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Oscillation and Periodicity of a Second Order Impulsive Delay Differential Equation with a Piecewise Constant Argument

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Abstract. This paper concerns with the existence of the solutions of a second order impulsive delay differential equation with a piecewise constant argument. Moreover, oscillation, nonoscillation and periodicity of the solutions are investigated.

1 Introduction

The differential equations with piecewise constant arguments has been studied widely in the literature. To understand the fundamental structure of these equations we refer the books [1] and [17]. Besides the theoretical investigations, these equations are also used to model some biological incidents [6], [8], the stabilization of hybrid control systems with feedback discrete controller [10], and damped oscillators [19]. Studies on such equations are motivated by the fact that they represent a hybrid of discrete and continuous dynamical systems and combine the properties of both differential and difference equations. However, solutions and qualitative properties of second order differential equations with piecewise constant arguments have received considerable attention by several papers such as [5], [7], [12], [13], [15], [16], [20] and references in these papers.

On the other hand, it is well known that impulsive effect plays an important role in the investigation of many biological, physical and economical models such as threshold phenomena, bursting rhythm models and optimal control problems. The studies on impulsive differential equations with piecewise constant arguments are quite new:

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In [11], Li and Shen considered the periodic value problem

$$y'(t) = f\big(t, y([t-k])\big), \quad t \neq n, \ t \in J,$$
$$\triangle y(n^+) = I_n(y(n)), \quad n = 1, 2, ..., p, \ y(0) = y(T),$$

and using the method of upper and lower solutions, they showed the existence of at least one solution of the given boundary value problem.

In [19], Wiener and Lakshmikantham proved existence and uniqueness of the initial value problem

$$x'(t) = f\big(x(t), x(g(t))\big), \quad x(0) = x_0,$$

and they gave some oscillation and stability results for the same problem, where f is a continuous function and $g\colon [0,\infty) \to [0,\infty)$, $g(t) \leq t$, is a step function.

There are also some papers of the authors: In [9], the oscillatory and periodic solutions of the first order linear scalar impulsive delay differential equation and in [4], existence and uniqueness and also oscillatory and periodic solutions of a class of first order nonhomogeneous advanced impulsive differential equations with piecewise constant arguments were studied. The asymptotic convergence of first order delay and advanced impulsive differential equations with piecewise constant arguments were also considered in [2], [3] and [14].

As far as we know there is not any paper which concerns a second order impulsive differential equations with piecewise constant arguments. With this paper we aim to extend our experiences in first order impulsive differential equations with piecewise constant arguments to the second order impulsive differential equations with piecewise constant arguments.

In 1999, Wiener and Lakshmikantham [18] studied the existence and some qualitative properties of the solutions of the second order differential equation of the type

$$x''(t) - a^2 x(t) = bx([t-1])\,. \tag{1}$$

Since the solutions of the equation $x'' - a^2 x = 0$ are nonoscillatory, they show that Eq. (1) has both oscillatory and nonoscillatory solutions. In this paper, we consider the Eq. (1) with an impulse condition.

We study the existence and some qualitative properties of the following second order impulsive differential equation with a piecewise constant argument:

$$x''(t) - a^2 x(t) = bx([t-1]), \quad t \neq n \in \mathbb{Z}^+ = \{1, 2, \ldots\}, \quad t \geq 0 \tag{2}$$
$$\Delta x'(n) = dx'(n), \quad n \in \mathbb{Z}^+ \tag{3}$$

where $a, b, d \in \mathbb{R} \setminus \{0\}$, $\Delta x'(n) = x'(n^+) - x'(n^-)$,

$$x'(n^+) = \lim_{t \to n^+} x'(t), \qquad x'(n^-) = \lim_{t \to n^-} x'(t),$$

$x(t) \in \mathbb{C}$ and $[\cdot]$ denotes the greatest integer function.

Now, we give the definition of a solution of (2)–(3).

