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Weak Solutions for Nonlinear Parabolic Equations with Variable Exponents

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Abstract. In this work, we study the existence and uniqueness of weak solutions of fourth-order degenerate parabolic equation with variable exponent using the difference and variation methods.

1 Introduction

The study of differential equations involving variable exponent conditions is a new and interesting topic in recent years. The interest in studying such problems is stimulated by their applications in elastic mechanics, fluid dynamics, nonlinear elasticity, electrorheological fluids etc. In particular, parabolic equations involving the $p(x)$-Laplacian appeared in the field of image restoration in [15], [19] and electrorheological fluids which are characterized by their ability to change the mechanical properties under the influence of the exterior electromagnetic field in [24]. Further, porous medium type equation with variable exponents is also studied in [4]. These physical problems are facilitated by the development of Lebesgue and Sobolev spaces with variable exponent. Recently, parabolic and elliptic equations which involves variable exponents has studied well in the literature, for example, see [1], [6], [7], [8], [5], [18], [19], [29] and also the references there in.

This paper is devoted to study the existence and uniqueness of weak solutions of the following fourth-order parabolic equation with variable exponents:

$$\left.\begin{array}{ll} \frac{\partial u}{\partial t} + \operatorname{div}\big(|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\big) = f - \operatorname{div} g, & (x,t)\in Q,\\ u|_\Gamma = \Delta u|_\Gamma = 0, & \\ u(x,0) = u_0(x), & x\in\Omega, \end{array}\right\} \tag{1}$$

where we denote the cylinder $Q \equiv \Omega \times (0,T]$, the lateral surface $\Gamma \equiv \partial\Omega \times (0,T]$ and $p\colon \overline{\Omega} \to (1,\infty)$ is a continuous function (called the variable exponent) and Ω is

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a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and T is a given positive number. Based on the physical consideration, as usual (1) is supplemented with the natural boundary conditions and the initial value condition. We assume that

$$u_0 \in H_0^1(\Omega), \quad f \in L^{p'(x)}(0,T;L^{(p^*(x))'}(\Omega)), \quad g \in (L^{p'(x)}(Q))^N. \tag{2}$$

When $N = 1$ and $p(x) = p$(constant), (1) is a generalized thin film equation, see [20], which has been extensively studied recently. Xu and Zhou [25] obtained the existence and uniqueness of weak solutions for such a kind of generalized thin film equation. Also when $p(x) = 2$, (1) is known as the Cahn-Hilliard equation (see [12]) which occurs in science and engineering. Indeed this equation also forms a base for the methods used to improve the sharpness of vague images in image analysis. Calderon and Kwembe [13] used the Cahn-Hilliard equation to model the long range effect of insects dispersal. Recently King [20] derived the homogeneous equation of type (1) for the case when $p > 1$ which is relevant to capillary driven flows of thin films of power-law fluids, where $u(x,t)$ denotes the height from the surface of the oil to the surface of the solid. Zhang and Zhou [28] established the existence, uniqueness and long-time behavior of weak solutions for fourth-order degenerate parabolic equation with variable exponents of nonlinearity. The exponent p is related to the rheological properties of the liquid: $p = 2$ corresponds to a Newtonian liquid, whereas $p \neq 2$ emerges when considering "power-law" liquids. When $p > 2$, the liquid is said to be shear-thinning. Xu and Zhou [26] studied the stability and regularity of weak solutions for a generalized thin film equation for the corresponding homogeneous equation of type (1) with p as a constant. In this connection, Bhuveneshwari et al. [10] established the existence of weak solutions for p-Laplacian equation. Bertsch et al. [9] proved the existence of weak solutions for a class of fourth-order degenerate equation. Moreover the existence, uniqueness and qualitative properties of solutions of (1) which are related to constant case have been studied in [2], [3], [20], [22], [23], [25] and references therein. Bowen et al. [11] investigated similiarity solutions of the thin film equation. As far as we know, there are few papers concerned with the fourth-order nonlinear parabolic equation involving multiple anisotropic exponents. It is not a trivial generalization of similar problems studied in the constant case. The main difficulties in studying the problem are caused by the complicated nonlinearities of homogeneous equation of type (1) and the lack of a maximum principle for fourth-order equations. Due to the degeneracy, problem (1) does not admit classical solutions in general.

The paper is organized as follows: In section 2, we introduce some basic results regarding the variable exponent spaces and notations. In section 3, we introduce the suitable time-independent equation of original parabolic equation and are proving that there exists a weak solution for time-independent equation. Further, using this result, we establish the existence of solutions of original equation. Finally, in section 4, we prove that the solutions obtained are unique.

2 Preliminaries

In this section, we recall some basic definitions, inequalities and the properties of the generalized Lebesgue and Sobolev spaces with variable exponents. However, for more detailed theory and the proofs of the following results, one can refer [16], [28].

2.1 Variable Exponent Spaces

Set

$$C_+(\overline{\Omega}) = \Big\{p \in C(\overline{\Omega}); \min_{x\in\overline{\Omega}} p(x) > 1\Big\}.$$

For any $p \in C_+(\overline{\Omega})$, we define $p^+ = \sup_{x\in\overline{\Omega}} p(x)$ and $p^- = \inf_{x\in\overline{\Omega}} p(x)$. We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u\colon \Omega \to \mathbb{R}$ such that $\int_\Omega |u|^{p(x)}\,\mathrm{d}x < \infty$ endowed with the norm

$$|u|_{p(\cdot)} = \inf\left\{\lambda > 0; \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1\right\}$$

called the Luxemburg norm. The space $L^{p(\cdot)}$ with the above norm is a separable and reflexive Banach space. The dual space of $L^{p(\cdot)}(\Omega)$ is isometric to $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and $p'(x)$ is the conjugate of $p(x)$.

