DE GRUYTER
OPEN

Communications in Mathematics 25 (2017) 149–157

Estimating the critical determinants of a class of three-dimensional star bodies

Werner Georg Nowak

Abstract. In the problem of (simultaneous) Diophantine approximation in $\mathbb{R}^3$ (in the spirit of Hurwitz's theorem), lower bounds for the critical determinant of the special three-dimensional body

$$K_2: \quad (y^2+z^2)(x^2+y^2+z^2) \le 1$$

play an important role; see [1], [6]. This article deals with estimates from below for the critical determinant $\Delta(K_c)$ of more general star bodies

$$K_c: \quad (y^2+z^2)^{c/2}(x^2+y^2+z^2) \le 1\,,$$

where c is any positive constant. These are obtained by inscribing into K_c either a double cone, or an ellipsoid, or a double paraboloid, depending on the size of c.

1 Introduction

During the last couple of decades, not much research has been done in the subfield of the *Geometry of Numbers* (see, e.g., the monograph by Gruber & Lekkerkerker [4]) which is concerned with the evaluation, or at least estimation, of *critical determinants* $\Delta(K)$ of starbodies K in $\mathbb{R}^s$, $s \ge 2$. These are defined as $\Delta(K) = \inf|\det A|$, where A ranges over all nonsingular real $(s\times s)$-matrices, such that the origin is the only point of the lattice $A\mathbb{Z}^s$ in the interior of K.

It is the author's aim to rouse new interest in this classic topic *a fortiori* in view of its close connection to *simultaneous Diophantine approximation* in the spirit of Hurwitz's theorem: This is discussed at length in the author's survey article [9], as well as in the author's papers [6], [7], [8], [10].

2010 MSC: 11J13, 11H16
Key words: Geometry of numbers; critical determinant; simultaneous Diophantine approximation
DOI: 10.1515/cm-2017-0012

In brief, for each positive integer $s \ge 2$, and $1 \le \nu \le \infty$, define $\theta_{s,\nu}$ as the supremum of all values C with the following property: *For every* $\alpha \in \mathbb{R}^s \setminus \mathbb{Q}^s$, *there exist infinitely many* $(\mathbf{p}, q) \in \mathbb{Z}^s \times \mathbb{Z}_+$ *with* $\gcd(\mathbf{p}, q) = 1$*, such that*

$$\left\| \alpha - \frac{1}{q}\mathbf{p} \right\|_\nu < \frac{1}{q(Cq)^{1/s}} . \tag{1}$$

Then it is known due to a famous result of Davenport [3] that

$$\theta_{s,\nu} = \Delta(K^{(s,\nu)}) , \tag{2}$$

where

$$K^{(s,\nu)} = \left\{ (x_0, \dots, x_s) \in \mathbb{R}^{s+1} : |x_0| \, \|(x_1, \dots, x_s)\|_\nu^s \le 1 \right\} .$$

However, the exact determination of $\theta_{s,\nu}$ has only been accomplished for $s = 1$ (Hurwitz's classic theorem: $\theta_{1,\nu} = \sqrt{5}$) and for $s = \nu = 2$: $\theta_{2,2} = \frac{1}{2}\sqrt{23}$ [2].

2 Objective of the present article

To fix notions, we concentrate on the most natural one of the unsolved cases concerning $\theta_{s,\nu}$, namely on our familiar three-dimensional space and the Euclidean norm. Armitage [1] proved that

$$\theta_{3,2} = \Delta(K^{(3,2)}) \ge (\Delta(K^*))^3 \Delta(K_2) ,$$

where

$$K^* : \quad x^2(x^2 + y^2)^3 \le 1$$

is a planar star body with $\Delta(K^*) \ge 1.159$, and

$$K_2 : \quad \left(y^2 + z^2\right)\left(x^2 + y^2 + z^2\right) \le 1 . \tag{3}$$

Armitage proceeded to estimate $\Delta(K_2)$ by inscribing an ellipsoid[1]

$$x^2 + 4y^2 + 4z^2 \le 2\sqrt{3} .$$

Thus he obtained

$$\theta_{3,2} = \Delta(K^{(3,2)}) \ge 1.774\ldots .$$

Around the turn of the millennium, the present author [6] replaced this ellipsoid by the double paraboloid

$$|x| \le (1 + \sqrt{2})\left(1 - y^2 - z^2\right) ,$$

and evaluated the critical determinant of the latter. This gave the overall improvement

$$\theta_{3,2} = \Delta(K^{(3,2)}) \ge 1.879\ldots .$$

[1] As it is common in the Geometry of Numbers, we will throughout use the terms *ellipsoid, paraboloid, cone, ...* for bodies, not for the boundary surfaces.