Definition 1. A function $x(t)$ defined on $\{-1\} \cup [0, \infty)$ is said to be a solution of (2)–(3) if it satisfies the following conditions:

1. $x\colon \{-1\} \cup [0,\infty) \to \mathbb{R}$ is continuous on $[0,\infty)$,

2. $x'(t)$ exists and continuous at the points $t \notin \mathbb{Z}^+$,

3. $x'(t)$ is right continuous and has left-hand limits at the points $t \in \mathbb{Z}^+$,

4. $x''(t)$ exists at each point $t \in [0,\infty)$ with the possible exception of the points $t \in \mathbb{Z}^+$ where one-sided derivatives exist,

5. $x(t)$ satisfies (2) for any $t \in (0,\infty)$ with the possible exception of the points $t \in \mathbb{Z}^+$,

6. $x'(t)$ satisfies (3) for every $t = n \in \mathbb{Z}^+$.

2 Existence and Uniqueness of Solutions

Theorem 1. *Let $d \neq 1$ and λ be a root of the characteristic equation*

$$\lambda^3 - \Big(\frac{2-d}{1-d}\cosh a\Big)\lambda^2 + \Big(\frac{a^2 - b(\cosh a - 1)(1-d)}{a^2(1-d)}\Big)\lambda - \frac{b(\cosh a - 1)}{a^2(1-d)} = 0\,. \quad (4)$$

Then the following function $x_\lambda(t)$ is a solution of (2)–(3) on $[0,\infty)$:

$$x_\lambda(t) = \frac{\lambda^{[t-1]}b}{a^2(e^{-a}-e^{a})}\Big(e^{a(\{t\}-1)} - e^{-a(\{t\}-1)} + e^{-a\{t\}} - e^{a\{t\}} - e^{-a} + e^{a}\Big)$$
$$+ \frac{\lambda^{[t]}}{e^{-a}-e^{a}}\Big(e^{a(\{t\}-1)} - e^{-a(\{t\}-1)}\Big) + \frac{\lambda^{[t+1]}}{e^{-a}-e^{a}}\Big(e^{-a\{t\}} - e^{a\{t\}}\Big) \quad (5)$$

where $\{t\}$ is the fractional part of t.

Moreover,

$$x(t) = c_1 x_{\lambda_1}(t) + c_2 x_{\lambda_2}(t) + c_3 x_{\lambda_3}(t) \quad (6)$$

is a general solution of (2)–(3) on $[0,\infty)$ where c_i, $i = 1,2,3$, are arbitrary constant and λ_i, $i = 1,2,3$, are different roots of (4).

Proof. Assume that $v\colon [0,1) \to \mathbb{R}$ is a continuous function with

$$v(0) = 1, \quad v(1) = \lambda\,. \quad (7)$$

Denote $x(t) = x_n(t) = \lambda^n v(s)$ is a solution of (2)–(3) on $[n, n+1)$ where $s := t-n$, $n \in \mathbb{Z}^+$, $0 \le s < 1$ and λ is a constant.

On the other hand, for $t \in [n, n+1)$ we have

$$x'(t) = \lambda^n \frac{\mathrm{d}v}{\mathrm{d}s}, \quad x''(t) = \lambda^n \frac{\mathrm{d}^2 v}{\mathrm{d}s^2} \text{ and } x([t-1]) = \lambda^{n-1}\,.$$

So, we can rewrite the Eq. (2) as

$$v'' - a^2 v = b\lambda^{-1}. \quad (8)$$

By solving the differential equation (8) with (7), we find

$$v(s) = \Big(\frac{a^2(e^{-a}-\lambda)+b\lambda^{-1}(e^{-a}-1)}{a^2(e^{-a}-e^{a})}\Big)e^{as} + \Big(\frac{a^2(\lambda-e^{a})-b\lambda^{-1}(e^{a}-1)}{a^2(e^{-a}-e^{a})}\Big)e^{-as} - \frac{b}{a^2}\lambda^{-1}. \quad (9)$$