The variable exponent $p\colon \overline{\Omega} \to [1,\infty)$ can be extended upto $\overline{Q}_T$ by setting $p(t,x) := p(x)$ for all $(t,x) \in \overline{Q}_T$; we may also consider the generalized Lebesgue space $L^{p(\cdot)}(Q_T)$ as the set of all measurable functions $u\colon Q_T \to \mathbb{R}$ such that $\int_{Q_T} |u(x,t)|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t < \infty$, endowed with the norm

$$|u|_{p(\cdot)} = \inf\Big\{\lambda > 0; \int_{Q_T} \left|\frac{u(x,t)}{\lambda}\right|^{p(x)} \mathrm{d}x\,\mathrm{d}t \le 1\Big\},$$

which also has the same properties as those of $L^{p(\cdot)}(\Omega)$. For any positive integer k, the variable exponent Sobolev space is given by

$$W^{k,p(\cdot)} = \big\{u \in L^{p(\cdot)}(\Omega); D^\alpha u \in L^{p(\cdot)}(\Omega)\big\},$$

endowed with the norm $\|u\|_{W^{k,p(\cdot)}} = \sum_{|\alpha|\le k} |D^\alpha u|_{L^{p(\cdot)}}$.

The exponent $p(x)$ is log-Hölder continuous function, that is, $|p(x) - p(y)| \le \frac{c}{-\log(x-y)}$ for all $x, y \in \Omega$ with $|x-y| < \frac{1}{2}$ with some constant c. Then the smooth functions are dense in variable exponent Sobolev spaces and the spaces $W_0^{1,p(\cdot)}(\Omega)$ are the completion of the $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$. For more details, see [29].

Lemma 1. *1) The space $L^{p(\cdot)}$ is a separable, uniform convex Banach space and its conjugate space is $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{(p^-)'}\right) \|u\|_{p(x)} \|v\|_{p'(x)} \le 2\|u\|_{p(x)}\|v\|_{p'(x)}.$$

2) *If* $p_1, p_2 \in C_+(\overline{\Omega}), p_1(x) \leq p_2(x)$ *for any* $x \in \Omega$. *Then there exists a continuous embedding* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ *whose norm does not exceed* $|\Omega| + 1$.

Lemma 2. *If we denote* $\mathcal{P}(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x$, *for every* $u \in L^{p(\cdot)}(\Omega)$, *then*

$$\min\{\|u\|_{p(x)}^{p-}, \|u\|_{p(x)}^{p+}\} \leq \mathcal{P}(u) \leq \max\{\|u\|_{p(x)}^{p-}, \|u\|_{p(x)}^{p+}\}.$$

Lemma 3. 1) $W^{k,p(\cdot)}(\Omega)$ *is a separable and reflexive Banach space.*

2) *For* $u \in W_0^{1,p(\cdot)}(\Omega)$ *with* $p \in C_+(\overline{\Omega})$ *satisfying log-Hölder continuity, the inequality,* $\|u\|_{L^{p(\cdot)}(\Omega)} \leq c\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ *holds, where the positive constant* c *depends on* p *and* Ω.

3) *For* $u \in W_0^{1,p(\cdot)}(\Omega)$ *with* $p \in C_+(\overline{\Omega}), 1 \leq p^- \leq p^+ \leq N$, *the Sobolev imbedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *holds for any measurable function* $r\colon \Omega \to [1,\infty)$ *such that* $\operatorname*{ess\,lim}_{x\in\Omega} \left(\frac{Np(x)}{1-p(x)} - r(x)\right) \geq 0$.

3 Existence of Weak Solutions

In this section, first we define the weak solutions of the degenerate parabolic problem (1). Further we introduce suitable time-independent equation of (1) and using the existence of solutions of time-independent equation, we establish the existence of weak solutions of the original parabolic equation (1).

Definition 1. [28] A function u is called a weak solution of the fourth-order parabolic equation (1) if the following conditions hold true, that is,

(i) $u \in C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega)) \cap L^{p^-}(0,T; W^{2,p(x)}(\Omega))$ with $\Delta u \in L^{p^-}(0,T; W_0^{1,p(x)}(\Omega))$ and $\nabla\Delta u \in (L^{p(x)}(Q))^N$,

(ii) For any $\varphi \in C^1(\overline{Q})$ with $\varphi(\cdot, T) = 0$, we have

$$-\int_\Omega u_0(x)\varphi(x,0)\,\mathrm{d}x - \int_0^T\int_\Omega \left[u\phi_t + |\nabla\Delta u|^{p(x)-2}\nabla\Delta u\cdot\nabla\varphi\right]\mathrm{d}x\,\mathrm{d}\tau = \int_0^T\int_\Omega f\varphi\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^T\int_\Omega g\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\tau. \quad (3)$$

Remark 1. Let u be a weak solution of (1). If $p(x)$ satisfies the log-Hölder continuity condition, then $u \in W^{1,p(x)}(\Omega) \cap H_0^1(\Omega) \subset W^{1,p(x)}(\Omega) \cap W_0^{1,1}(\Omega)$ and thus $u \in W^{1,p(x)}(\Omega) = W_0^{1,p(x)}(\Omega)$. By using the approximation technique, we have, for each $t \in [0,T]$ and every $\varphi \in C^1(\overline{Q})$,

$$\int_\Omega u\varphi\,\mathrm{d}x\Big|_0^t - \int_0^t\int_\Omega \left[u\varphi_t + |\nabla\Delta u|^{p(x)-2}\nabla\Delta u\cdot\nabla\varphi\right]\mathrm{d}x\,\mathrm{d}\tau = \int_0^t\int_\Omega f\varphi\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_\Omega g\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\tau. \quad (4)$$

