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A Note on Transcendental Power Series Mapping the Set of Rational Numbers into Itself

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Abstract. In this note, we prove that there is no transcendental entire function $f(z) \in \mathbb{Q}[[z]]$ such that $f(\mathbb{Q}) \subseteq \mathbb{Q}$ and $\operatorname{den} f(p/q) = F(q)$, for all sufficiently large q, where $F(z) \in \mathbb{Z}[z]$.

1 Introduction

A real number ξ is called a *Liouville number*, if for any real number $\omega > 0$ there exists a rational number p/q, with $q > 1$, such that

$$0 < \left|\xi - \frac{p}{q}\right| < q^{-\omega}.$$

In his pioneer book, Maillet [3, Chapter III] proved that the set of the Liouville numbers is preserved under rational functions with rational coefficients. Based on this result, in 1984, Mahler [2] posed the following question

Question 1. *Are there transcendental entire functions $f(z)$ such that if ξ is any Liouville number, then so is $f(\xi)$?*

Recently, some authors (see [4], [5], [7]) constructed classes of Liouville numbers which are mapped into Liouville numbers by transcendental entire functions. For example, to prove this, Marques and Moreira [4] showed the existence of transcendental entire functions f, such that $f(\mathbb{Q}) \subseteq \mathbb{Q}$ and $\operatorname{den} f(p/q) < q^{8q^2}$, for all $p/q \in \mathbb{Q}$, with $q > 1$ (where $\operatorname{den} z$ denotes the denominator of the rational number z). Moreover, their proof implies that the Mahler's question has an affirmative answer if the answer to the below question is also 'yes' (see also [7, Theorem 2.1]).

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Question 2. *Are there transcendental entire functions $f(z)$ such that $f(\mathbb{Q}) \subseteq \mathbb{Q}$ and*

$$\operatorname{den} f(p/q) \leq F(q),$$

for some fixed polynomial $F(z) \in \mathbb{Z}[z]$ and for all sufficiently large q?

In 2015, Marques, Ramirez and Silva [6] proved that the answer for the previous question is 'no' for lacunary power series in $\mathbb{Q}[[z]]$ (see [1] for the definition of lacunary power series as well as some results related to their arithmetic properties). Moreover, their proof also implies that there is no transcendental entire function $f(z) \in \mathbb{Q}[[z]]$ such that $f(\mathbb{Q}) \subseteq \mathbb{Q}$ and

$$\operatorname{den} f(p/q) = o(q).$$

In an attempt of answering the previous question, a natural question arises: Could $\operatorname{den} f(p/q)$ be a polynomial in q for all sufficiently large q?

In this paper, we shall answer this previous question by proving that

Theorem 1. *There is no transcendental entire function $f(z) \in \mathbb{Q}[[z]]$ such that $f(\mathbb{Q}) \subseteq \mathbb{Q}$ and*

$$\operatorname{den} f(p/q) = F(q),$$

for all sufficiently large q, where $F(z) \in \mathbb{Z}[z]$.

2 The proof

Suppose, towards a contradiction, that for some $F(z) \in \mathbb{Z}[z]$ with degree $m \geq 1$ (the case $m = 0$ was solved in Remark 2.1 of [6]), there exists such a function, say $f(z) = \sum_{k \geq 0} a_k z^k \in \mathbb{Q}[[z]]$.

Thus, for all sufficiently large q, we have that $f(1/q) = n(q)/F(q)$, where $n(q)$ is an integer. Then

$$f\left(\frac{1}{q}\right) - \left(a_0 + \frac{a_1}{q} + \cdots + \frac{a_{m-1}}{q^{m-1}}\right) = \sum_{k \geq m} \frac{a_k}{q^k}.$$

By setting $A = \prod_{i=0}^{m-1} \operatorname{den}(a_i)$, we have

$$A\frac{n(q)}{F(q)} - \left(b_0 + \frac{b_1}{q} + \cdots + \frac{b_{m-1}}{q^{m-1}}\right) = \frac{Aa_m}{q^m} + A \sum_{k \geq m+1} \frac{a_k}{q^k},$$

where $b_i = Aa_i \in \mathbb{Z}$. Therefore,

$$A\frac{n(q)}{F(q)} - \frac{C(q)}{q^{m-1}} = \frac{Aa_m}{q^m} + A \sum_{k \geq m+1} \frac{a_k}{q^k},$$