Since $x_n(t) = \lambda^n v(s)$ and $s = t-n$, we get

$$x_n(t) = \frac{\lambda^{n-1}b}{a^2(e^{-a}-e^{a})}\Big(e^{a(t-n-1)} - e^{-a(t-n-1)} + e^{-a(t-n)} - e^{a(t-n)} - e^{-a} + e^{a}\Big) + \frac{\lambda^n}{e^{-a}-e^{a}}\Big(e^{a(t-n-1)} - e^{-a(t-n-1)}\Big) + \frac{\lambda^{n+1}}{e^{-a}-e^{a}}\big(e^{-a(t-n)} - e^{a(t-n)}\big), \quad (10)$$

where $n \le t < n+1$. If we take the derivative of (10), we obtain

$$x_n'(t) = \frac{\lambda^{n-1}b}{a(e^{-a}-e^{a})}\Big(e^{a(t-n-1)} - e^{-a(t-n-1)} - e^{-a(t-n)} - e^{a(t-n)}\Big) + \frac{\lambda^n a}{e^{-a}-e^{a}}\Big(e^{a(t-n-1)} + e^{-a(t-n-1)}\Big) - \frac{\lambda^{n+1}a}{e^{-a}-e^{a}}\Big(e^{-a(t-n)} + e^{a(t-n)}\Big), \quad (11)$$

where $n \le t < n+1$. Using the impulse condition (3) at $t = n+1$, we have

$$x_n'(n+1) = (1-d)x_{n+1}'(n+1)$$

and this equality yields us to the characteristic equation (4). On the other hand, if λ_1, λ_2 and λ_3 are different roots of Eq. (4), then it is trivial that $x_{\lambda_1}(t)$, $x_{\lambda_2}(t)$ and $x_{\lambda_3}(t)$ are independent solutions of (2)–(3). Thus, a general solution of (2)–(3) is in the form of (6). □

Theorem 2. *If λ_1, λ_2 and λ_3 are different roots of Eq. (4) and*

$$\lambda_2\lambda_3(\lambda_2-\lambda_3) + \lambda_1\lambda_2(\lambda_1-\lambda_2) \neq \lambda_1\lambda_3(\lambda_1-\lambda_3), \quad (12)$$

then (2)–(3) with the initial conditions

$$x(-1) = x_{-1}, \quad x(0) = x_0, \quad x'(0) = y_0 \quad (13)$$

has a unique solution on $[0,\infty)$ where x_{-1}, x_0 and y_0 are given numbers.

Proof. Substituting the initial conditions (13) into (6) we have the system

$$\begin{cases} x_{-1} = c_1x_{\lambda_1}(-1) + c_2x_{\lambda_2}(-1) + c_3x_{\lambda_3}(-1) \\ x_0 = c_1x_{\lambda_1}(0) + c_2x_{\lambda_2}(0) + c_3x_{\lambda_3}(0) \\ y_0 = c_1x'_{\lambda_1}(0) + c_2x'_{\lambda_2}(0) + c_3x'_{\lambda_3}(0) \end{cases} \quad (14)$$

It can be easily seen that if the determinant of coefficients of the system (14)

$$D = \Big(\frac{a}{\cosh a}\Big)\Big(\frac{\lambda_2\lambda_3(\lambda_2-\lambda_3) + \lambda_1\lambda_2(\lambda_1-\lambda_2) - \lambda_1\lambda_3(\lambda_1-\lambda_3)}{\lambda_1\lambda_2\lambda_3}\Big)$$

is different from zero, then the system (14) has a unique solution $c_1,\ c_2,\ c_3$. By the virtue of the hypothesis (12) and $a \neq 0$, we get $D \neq 0$. This proves the theorem. □

The above theorems are given in the case of the roots of Eq. (4) are different from each other. In the case of two or all of them are equal, we stated the existence and uniqueness theorems as follows:

Theorem 3. *(a) If $\lambda_1 = \lambda_2 = \lambda$ and λ_3 is different, then a general solution of (2)–(3) is*