Remark 2. Since $C_0^\infty(\Omega)$ is dense in $W^{1,p(x)}(\Omega)$ due to the log-Hölder continuity condition, we can choose Δu as a test function in (3) and (4). Indeed we may use the Steklov averages

$$\left.\begin{aligned} [v]_h(x,t) &= \tfrac{1}{h}\int_t^{t+h} v(x,\tau)\,\mathrm{d}\tau,\\ [f]_h(x,t) &= \tfrac{1}{h}\int_t^{t+h} f(x,\tau)\,\mathrm{d}\tau,\\ [g]_h(x,t) &= \tfrac{1}{h}\int_t^{t+h} g(x,\tau)\,\mathrm{d}\tau. \end{aligned}\right\} \tag{5}$$

of the function $v(x,t)$ to replace the corresponding function and then pass to the limits. Therefore we obtain from (4) an energy type estimate

$$\begin{aligned} &\frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega |\nabla\Delta u|^{p(x)}\,\mathrm{d}x\mathrm{d}\tau\\ &\qquad = \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega)} + C\|f\|^{p'(x)}_{L^{p'(x)}(0,T;L^{p^{(*)'}(x)}(\Omega))} + C\|g\|^{p'(x)}_{L^{p'(x)}(Q)} \end{aligned} \tag{6}$$

Lemma 4. *Suppose that $p(x) > 1$. Then, for every $v \in W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$,*

$$\|v\|_{W^{2,p(x)}} \le C|\Delta v|_{p(x)}. \tag{7}$$

where positive constant $C > 0$ depends only on p, N, Ω.

Proof. Using the definition of the space $W^{2,p(x)}$, Lemma 3 and the theorem in [27], we get

$$\|v\|_{W^{2,p(x)}(\Omega)} \le C\left(|\nabla v|_{p(x)} + |\Delta v|_{p(x)}\right), \tag{8}$$

where $C > 0$ is a constant. To prove the desired result of the lemma, we want to show that

$$\|v\|_{W^{1,p(x)}(\Omega)} \le C|\Delta v|_{p(x)}. \tag{9}$$

Suppose we assume that (9) is not true, then there exists a sequence $\{v_n\}_{n=1}^\infty$ in the space $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$ such that

$$\|v_n\|_{W^{1,p(x)}(\Omega)} > n|\Delta v_n|_{p(x)}. \tag{10}$$

Without loss of generality, we assume that $\|v_n\|_{W^{1,p(x)}(\Omega)} = 1$. Then it follows from (8) and (10), $\|v_n\|_{W^{2,p(x)}(\Omega)} \le C$, $|\Delta v_n|_{p(x)} \le \frac{1}{n}$. Now consider a subsequence (still denoted by $\{v_n\}_{n=1}^\infty$) and a function $v \in W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$ such that $v_n \rightharpoonup v$ weakly in $W^{2,p(x)}(\Omega)$, which implies that $v_n \to v$ strongly in $W_0^{1,p(x)}(\Omega)$. Therefore we get

$$\|v\|_{W^{1,p(x)}(\Omega)} = 1. \tag{11}$$

On the other hand, by the weak convergence of second derivative of v_n, we get,

$$|\Delta v|_{p(x)} \le \liminf_{n\to\infty} |\Delta v_n|_{p(x)} = 0,$$

which implies that $\Delta v = 0$. Since $v \in W_0^{1,p(x)}(\Omega)$, we conclude that $v = 0$ a.e. in Ω which contradicts (11). This completes proof of the theorem. □

Now let n be a positive integer and $h = \frac{T}{n}$. Then consider the following time-discrete problem of (1)

$$\left.\begin{aligned} &\tfrac{u_k - u_{k-1}}{h} + \nabla \cdot (|\nabla \Delta u_k|^{p(x)-2} \nabla \Delta u_k) \\ &\qquad = [f]_h((k-1)h) - \operatorname{div} [g]_h((k-1)h) \qquad \text{in } \Omega, \\ &u_k = \Delta u_k = 0, k = 1, 2, \dots, n \qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{12}$$

where $[f]_h, [g]_h$ are respectively as defined in (5). Clearly, from the definition, $[f]_h(\cdot) \in L^{(p^*(x))'}(\Omega)$ and $[g]_h(\cdot) \in \left(L^{p'(x)}(\Omega)\right)^N$ where p^* is the Sobolev conjugate exponent of p given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ q & \text{if } p = N, q \in (N, +\infty), \\ +\infty & \text{if } p > N. \end{cases} \tag{13}$$

Before we prove the existence of weak solutions of (12), first we establish the existence of weak solutions of the following elliptic problem, that is, the case $k = 1$ in (12),

$$\left.\begin{aligned} &\tfrac{u-u_0}{h} + \operatorname{div}\left(|\nabla \Delta u|^{p(x)-2} \nabla \Delta u\right) = [f]_h(0) - \operatorname{div} [g]_h(0), \quad \text{in } \Omega, \\ &u = \Delta u = 0, k = 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{aligned}\right\} \tag{14}$$

Now we consider the space

$$W = \left\{u \in H_0^1(\Omega) \cap W^{2,p(x)}(\Omega) \;\middle|\; \Delta u \in W_0^{1,p(x)}(\Omega)\right\}$$

with the norm $\|u\|_W = \|u\|_{H_0^1(\Omega)} + \|u\|_{W^{2,p(x)}(\Omega)} + \|\Delta u\|_{W_0^{1,p(x)}(\Omega)}$. It is easy to verify that W is a Banach space.

