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On the critical determinants of certain star bodies

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Abstract. In a classic paper [14], W.G. Spohn established the to-date sharpest estimates from below for the simultaneous Diophantine approximation constants for three and more real numbers. As a by-result of his method which used Blichfeldt's Theorem and the calculus of variations, he derived a bound for the critical determinant of the star body

$$|x_1|\left(|x_1|^3+|x_2|^3+|x_3|^3\right)\le 1\,.$$

In this little note, after a brief exposition of the basics of the geometry of numbers and its significance for Diophantine approximation, this latter result is improved and extended to the star body

$$|x_1|\left(|x_1|^3+\left(x_2^2+x_3^2\right)^{3/2}\right)\le 1\,.$$

1 Introduction

In a recent survey paper [12], the author has expressed his regret that at least in certain parts of the *geometry of numbers*, research has essentially been terminated in the second half of the 20th century. In particular, this concerns evaluations, resp., estimations of critical determinants of star bodies, and applications thereof to Diophantine approximation. In the author's opinion, this is even more regrettable, since only a short time later personal computers became available, along with a wealth of software for symbolic calculations and for graphics. At least for the heuristic part of attacking problems in the area, these are very helpful.

Therefore, the present little note is intended to continue the tradition from the "Golden Age" of the geometry of numbers, making use of symbolic computation support and graphics tools, and appealing to an earlier result established by the author around the turn of the millennium [11].

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We indicate briefly how the present little problem is linked to classic work in the field. Before doing so, we recall the basic concepts of the geometry of numbers: A *lattice* in $\mathbb{R}^s$, $s \geq 2$, is defined as $\Gamma = A\mathbb{Z}^s$, where A is a non-singular real $(s \times s)$-matrix. Its *lattice constant* is $d(\Gamma) = |\det A|$.

A *star body* K in $\mathbb{R}^s$, is a non-empty $\mathbf{o}$-symmetric closed subset of $\mathbb{R}^s$, with the property that, for any point $\mathbf{p} \in K$, the closed straight line segment joining $\mathbf{p}$ with $\mathbf{o}$ is contained in the interior of K, with the possible exception of $\mathbf{p}$ itself.

Further, a lattice Γ in $\mathbb{R}^s$ is called *admissible* for a star body K, if $\mathbf{o}$ is the only lattice point of Γ contained in the interior of K.

Finally, the *critical determinant* $\Delta(K)$ of a star body K is the infimum of all lattice constants $d(\Gamma)$, where Γ ranges over all lattices which are admissible for K. See also the monograph by Gruber and Lekkerkerker [5].

2 Simultaneous Diophantine approximation, and its connection to the geometry of numbers

The classic *Hurwitz's Theorem* tells us that, for every irrational number α, there exist infinitely many reduced fractions $\frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{Z}_+$, such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\,q^2}\,, \tag{1}$$

and that the constant $\sqrt{5}$ is best possible. See, e.g., [9, p. 189 and p. 221].

More generally, for each positive integer s, one can define θ_s as the supremum of all values c with the following property: *For every* $\alpha \in \mathbb{R}^s \setminus \mathbb{Q}^s$, *there exist infinitely many* $(\mathbf{p}, q) \in \mathbb{Z}^s \times \mathbb{Z}_+$ *with* $\gcd(\mathbf{p}, q) = 1$, *such that*

$$\left\|\alpha - \frac{1}{q}\mathbf{p}\right\|_\infty < \frac{1}{q(cq)^{1/s}}\,. \tag{2}$$

However, the determination of θ_s remains an open problem to date, for any $s \geq 2$. In fact, on the case $s = 2$ already Charles Hermite (1822–1901) has written:

> La recherche des fractions p'/p, p''/p qui approchent le plus de deux nombres donnés n'a cessé depuis plus de 50 ans de me préoccuper et aussi de désespérer.

In 1955, H. Davenport [4] established a result connecting the simultaneous Diophantine approximation problem with the geometry of numbers: For any positive integer s,

$$\theta_s = \Delta(K_s)\,, \tag{3}$$

where

$$K_s = \left\{(x_0, \dots, x_s) \in \mathbb{R}^{s+1} \colon\ |x_0|\,\|(x_1, \dots, x_s)\|_\infty^s \leq 1\right\}\,.$$

The case $s = 2$ is somewhat exceptional, since it admits some finer analysis, using intrinsic geometric considerations of very special planar domains. Based on earlier work by Mullender [8], Davenport [3] showed that $\theta_2 = \Delta(K_2) \geq \sqrt[4]{46} = 2.604\ldots$. This was refined later by Mack [6] and the author [10] who obtained $\theta_2 = \Delta(K_2) \geq \left(\frac{13}{8}\right)^2 = 2.64062\ldots$. In the opposite direction, Cassels [2] showed that $\theta_2 = \Delta(K_2) \leq 3.5\,$.