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t) \tag{15}$$

where c_i, $i = 1, 2, 3$ are arbitrary constant and

$$\begin{aligned} x_1(t) &= A_1([t])v_1(\{t\}) + A_2([t])v_2(\{t\}) + A_3([t])v_3(\{t\}) \\ x_2(t) &= [t-1]A_1([t])v_1(\{t\}) + [t]A_2([t])v_2(\{t\}) + [t+1]A_3([t])v_3(\{t\}) \\ x_3(t) &= x_{\lambda_3}(t) \end{aligned}$$

where $\{t\}$ is the fractional part of t;

$$A_1([t]) = \frac{\lambda^{[t-1]}b}{a^2(e^{-a} - e^a)}, \quad A_2([t]) = \frac{\lambda^{[t]}}{e^{-a} - e^a}, \quad A_3([t]) = \frac{\lambda^{[t+1]}}{e^{-a} - e^a}; \tag{16}$$

$$\begin{aligned} v_1(\{t\}) &= e^{a(\{t\}-1)} - e^{-a(\{t\}-1)} + e^{-a\{t\}} - e^{a\{t\}} - e^{-a} + e^a, \\ v_2(\{t\}) &= e^{a(\{t\}-1)} - e^{-a(\{t\}-1)} \\ v_3(\{t\}) &= e^{-a\{t\}} - e^{a\{t\}}; \end{aligned} \tag{17}$$

and $x_{\lambda_3}(t)$ is the same as (5) provided that replacing λ with λ_3.

(b) If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then a general solution of (2)–(3) is again given by (15). But in this case

$$\begin{aligned} x_1(t) &= A_1([t])v_1(\{t\}) + A_2([t])v_2(\{t\}) + A_3([t])v_3(\{t\}) \\ x_2(t) &= [t-1]A_1([t])v_1(\{t\}) + [t]A_2([t])v_2(\{t\}) + [t+1]A_3([t])v_3(\{t\}) \\ x_3(t) &= [t-1]^2A_1([t])v_1(\{t\}) + [t]^2A_2([t])v_2(\{t\}) + [t+1]^2A_3([t])v_3(\{t\}) \end{aligned}$$

where $\{t\}$ is the fractional part of t, A_i and v_i, $i = 1, 2, 3$ are given by (16) and (17), respectively.

Theorem 4. *(a) Assume that $\lambda_1 = \lambda_2 = \lambda$ and λ_3 is different. If*

$$\lambda^2 - 2\lambda\lambda_3 + \lambda_3^2 \neq 0,$$

then (2)–(3) with the initial conditions (13) has a unique solution on $[0, \infty)$.

(b) If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then (2)–(3) with the initial conditions (13) has a unique solution on $[0, \infty)$.

The proofs of Theorem 3 and 4 are similar to the proofs of Theorem 1 and 2 , respectively.

Corollary 1. *If $b = -a^2$, there exists constant solutions of (2)–(3). Since constant solutions has already satisfied the impulse condition (3), we do not need extra condition on impulse condition.*

Corollary 2. *If $d = 0$, then the impulse condition (3) is removed and Eq. (2)–(3) reduces to the Eq. (1.1) in [18].*

Corollary 3. *If $a = 0$, then*

$$x_\lambda(t) = \frac{b}{2}\{t\}(\{t\}-1)\lambda^{[t-1]} - (\{t\}-1)\lambda^{[t]} + \{t\}\lambda^{[t+1]} \tag{18}$$

is a solution of (2)–(3) on the interval $[0,\infty)$ where $\{t\}$ is the fractional part of t.

Remark 1. The solution (18) is also a limiting case of (5) as $t \to \infty$.

Remark 2. If the roots of the Eq. (4) are real, then the solutions of the Eq. (2)–(3) are real.