Definition 2. A function $u \in W$ is called a weak solution of the problem (14), if, for any $\varphi \in C^1(\overline{\Omega}) \cap W_0^{1,p(x)}(\Omega)$, we have

$$\int_\Omega \frac{u - u_0}{h} \varphi \, \mathrm{d}x - \int_\Omega \left(|\nabla \Delta u|^{p(x)-2} \nabla \Delta u\right) \cdot \nabla \varphi \, \mathrm{d}x \\ = \int_\Omega [f]_h(0) \varphi \, \mathrm{d}x - \int_\Omega \operatorname{div} [g]_h(0) \varphi \, \mathrm{d}x. \tag{15}$$

Theorem 1. *Under the assumptions of $u_0 \in H_0^1(\Omega), f \in L^{p'}\left(0, T; L^{(p^*(x))'}\right)$ and $g \in \left(L^{p'(x)}(Q)\right)^N$, there exists at least one weak solution for (14).*

Proof. Consider the variational problem,

$$\min \{J(u) \mid u \in W\}$$

where the functional J is defined by

$$J(u) = \frac{1}{2h} \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_\Omega \frac{1}{p(x)} |\nabla \Delta u|^{p(x)} \, \mathrm{d}x \\ - \int_\Omega [f]_h(0) \Delta u \, \mathrm{d}x - \int_\Omega [g]_h(0) \cdot \nabla \Delta u \, \mathrm{d}x, \tag{16}$$

where $f \in L^{p'}\big(0,T; L^{(p^*(x))'}(\Omega)\big)$, $g \in \big(L^{p'(x)}(Q)\big)^N$ are known functions.

Now we establish that $J(u)$ has a minimizer $u_1(x)$ in W. Therefore the function u_1 is a weak solution of the corresponding Euler-Lagrange equation of $J(u)$ which is (14) in the case $k=1$.

Using Lemma 1 with Hölder's and Young's inequalities, we have

$$\begin{aligned}\left|\int_\Omega [f]_h(0)\Delta u\,\mathrm{d}x\right| &\le \left(\int_\Omega |[f]_h(0)|^{(p^*(x))'}\,\mathrm{d}x\right)^{\frac{1}{(p^*(x))'}}\left(\int_\Omega |\Delta u|^{p^*(x)}\,\mathrm{d}x\right)^{\frac{1}{p^*(x)}}\\ &\le \|[f]_h(0)\|_{L^{(p^*(x))'}(\Omega)}\|\Delta u\|_{L^{p^*(x)}(\Omega)}\\ &\le \|[f]_h(0)\|_{L^{(p^*(x))'}(\Omega)}C\|\nabla\Delta u\|_{L^{p(x)}(\Omega)}\\ &\le \epsilon\|\nabla\Delta u\|^{p(x)}_{L^{p(x)}(\Omega)} + C(\epsilon)\|[f]_h(0)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)}. \end{aligned} \tag{17}$$

where C is a small positive constant. Similarly we get

$$\left|\int_\Omega [g]_h(0)\nabla\Delta u\,\mathrm{d}x\right| \le \epsilon\|\nabla\Delta u\|^{p(x)}_{L^{p(x)}(\Omega)} + C(\epsilon)\|[g]_h(0)\|^{p'(x)}_{L^{p'(x)}(\Omega)}, \tag{18}$$

where C is a small positive constant. For sufficiently small ϵ, we get

$$\begin{aligned} J(u) &\ge \frac{1}{2h}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}\,\mathrm{d}x + \epsilon\|\nabla\Delta u\|^{p(x)}_{L^{p(x)}(\Omega)}\\ &\quad + C(\epsilon)\|[f]_h(0)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)} + \epsilon\|\nabla\Delta u\|^{p(x)}_{L^{p(x)}(\Omega)} + C(\epsilon)\|[g]_h(0)\|^{p'(x)}_{L^{p'(x)}(\Omega)}\\ &\ge \frac{1}{2h}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}\,\mathrm{d}x\\ &\quad + 2\epsilon\|\nabla\Delta u\|^{p(x)}_{L^{p(x)}(\Omega)} + C(\epsilon)\|[f]_h(0)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)} + C(\epsilon)\|[g]_h(0)\|^{p'(x)}_{L^{p'(x)}(\Omega)}\\ &\ge \frac{1}{2h}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_\Omega \min\left\{\frac{1}{p^+}, 2\epsilon\right\}|\nabla\Delta u|^{\min\{p(x),p\}}\,\mathrm{d}x\\ &\quad + C(\epsilon)\|[f]_h(0)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)} + C(\epsilon)\|[g]_h(0)\|^{p'(x)}_{L^{p'(x)}(\Omega)}. \end{aligned} \tag{19}$$

Recalling $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ and Lemma 3, we have

$$\|u\|_{W^{2,p(x)}(\Omega)} \le C\|\Delta u\|_{p(x)}. \tag{20}$$

Since $\Delta u \in W_0^{1,p(x)}(\Omega)$, we get

$$|\Delta u|_{p(x)} \le C|\nabla\Delta u|_{p(x)}. \tag{21}$$

Therefore

$$\begin{aligned} \|u\|_W &\le \|u\|_{H_0^1(\Omega)} + C|\Delta u|_{p(x)} + \|\Delta u\|_{W_0^{1,p(x)}(\Omega)}\\ &\le \|u\|_{H_0^1(\Omega)} + C|\nabla\Delta u|_{p(x)} + \|\Delta u\|_{W_0^{1,p(x)}(\Omega)}\\ &\le \|u\|_{H_0^1(\Omega)} + C|\nabla\Delta u|_{p(x)} + C|\nabla\Delta u|_{p(x)}\\ &\le C\left[\|u\|_{H_0^1(\Omega)} + |\nabla\Delta u|_{p(x)}\right]. \end{aligned} \tag{22}$$