3 The approach by Blichfeldt and Spohn's Theorem

For $s \geq 3$, however, the sharpest lower bounds for θ_s are contained in a deep and (among experts) celebrated result by Spohn [14]. This in turn was based on a classic theorem of Blichfeldt [1] (see also [5, p. 123]): For any starbody K in $\mathbb{R}^s$ and any measurable set $M \subseteq \mathbb{R}^s$ whose *difference set*

$$\mathcal{D}M = \{\mathbf{m}_1 - \mathbf{m}_2 : \mathbf{m}_1, \mathbf{m}_2 \in M\}$$

is contained in K, it follows that

$$\Delta(K) \geq \operatorname{vol}(M)\,. \tag{4}$$

Blichfeldt's approach was sharpened by Mullender [7] and made perfect by Spohn [14] who used the calculus of variations to determine, for each star body K_s, the set $M_{s+1} \subset \mathbb{R}^{s+1}$ with maximal volume such that $\mathcal{D}M_{s+1} \subseteq K_s$. Evaluating this volume and using (3) and (4), one obtains

$$\theta_s \geq s2^{s+1} \int_0^1 \frac{w^{s-1}}{(1+w)^s(1+w^s)}\,dw\,.$$

Numerically,

$$\theta_3 \geq 2.449\ldots,\ \theta_4 \geq 2.559\ldots,\ \theta_5 \geq 2.638\ldots\,.$$

For $s \geq 3$, these are the sharpest explicit lower bounds for the constants θ_s known to date.

4 Another special star body

In his paper [14, p. 887], Spohn makes the remark that his method, based on Blichfeldt's Theorem and the calculus of variations, can also be applied to the three-dimensional star body

$$K^* : \quad |x|\left(|x|^3 + |y|^3 + |z|^3\right) \leq 1\,, \tag{5}$$

to obtain the estimate

$$\Delta(K^*) \geq 1.425\,. \tag{6}$$

This improves upon the bound $\Delta(K^*) \geq \sqrt{2}$ which follows by inscribing an ellipsoid into K^* and using the value $\Delta(S_3) = \frac{1}{\sqrt{2}}$ of the unit ball S_3 in $\mathbb{R}^3$. The details of Spohn's analysis can be found in another one of his works [13].

The aim of this little note is to improve upon the estimate (6). Our main tool is a result proved some time ago by the author [11]: The critical determinant of the double paraboloid in $\mathbb{R}^3$

$$\mathcal{P} : \quad |x| + y^2 + z^2 \leq 1$$

equals

$$\Delta(\mathcal{P}) = \frac{1}{2}\,. \tag{7}$$

The idea is to inscribe into K^* a suitable body of rotation K^{**}, and to fit into K^{**} a double paraboloid

$$\mathcal{P}_a: \quad |x| + \left(\frac{y}{a}\right)^2 + \left(\frac{z}{a}\right)^2 \le 1\,, \tag{8}$$

with optimal $a > 0$.

Theorem 1. *Let K^* be defined as in* (5), *and let*

$$K^{**}: \quad |x|\left(|x|^3 + \left(y^2+z^2\right)^{3/2}\right) \le 1\,.$$

Then $K^{**} \subseteq K^*$ and[1]

$$\Delta(K^*) \ge \Delta(K^{**}) \ge 1.5044\ldots$$

Proof. The first part is obvious, by the elementary inequality

$$\left(|y|^3 + |z|^3\right)^2 \le \left(y^2+z^2\right)^3$$

which is true for all reals y, z. To estimate $\Delta(K^{**})$ from below, we inscribe into K^{**} a double paraboloid $\mathcal{P}_a$ with maximal a. The critical determinant of the latter equals $\frac{a^2}{2}$. Since both are bodies of rotation about the x-axis, it suffices to consider the intersections of K^{**}, $\mathcal{P}_a$ with the (x,y)-plane: Call these planar domains

$$\begin{aligned} R^{**}: &\quad |x|\left(|x|^3 + |y|^3\right) \le 1\,,\\ R_a: &\quad |x| + \left(\frac{y}{a}\right)^2 \le 1\,. \end{aligned} \tag{9}$$

See Figure 1 below.