3 Oscillation and Periodicity of Solutions

In this section, we study the qualitative aspect of the given Eq. (2)–(3) such as oscillation and periodicity. The following theorem is obtained from (6) easily:

Theorem 5. *All solutions of (2)–(3) approaches to zero as $t \to +\infty$ if and only if the roots of Eq. (4) λ_i, $i = 1, 2, 3$ satisfy the inequalities*

$$|\lambda_i| < 1, \quad i = 1, 2, 3.$$

Theorem 6. *If $b < 0$ and $d < 1$, then (2)–(3) has oscillatory solutions.*

Proof. Considering (5) for $t \in [n, n+1)$, we have

$$x(n)x(n+1) = \lambda^{2n+1} \tag{19}$$

where λ is a root of (4).

On the other hand, using Descartes' rule of signs method one can easily see that the function

$$f(-\lambda) = -\lambda^3 - \Big(\frac{2-d}{1-d}\cosh a\Big)\lambda^2 - \Bigg(\frac{a^2 - b(\cosh a - 1)(1-d)}{a^2(1-d)}\Bigg)\lambda - \frac{b(\cosh a - 1)}{a^2(1-d)}$$

has a single negative root which implies that the characteristic equation (4) has a negative root. So, by (19)

$$x(n)x(n+1) < 0$$

hence (2)–(3) has oscillatory solutions. □

Remark 3. If $d = 0$, then Theorem 6 reduces to Theorem 3.1 in [18].

Theorem 7. *Assume that* $d < 1$ *and* $0 < b < \dfrac{a^2}{(\cosh a - 1)(1-d)}$, *then there exists nonoscillatory solutions of* (2)–(3).

Proof. Let us rewrite Eq. (4) as

$$\lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma \tag{20}$$

where

$$\alpha = \frac{d-2}{1-d}\cosh a, \quad \beta = \frac{a^2 - b(\cosh a - 1)(1-d)}{a^2(1-d)}, \quad \gamma = -\frac{b(\cosh a - 1)}{a^2(1-d)}. \tag{21}$$

By using the hypotheses of theorem, we obtain that $\alpha < 0$, $\beta > 0$ and $\gamma < 0$. Therefore, from Descartes' rule of signs method, there exists at least one positive root of Eq. (4). So, (2)–(3) has nonoscillatory solutions. □

Theorem 8. *If* $d < 1$ *and* $b > \frac{a^2}{(\cosh a-1)(1-d)}$, *then* (2)–(3) *has both oscillatory and nonoscillatory solutions.*

Proof. Hypotheses of the theorem give us that $\alpha < 0$, $\beta < 0$ and $\gamma < 0$ where α, β, γ are the same notations in (21). By using Descartes' rule of signs method, we conclude that there exists a unique positive root of Eq. (4). So other roots must be negative or complex. Therefore, Eq. (4) has both oscillatory and nonoscillatory solutions. □

Remark 4. If $d = 0$, then hypotheses of Theorem 8 reduces to $b > \dfrac{a^2}{\cosh a - 1}$ which is the same in Eq. (3.4) in [18].

Theorem 9. *If* $d < 1$, $b < 0$ *and*

$$\alpha^2 - 4\beta < 0,$$

then each solution of (2)–(3) *is oscillatory where* α *and* β *are defined as in* (21).

Proof. The characteristic equation (4) can be written as

$$g(\lambda) = h(\lambda)$$

where $g(\lambda) = \lambda^2 + \alpha\lambda + \beta$ is a parabola and $h(\lambda) = \dfrac{\gamma}{\lambda}$ is a hyperbola. On the other hand, the minimum point of the parabola $g(\lambda)$ is $\lambda_{\min} = -\frac{\alpha}{2}$ and by using the hypotheses of theorem, we get

$$h(\lambda_{\min}) < g(\lambda_{\min}).$$

Thus, the parabola $g(\lambda)$ intersects the hyperbola $h(\lambda)$ at a single point with a negative absissa. This means that Eq. (4) has no positive roots which implies that each solution of (2)–(3) is oscillatory. □

Theorem 10. *Every solution of* (2)–(3) *is* k*-periodic if and only if*

$$\lambda^k = 1 \tag{22}$$

where λ *is the root of characteristic equation* (4) *and* k *is a positive integer.*

Proof. Assume that (2)–(3) has k-periodic solution. So,

$$x_\lambda(t+k) = x_\lambda(t) \tag{23}$$

where $x_\lambda(t)$ is given by (5). It is obvious that

$$[t+k-1] = [t-1]+k, \quad [t+k] = [t]+k, \quad \{t+k\} = \{t\}, \quad \{t+k\} = \{t\}-1$$

here $\{t\}$ denotes the fractional part of t. By using (5) and (23) we obtain

$$x_\lambda(t)(\lambda^k - 1) = 0$$

and this implies (22).