where $C(z) \in \mathbb{Z}[z]$ has degree $\leq m - 1$. After multiplying by $F(q)$, we obtain

$$An(q) - \frac{D(q)}{q^{m-1}} = \frac{Aa_m F(q)}{q^m} + A \sum_{k \geq m+1} \frac{a_k F(q)}{q^k},$$

where $D(z) \in \mathbb{Z}[z]$ has degree $\leq 2m-1$. However, we can write $D(q)/q^{m-1} = E(q)/q^{m-1} + G(q)$, where $E, G \in \mathbb{Z}[z], \deg E \leq m-2$ and $\deg G \leq m$. Now, write

$$An(q) - G(q) = \frac{Aa_m F(q)}{q^m} + A \sum_{k \geq m+1} \frac{a_k F(q)}{q^k} + \frac{E(q)}{q^{m-1}}. \tag{1}$$

Now, we want to evaluate the limit in the right-hand side above when $q \to \infty$. Let ϵ be the leading coefficient of $F(z)$ (which we can assume to be ≥ 1). Note that $\lim_{q\to\infty} F(q)/q^m = \epsilon$ and $\lim_{q\to\infty} E(q)/q^{m-1} = 0$. Now, we need to calculate the limit of the summatory. For that, take a real number δ such that $0 < \delta < 1/\epsilon \leq 1$. Then $\delta^k < 1/\epsilon$, for all $k \geq m+1$. Thus,

$$\frac{q^k}{\delta^k} \geq \frac{q^{m+1}}{\delta^k} > qF(q)$$

for all sufficiently large q (since the degree of $zF(z)$ is $m+1$ and its leading coefficient is ϵ). Hence,

$$\frac{|F(q)|}{q^k} = \frac{F(q)}{q^k} < \frac{1}{q\delta^k}$$

(for all sufficiently large q and for all $k \geq m+1$) and so

$$\left| \sum_{k \geq m+1} \frac{a_k F(q)}{q^k} \right| \leq \frac{1}{q} \sum_{k \geq m+1} \frac{|a_k|}{\delta^k}.$$

Since $\sum_{k \geq m+1} |a_k|/\delta^k < \infty$ (by the absolute convergence of $\sum_{k\geq 0} a_k z^k$ in $\mathbb{C}$), then

$$\lim_{q\to\infty} \sum_{k \geq m+1} \frac{a_k F(q)}{q^k} = 0.$$

In conclusion, the right-hand side of (1) tends to $Aa_m\epsilon$ as $q \to \infty$. Therefore, for all sufficiently large q, it holds that

$$0 \leq |An(q) - G(q)| \leq Aa_m\epsilon + 1.$$

Since $An(q) - G(q)$ is an integer, then there exist $t \in \mathbb{Z}$ and an infinite set $S \subseteq \mathbb{N}$ such that $An(q) - G(q) = t$ for all $q \in S$. Thus

$$f\left(\frac{1}{q}\right) = \frac{n(q)}{F(q)} = \frac{G(q)+t}{AF(q)} = \frac{e_0 + e_1 q + \cdots + e_m q^m}{d_0 + d_1 q + \cdots + d_m q^m} = \frac{P(1/q)}{Q(1/q)}, \tag{2}$$

where $P(z) = \sum_{i=0}^m e_i z^{m-i}$ and $Q(z) = \sum_{i=0}^m d_i z^{m-i}$.

Let r be a positive real number such that $r < \min\{|z| : Q(z) = 0\}$ (observe that $Q(0) = d_m = A\epsilon \neq 0$). Then the function $h(z)$ given by

$$h(z) = \frac{P(z)}{Q(z)}$$

is analytic on the interval $(-r, r)$. Moreover, by (2), we have that the analytic functions $f(z)$ and $h(z)$ coincide on the set $\{1/q : q \in S \cap (1/r, \infty)\} \subseteq (-r, r)$ which has a limit point in $(-r, r)$. Thus, by the identity principle for analytic functions, we have that $f(z) = h(z)$ on $(-r, r)$. In particular, the entire functions $Q(z)f(z)$ and $P(z)$ coincide on $(-r, r)$ yielding, by the same principle, that they have equal values for all $z \in \mathbb{C}$. Hence the function $f(z)$ satisfies $P(z, f(z)) = 0$, for all $z \in \mathbb{C}$, where $P(x, y) = Q(x)y - P(x)$ (which is a nonzero polynomial). However, this contradicts the transcendence of f. The proof is then complete. $\square$

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