Hence (22) assures that $J(u) \to +\infty$, if $\|u\|_W \to +\infty$. On the other hand, $J(u)$ is clearly weakly lower semi continuous on W. So it follows from the critical point theory, see [14], there exists $u_1 \in W$ such that

$$J(u_1) = \inf_{w \in W} J(w).$$

Therefore, from the above equation, we conclude that the function u_1 is a weak solution of the corresponding Euler-Lagrange equation of $J(u)$. □

Theorem 2. *Under assumptions of* $u_0 \in H_0^1(\Omega)$, $f \in L^{p'}\big(0,T;L^{(p^*(x))'}\big)$, $g \in \big(L^{p'(x)}(Q)\big)^N$ *and* $p(x) \in C_+(\overline{\Omega})$, $p(x)$ *satisfies the log-Hölder continuity condition, the initial-boundary value problem* (1) *admits a unique weak solution.*

Proof. Construct a suitable approximation solution sequence $\{u_h\}$ for the parabolic problem (1). For $k = 1$, from the above theorem, there exists a weak solution $u_1 \in W$. Continuing the same procedures, one find weak solutions $u_k \in W$ of (12), $k = 2, \dots, n$. It follows that, for every $\eta \in W$,

$$\begin{aligned} -\frac{1}{h}\int_\Omega \nabla(u_k - u_{k-1}) \cdot \nabla\eta \,\mathrm{d}x - \int_\Omega (|\nabla\Delta u_k|^{p(x)-2}\nabla\Delta u_k) \cdot \nabla\Delta\eta \,\mathrm{d}x \\ = \int_\Omega [f]_h((k-1)h)\Delta\eta \,\mathrm{d}x + \int_\Omega [g]_h((k-1)h) \cdot \nabla\Delta\eta \,\mathrm{d}x. \end{aligned} \tag{23}$$

Take $\eta = u_k$ as a test function in (23) and using Young's inequality, we get

$$\begin{aligned} \frac{1}{2}\|\nabla u_k\|^2_{L^2(\Omega)} + h\int_\Omega |\nabla\Delta u_k|^{p(x)} \,\mathrm{d}x \le \frac{1}{2}\|\nabla u_{k-1}\|^2_{L^2(\Omega)} \\ + Ch\|[f]_h((k-1)h)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)} + Ch\|[g]_h((k-1)h)\|^{p'(x)}_{L^{p'(x)}(\Omega)}. \end{aligned} \tag{24}$$

From the above, we get

$$\begin{aligned} \|\nabla u_k\|^2_{L^2(\Omega)} \le \|\nabla u_0\|^2_{L^2(\Omega)} + Ch\sum_{i=0}^{k-1} \|[f]_h(ih)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)} \\ + Ch\sum_{i=0}^{k-1} \|[g]_h(ih)\|^{p'(x)}_{L^{p'(x)}(\Omega)}, \end{aligned} \tag{25}$$

where

$$\begin{aligned} h\sum_{i=0}^{k-1} \|[f]_h(ih)\|^{p'}_{L^{(p^*(x))'}(\Omega)} &\le \sum_{i=0}^{k-1}\int_{ih}^{ih+h} \|f(\tau)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)} \,\mathrm{d}\tau \\ &= \int_0^{kh} \|f(\tau)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)} \,\mathrm{d}\tau, \\ h\sum_{i=0}^{k-1} \|[g]_h(ih)\|^{p'(x)}_{L^{p'(x)}(\Omega)} &\le \sum_{i=0}^{k-1}\int_{ih}^{ih+h} \|g(\tau)\|^{p'(x)}_{L^{p'(x)}(\Omega)} \,\mathrm{d}\tau \\ &= \int_0^{kh} \|g(\tau)\|^{p'(x)}_{L^{p'(x)}(\Omega)} \,\mathrm{d}\tau. \end{aligned}$$

For every $h = \frac{T}{n}$, we define

$$u_h(x,t) = \begin{cases} u_0(x), & t = 0, \\ u_1(x), & 0 < t \le h, \\ \cdots, & \cdots, \\ u_j(x), & (j-1)h < t \le jh, \\ \cdots, & \cdots, \\ u_n(x), & (n-1)h < t \le nh = T. \end{cases} \tag{26}$$

For each $t \in (0,T]$, there exists some $k \in \{1,2,\dots,n\}$ such that $t \in ((k-1)h, kh]$. Now by (25), we have

$$\begin{aligned} \|\nabla u_h(t)\|^2_{L^2(\Omega)} &\le \|\nabla u_0\|^2_{L^2(\Omega)} + C\int_0^{kh} \|f(\tau)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)}\,\mathrm{d}\tau \\ &\quad + C\int_0^{kh} \|g(\tau)\|^{p'}_{L^{p'(x)}(\Omega)}\,\mathrm{d}\tau \\ &\le \|\nabla u_0\|^2_{L^2(\Omega)} + C\int_0^{T} \|f(\tau)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)}\,\mathrm{d}\tau \\ &\quad + C\int_0^{T} \|g(\tau)\|^{p'(x)}_{L^{p'(x)}(\Omega)}\,\mathrm{d}\tau. \end{aligned}$$