Conversely, let (22) is true. Then, from (5) we have

$$x_\lambda(t+k) - x_\lambda(t) = 0.$$

This means that (2)–(3) has k-periodic solution. □

Corollary 4. *If* $d = 2$ *and*

$$b = \frac{a^2(1+\cosh a(2-d))}{(\cosh a - 1)(1-d)}, \tag{24}$$

then (2)–(3) *has* 3*-periodic solutions.*

Proof. Substituting (24) in Eq. (4), we have

$$(\lambda^2+\lambda+1)\left(\lambda - \frac{1+\cosh a(2-d)}{(1-d)^2}\right) = 0. \tag{25}$$

The roots of (25) are

$$\lambda_1 = \frac{-1+i\sqrt{3}}{2}, \quad \lambda_2 = \frac{-1-i\sqrt{3}}{2} \quad \text{and} \quad \lambda_3 = \frac{1+\cosh a(2-d)}{(1-d)^2}.$$

Since

$$\lambda_1^3 = \lambda_2^3 = 1,$$

according to Theorem 10, it is said that Eq. (2)–(3) has 3-periodic solutions. □

Remark 5. If $d = 0$, then the condition (24) reduces to

$$\frac{b}{a^2} = \frac{2\cosh a + 1}{\cosh a - 1}$$

which is the same as in Theorem 3.7 in [18].

Example 1. Let us consider the following equation

$$x''(t) - x(t) = \frac{1}{1-\cosh 1} x([t-1]), \quad t \neq n \in \mathbb{Z}^+ = \{1, 2, \ldots\}, \quad t \geq 0 \tag{26}$$

$$\Delta x'(n) = 2x'(n), \quad n \in \mathbb{Z}^+. \tag{27}$$

This is a special case of Eq. (2)–(3) with $a = 1$, $b = \frac{1}{(1-\cosh 1)}$ and $d = 2$.

In this example, hypotheses of Corollary 4 are satisfied, so it is said that this equation has 3-periodic solutions. Indeed, the characteristic roots of the equation (4) are

$$\lambda_1 = \frac{-1+i\sqrt{3}}{2}, \quad \lambda_2 = \frac{-1-i\sqrt{3}}{2} \quad \text{and} \quad \lambda_3 = 1,$$

and it is easily seen $\lambda_1^3 = \lambda_2^3 = \lambda_3^3 = 1$. We can also confirm these periodic solutions by using the formula (5) for the solutions of the Eq. (2)–(3). The following figure shows 3-periodic solutions of Eq. (26)–(27) for $\lambda = \frac{-1+i\sqrt{3}}{2}$.

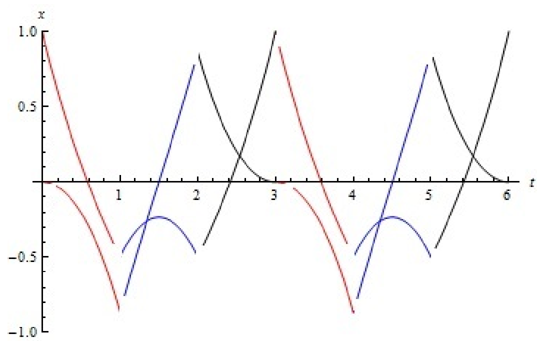

Figure 1: Real and imaginary parts of the 3-Periodic solutions of Eq. (26)–(27) for $\lambda = \frac{-1+i\sqrt{3}}{2}$.

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