Of course, we may restrict the calculations to the first quadrant. Our aim is to calculate a maximal so that $R_a \subseteq R^{**}$, and the point $(x_0, y_0) \in \mathbb{R}_+^2$, $x_0 < 1$, which is common to both ∂R_a and ∂R^{**}. In this point, both curves obviously have the same slope y'. We conclude that

$$y_0^3 = -x_0^3 + \frac{1}{x_0}\,, \quad y_0^2 = a^2(1-x_0)\,, \tag{10}$$

and, taking derivatives,

$$3y_0^2 y' = -3x_0^2 - \frac{1}{x_0^2}\,, \quad 2y_0 y' = -a^2\,. \tag{11}$$

Dividing the first eq. of (11) by the second one, we get

$$\frac{3}{2} y_0 = \frac{1}{a^2}\left(3x_0^2 + \frac{1}{x_0^2}\right)\,. \tag{12}$$

[1] As will be evident from the proof, the numerical constant in principle can be expressed explicitly by radicals (nested roots of integers).

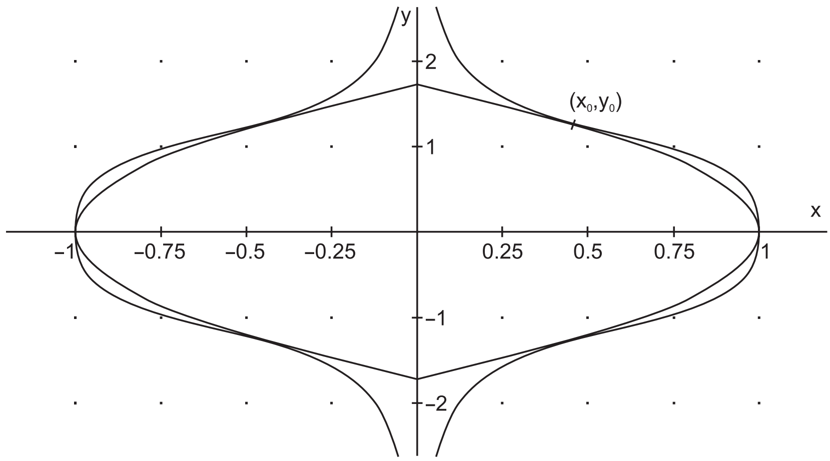

Figure 1: The intersections of the non-convex body K^{**} and the double paraboloid $\mathcal{P}_a$ with the (x, y)-plane

Multiplying this by the second part of (10) gives

$$y_0^3 = \frac{2}{3}(1 - x_0)\left(3x_0^2 + \frac{1}{x_0^2}\right). \tag{13}$$

Setting this equal to the first part of (10), multiplying by x_0^2, dividing by $1 - x_0$, and simplifying, we arrive at

$$3x_0^4 - 3x_0^3 - 3x_0^2 - 3x_0 + 2 = 0. \tag{14}$$

This biquadratic polynomial equation has only one real positive root < 1, namely

$$x_0 = 0.43306989\ldots. \tag{15}$$

From this and the first part of (10),

$$y_0 = 1.3060612\ldots. \tag{16}$$

Finally, from the second part of (10),

$$a = 1.7345976\ldots. \tag{17}$$

It remains to show that actually $R_a \subseteq R^{**}$. Going back again to the definitions of the curves ∂R_a and ∂R^{**}, it obviously suffices to verify that, for $0 < x \leq 1$,

$$\left(\left(-x^3 + \frac{1}{x}\right)^2 - a^6(1-x)^3\right)x^2 \geq 0. \tag{18}$$

By construction, the left-hand side here is a polynomial of degree eight which has double zeros at $x = x_0$ and also at $x = 1$. Hence, the left-hand side of (18) can be written in the form

$$(x - x_0)^2 (x - 1)^2 p_4(x - x_0) \,. \tag{19}$$

The calculation shows that the biquadratic polynomial $p_4(x - x_0)$ has only positive coefficients. In fact, with the numerical values rounded,

$$p_4(x - x_0) = (x - x_0)^4 + 4.598(x - x_0)^3 + 10.14(x - x_0)^2 + 41.81(x - x_0) + 21.87 \,.$$

This establishes the assertion (18). The conclusion that $\Delta(K^{**}) \geq \Delta(\mathcal{P}_a) = \frac{1}{2}a^2$, together with (17), completes the proof of the theorem. $\square$

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