Hence the above inequality shows that

$$\|\nabla u_h\|^2_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{27}$$

where $C > 0$ is a constant. Summing up the inequalities in (24), we obtain

$$\begin{aligned} \sum_{k=1}^n \frac{1}{2}\|\nabla u_k\|^2_{L^2(\Omega)} &+ h\sum_{k=1}^n \int_\Omega |\nabla\Delta u_k|^{p(x)}\,\mathrm{d}x \\ &\le \frac{1}{2}\sum_{k=1}^n \|\nabla u_{k-1}\|^2_{L^2(\Omega)} + Ch\sum_{k=1}^n \|[f]_h((k-1)h)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)} \\ &\quad + Ch\sum_{k=1}^n \|[g]_h((k-1)h)\|^{p'(x)}_{L^{p'(x)}(\Omega)} \\ &\le \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega)} + \cdots + \frac{1}{2}\|\nabla u_{n-1}\|^2_{L^2(\Omega)} \\ &\quad + C\int_0^T \|f(\tau)\|^{p'(x)}_{L^{(p^*(x))'}(\Omega)}\,\mathrm{d}\tau \\ &\quad + C\int_0^T \|g(\tau)\|^{p'(x)}_{L^{p'(x)}(\Omega)}\,\mathrm{d}\tau. \end{aligned} \tag{28}$$

From the above inequality, we get

$$\begin{aligned}\int_0^T \int_\Omega |\nabla\Delta u_h|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t &= h\sum_{k=1}^n \int_\Omega |\nabla\Delta u_k|^{p(x)}\,\mathrm{d}x \\ &\le \|\nabla u_0\|^2_{L^2(\Omega)} + C\int_0^T \|f(\tau)\|^{p'}_{L^{(p^*(x))'}(\Omega)}\,\mathrm{d}\tau \\ &\quad + C\int_0^T \|g(\tau)\|^{p'}_{L^{p'(x)}(\Omega)}\,\mathrm{d}\tau \\ &\le C,\end{aligned}$$

where $C > 0$ is a constant. By Lemma 2 we have

$$\int_0^T \min\left\{|\nabla\Delta u_h|^{p+}_{p(x)}, |\nabla\Delta u_h|^{p-}_{p(x)}\right\}\,\mathrm{d}t \le \int_0^T \int_\Omega |\nabla\Delta u_h|^{p(x)}\,\mathrm{d}x\,\mathrm{d}t \le C.$$

Thus we conclude that

$$\nabla\Delta u_h \in (L^{p(x)}(Q))^N \quad \text{and} \quad \|\nabla\Delta u_h\|_{L^{p^-}(0,T;L^{p(x)}(\Omega))} \le C. \tag{29}$$

Employing the same technique as in the proof of (22), we obtain

$$\|u_h\|_{L^\infty(0,T;H_0^1(\Omega))} + \|u_h\|_{L^{p^-}(0,T;W^{2,p(x)}(\Omega))} + \|\Delta u_h\|_{L^p(0,T;W_0^{1,p(x)}(\Omega))} \le C. \tag{30}$$

Therefore, from (29) and (30), there exists a subsequence u_h (which also denoted by u_h) such that

$$\begin{aligned} u_h &\rightharpoonup u, \text{ weakly}^* \text{ in } L^\infty(0,T;H_0^1(\Omega)), \\ u_h &\rightharpoonup u, \text{ weakly in } L^{p^-}(0,T;W^{2,p(x)}(\Omega)), \\ \nabla\Delta u_h &\rightharpoonup \nabla\Delta u, \text{ weakly in } (L^{p(x)}(Q))^N, \\ |\nabla\Delta u_h|^{p(x)-2}\nabla\Delta u_h &\rightharpoonup \zeta, \text{ weakly in } L^{(p')^-}(0,T;L^{p'(x)}(\Omega)),\end{aligned}$$

which follows (see [17]) that

$$\|u\|_{L^\infty(0,T;H_0^1(\Omega))} + \|u\|_{L^{p^-}(0,T;W^{2,p(x)}(\Omega))} + \|\Delta u\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} \le C.$$

Next we prove that the function u is a weak solution of problem (1). For each $\varphi \in C^1(\overline{Q})$ with $\varphi(\cdot,T) = 0$ and $\varphi(x,t)|_\Gamma = 0$ and for every $k \in \{1,2,\ldots,n\}$, we solve the equation $-\Delta\eta_k(x) = \varphi(x,kh)$ to find a function $\eta_k \in W$ and let it be a test function in (23) to have

$$\begin{aligned}\frac{1}{h}\int_\Omega (u_k - u_{k-1})\varphi(x,kh)\,\mathrm{d}x - \int_\Omega |\nabla\Delta u_k|^{p(x)-2}\nabla\Delta u_k\cdot\nabla\varphi(x,kh)\,\mathrm{d}x \\ = \int_\Omega \big[[f]_h(x,(k-1)h)\varphi(x,kh)\,\mathrm{d}x + [g]_h(x,(k-1)h)\cdot\nabla\varphi(x,kh)\big]\,\mathrm{d}x.\end{aligned}$$

Summing up all the equalities and using the definition of $u_h(x,t)$, we have

$$h\sum_{k=1}^{n-1}\int_\Omega u_h(x,kh)\frac{\varphi(x,kh)-\varphi(x,(k+1)h)}{h}\,\mathrm{d}x-\int_\Omega u_0(x)\varphi(x,h)\,\mathrm{d}x$$
$$-\,h\sum_{k=1}^{n}\int_\Omega(|\nabla\Delta u_h|^{p(x)-2}\nabla\Delta u_h)(x,kh)\cdot\nabla\varphi(x,kh)\,\mathrm{d}x$$
$$=h\sum_{k=1}^{n}\int_\Omega\big[[f]_h(x,(k-1)h)\varphi(x,kh)+[g]_h(x,(k-1)h)\cdot\nabla\varphi(x,kh)\big]\,\mathrm{d}x. \quad (31)$$

