s^l F^{ir}\right).$$

This tensor is not symmetric but it can be symmetrized without the influence on the total momentum. Moreover, the reasoning leading to symmetrization guarantees the conservation of angular momentum as well.

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