From the above convergence results and $\varphi\in C^1(\overline{Q})$, we have

$$h\sum_{k=1}^{n}\int_\Omega\big(|\nabla\Delta u_h|^{p(x)-2}\nabla\Delta u_h\big)(x,kh)\cdot\nabla\varphi(x,kh)\,\mathrm{d}x$$
$$=\int_0^T\int_\Omega\big(|\nabla\Delta u_h|^{p(x)-2}\nabla\Delta u_h\big)(x,\tau)\cdot\nabla\varphi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\sum_{k=1}^{n}\int_{(k-1)h}^{kh}\int_\Omega\big(|\nabla\Delta u_h|^{p(x)-2}\nabla\Delta u_h\big)(x,\tau)(\nabla\varphi(x,kh)-\nabla\varphi(x,\tau))\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\to\int_0^T\int_\Omega\zeta\cdot\nabla\varphi(x,\tau)\,\mathrm{d}x\mathrm{d}\tau,\ \text{ as } h\to 0.$$

Since

$$h\sum_{k=1}^{n}\int_\Omega[f]_h(x,(k-1)h)\varphi(x,kh)\,\mathrm{d}x=\sum_{k=1}^{n}\int_{(k-1)h}^{kh}\int_\Omega f(x,\tau)\varphi(x,kh)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\to\int_0^T\int_\Omega f\varphi\,\mathrm{d}x\,\mathrm{d}\tau$$

and

$$h\sum_{k=1}^{n}\int_\Omega[g]_h(x,(k-1)h)\cdot\nabla\varphi(x,kh)\,\mathrm{d}x=\sum_{k=1}^{n}\int_{(k-1)h}^{kh}\int_\Omega g(x,\tau)\cdot\nabla\varphi(x,kh)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\to\int_0^T\int_\Omega g\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\tau,$$

as $h\to0$, we obtain from (31), we get, as $h\to 0$,

$$-\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}\,\mathrm{d}\,x\mathrm{d}\tau-\int_\Omega u_0(x)\varphi(x,0)\,\mathrm{d}x-\int_0^T\int_\Omega\zeta\cdot\nabla\varphi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$=\int_0^T\int_\Omega(f\varphi+g\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}\tau \quad (32)$$

The above equation proves that $\frac{\partial u}{\partial t} \in L^{(p')^-}\big(0,T;W^{-1,p'(x)}(\Omega)\big)$. For some larger integer s such that $W^{-1,p'(x)}(\Omega) \subset H^{-s}(\Omega)$ we obtain

$$\frac{\partial u}{\partial t} \in L^{(p')^-}\big(0,T;H^{-s}(\Omega)\big),$$

and it follows [30] that

$$u \in C\big([0,T];H^{-s}(\Omega)\big).$$

For each $\epsilon > 0$ and all $t, t_0 \in [0,T]$, by (30), there exists a positive number $\delta > 0$ such that

$$\delta\|\nabla u(t) - \nabla u(t_0)\|_{L^2(\Omega)} \leq \frac{\epsilon}{2}.$$

From the compact imbedding relation $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$, we have, for all $t, t_0 \in [0,T]$,

$$\begin{aligned}\|u(t) - u(t_0)\|_{L^2(\Omega)} &\leq \delta\|u(t) - u(t_0)\|_{H_0^1(\Omega)} + C(\delta)\|u(t) - u(t_0)\|_{H^{-s}(\Omega)}\\ &\leq \delta\|\nabla u(t) - \nabla u(t_0)\|_{L^2(\Omega)} + C(\delta)\|u(t) - u(t_0)\|_{H^{-s}(\Omega)}\\ &\leq \frac{\epsilon}{2} + C(\delta)\|u(t) - u(t_0)\|_{H^{-s}(\Omega)},\end{aligned}$$

where the first inequality is guaranteed by Lemma 5.1 in Chapter 1 of [21]. Hence we proved that

$$u \in C\big([0,T];L^2(\Omega)\big).$$

Finally we show that $\zeta = |\nabla\Delta u|^{p(x)-2}\nabla\Delta u$, a.e in Q to prove the existence of weak solutions. Considering Δu as a test function in (32), we have

$$\begin{aligned}&\frac{\|\nabla u_0\|^2_{L^2(\Omega)} - \|\nabla u(T)\|^2_{L^2(\Omega)}}{2} - \int_0^T\int_\Omega \zeta\cdot\nabla\Delta u\,\mathrm{d}x\,\mathrm{d}\tau\\ &\qquad = \int_0^T\int_\Omega [f\Delta u + g\cdot\nabla\Delta u]\,\mathrm{d}x\,\mathrm{d}\tau. \quad (33)\end{aligned}$$

Denote $Au = |\nabla\Delta u|^{p(x)-2}\nabla\Delta u$ and, by the monotonicity assumption of the operator,

$$\big(|\zeta|^{p(x)-2}\zeta - |\eta|^{p(x)-2}\eta\big)(\zeta - \eta) \geq 0,$$

for all $\zeta, \eta \in \mathbb{R}^N$, we have

$$\int_\Omega \big(Au_k - Av(\tau)\big)\cdot\big(\nabla\Delta u_k - \nabla\Delta v(\tau)\big)\,\mathrm{d}x \geq 0, \quad (34)$$

for each $k = 1, 2, \ldots, n$ and every $v \in L^{p^-}\big(0,T;W^{2,p(x)}(\Omega)\big)$ with $\Delta v \in L^{p^-}\big(0,T;W_0^{1,p(x)}(\Omega)\big)$. Considering u_k as a test function in (23), we have

$$\begin{aligned}&-\frac{1}{h}\int_\Omega \nabla(u_k - u_{k-1})\cdot\nabla u_k\,\mathrm{d}x - \int_\Omega Au_k\cdot\nabla\Delta u_k\,\mathrm{d}x\\ &\qquad = \int_\Omega [f]_h((k-1)h)\Delta u_k\,\mathrm{d}x + \int_\Omega [g]_h((k-1)h)\cdot\nabla\Delta u_k\,\mathrm{d}x. \quad (35)\end{aligned}$$

From (34), we obtain

$$-\frac{1}{h}\int_\Omega \nabla(u_k-u_{k-1}).\nabla u_k\,\mathrm{d}x-\int_\Omega Au_k\cdot\nabla\Delta v(\tau)\,\mathrm{d}x-\int_\Omega Av\cdot(\nabla\Delta u_k-\nabla\Delta v(\tau)\,\mathrm{d}x$$
$$-\int_\Omega [f]_h((k-1)h)\Delta u_k\,\mathrm{d}x-\int_\Omega [g]_h((k-1)h)\cdot\nabla\Delta u_k\,\mathrm{d}x\geq 0. \quad (36)$$

Now, by Young's inequality, we obtain

$$-\int_\Omega \nabla(u_k-u_{k-1})\cdot\nabla u_k\,\mathrm{d}x\leq \frac{\|\nabla u_{k-1}\|^2_{L^2(\Omega)}-\|\nabla u_k\|^2_{L^2(\Omega)}}{2}. \quad (37)$$

Integrating (35) over $((k-1)h,kh)$ and using the above result, we get

$$\frac{\|\nabla u_{k-1}\|^2_{L^2(\Omega)}-\|\nabla u_k\|^2_{L^2(\Omega)}}{2}-\int_{(k-1)h}^{kh}\int_\Omega Au_k\cdot\nabla\Delta v\,\mathrm{d}x\,\mathrm{d}\tau$$
$$-\int_{(k-1)h}^{kh}\int_\Omega Av\cdot(\nabla\Delta u_k-\nabla\Delta v)\,\mathrm{d}x\,\mathrm{d}\tau\geq 0.$$

Summing up the above inequalities for $k=1,2,\dots,n$, we obtain

$$\frac{\|\nabla u_0\|^2_{L^2(\Omega)}-\|\nabla u_h(T)\|^2_{L^2(\Omega)}}{2}-\int_0^T\int_\Omega Au_h.\nabla\Delta v\,\mathrm{d}x\,\mathrm{d}\tau$$
$$-\int_0^T\int_\Omega Av\cdot(\nabla\Delta u_h-\nabla\Delta v)\mathrm{d}x\,\mathrm{d}\tau-\int_0^T\int_\Omega(f\Delta u_h-g\cdot\nabla\Delta u_h)\,\mathrm{d}x\,\mathrm{d}\tau\geq 0.$$

Passing to limits as $h\to 0$, we get

$$\frac{\|\nabla u_0\|^2_{L^2(\Omega)}-\|\nabla u(T)\|^2_{L^2(\Omega)}}{2}-\int_0^T\int_\Omega \zeta\cdot\nabla\Delta v\,\mathrm{d}x\,\mathrm{d}\tau$$
$$-\int_0^T\int_\Omega Av\cdot(\nabla\Delta u-\nabla\Delta v)\,\mathrm{d}x\,\mathrm{d}\tau-\int_0^T\int_\Omega(f\Delta u-g\cdot\nabla\Delta u)\,\mathrm{d}x\,\mathrm{d}\tau\geq 0. \quad (38)$$

Combining (38) with (33), we have

$$\int_0^T\int_\Omega(\zeta-Av)\cdot(\nabla\Delta u-\nabla\Delta v)\,\mathrm{d}x\,\mathrm{d}\tau\geq 0 \quad (39)$$

We choose $v=u-\lambda w$ for any $\lambda>0,\nabla\Delta w\in(L^{p(x)}(Q))^N$ in the above inequality to have

$$\int_0^T\int_\Omega(\zeta-A(u-\lambda w))\cdot\nabla\Delta w\,\mathrm{d}x\,\mathrm{d}\tau\geq 0.$$

Passing to limits as $\lambda\to 0^+$ and, using Lebesgue's dominated convergence theorem, we obtain

$$\int_0^T\int_\Omega(\zeta-Au)\cdot\psi\,\mathrm{d}x\,\mathrm{d}\tau\geq 0,\quad \forall\psi\in(L^{p(x)}(Q))^N.$$

Hence we conclude that $\zeta=Au$, a.e. in Q. □

4 Uniqueness of Weak Solutions

Theorem 3. *The solutions of the given degenerate parabolic fourth-order equation* (1) *are unique.*

Proof. Suppose there exist two weak solutions u and v of problem (1). Using Remark 1, we have

$$\int_\Omega (u-v)\varphi \,\mathrm{d}x\Big|_0^t - \int_0^t \int_\Omega \big[(u-v)\varphi_t + \big(|\nabla\Delta u|^{p(x)-2}\nabla\Delta u - |\nabla\Delta v|^{p(x)-2}\nabla\Delta v\big)\cdot\nabla\varphi\big]\,\mathrm{d}x\,\mathrm{d}s = 0.$$

Choosing $\Delta(u-v)$ as a test function in the above equality and using Remark 2, we have, for every $t \in (0,T)$,

$$\int_\Omega \frac{|\nabla u - \nabla v|^2(t)}{2}\,\mathrm{d}x + \int_0^t \int_\Omega \big[|\nabla\Delta u|^{p(x)-2}\nabla\Delta u - |\nabla\Delta v|^{p(x)-2}\nabla\Delta v\big]\cdot(\nabla\Delta u - \nabla\Delta v)\,\mathrm{d}x\,\mathrm{d}s = 0. \tag{40}$$

Since the two terms on the left-hand side are nonnegative, we have $\nabla u = \nabla v$ a.e. in Q, Since $u - v = 0$ on Γ, we conclude $u - v = 0$ a.e. in Q, which implies $u = v$ a.e. in Q. Thus we obtain the uniqueness of weak solutions. □

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