It is the aim of the present article to view the body K_2 as a member of a more general family of star-bodies[2]

$$K_c : \quad \left(y^2 + z^2\right)^{c/2} \left(x^2 + y^2 + z^2\right) \leq 1\,, \tag{4}$$

where c is an arbitrary fixed positive constant. Our objective is to deduce a lower bound for $\Delta(K_c)$, depending on c, for every $c > 0$.

We start with a brief survey of the bounds established, postponing a more detailed representation of the results to Table 2 at the end.

c	1	1.2	1.4	1.6	1.8	2
$\Delta(K_c) \geq$	0.9186	0.9612	1.0130	1.0780	1.1428	1.2071
c	2.2	2.4	2.6	2.8	3	
$\Delta(K_c) \geq$	1.2712	1.3358	1.4139	1.4917	1.5693	

Table 1: Lower bounds for $\Delta(K_c)$ obtained, for a couple of values c.

3 Strategy of proof and auxiliary results

There is no direct approach to estimate the critical determinant of a non-convex unbounded starbody like K_c. However, for convex (and **o**-symmetric) bodies in $\mathbb{R}^3$ the situation is considerably better. For this case, Minkowski [5] has established a general theorem which tells us how in this case the critical lattices[3] necessarily look like; see also [4, p. 342, Theorem 3]. On the basis of this result, Minkowski was able to evaluate $\Delta(\mathcal{O}) = \frac{19}{108}$ for the octahedron

$$\mathcal{O} : \quad |x| + |y| + |z| \leq 1\,.$$

Similarly, Ollerenshaw [11] showed that

$$\Delta(\mathcal{B}_3) = \frac{1}{\sqrt{2}} \tag{5}$$

for the origin-centered unit ball $\mathcal{B}_3$ in $\mathbb{R}^3$. Furthermore, Whitworth [12] considered the double cone

$$\mathcal{C} : \quad |x| + \sqrt{y^2 + z^2} \leq 1$$

and obtained

$$\Delta(\mathcal{C}) = \frac{\sqrt{6}}{8}\,. \tag{6}$$

Finally, the author [6] was able to show for the double paraboloid

$$\mathcal{P} : \quad |x| + y^2 + z^2 \leq 1$$

[2] Obviously, no loss of generality is implied by the fact that only one of the exponents of the two brackets is assumed to vary.

[3] A lattice $A\mathbb{Z}^3$ is called *critical* for a body B if $|\det A| = \Delta(B)$ and **o** is the only lattice point in the interior of B.

that

$$\Delta(\mathcal{P}) = \tfrac{1}{2}\,. \tag{7}$$

Our argument will be based on the idea to inscribe into K_c one of the three last-mentioned convex bodies, depending on the value of c, and to use the results (5)–(7). In fact, for a certain interval around $c = 2$, the choice of a paraboloid will turn out to be optimal, while for smaller values of c an ellipsoid will be the best choice, and for larger c the double cone will be most appropriate.

4 The details of the analysis

Lemma 1. *For fixed c, $0 < c < 4$, let*

$$\lambda_0 := \left(\frac{6}{4-c}\right)^{2/(2+c)}\,. \tag{8}$$

For any $\lambda > 0$, the ellipsoid

$$\mathcal{E}_c(\lambda): \quad \frac{x^2}{(1+\frac{1}{2}c)\lambda^{c/2}} + \frac{(y^2+z^2)}{1+\frac{1}{2}c}\left(\tfrac{1}{2}c\lambda + \lambda^{-c/2}\right) \le 1$$

is completely contained in K_c and has critical determinant

$$\Delta(\mathcal{E}_c(\lambda)) = \frac{(1+\frac{1}{2}c)^{3/2}}{\sqrt{2}}\,\frac{\lambda^{c/4}}{\frac{1}{2}c\lambda + \lambda^{-c/2}}\,. \tag{9}$$

For any fixed c, $0 < c < 4$, this expression attains its maximum for $\lambda = \lambda_0$, as given in (8). Hence $\Delta(K_c) \ge \Delta(\mathcal{E}_c)$ with $\mathcal{E}_c := \mathcal{E}_c(\lambda_0)$.

Proof. Let $r = \sqrt{y^2+z^2}$ for short. Then, by the mean inequality with weights,

$$r^c(r^2+x^2) = (\lambda r^2)^{c/2}\,\frac{r^2+x^2}{\lambda^{c/2}} \le \left(\frac{\frac{1}{2}c\lambda r^2 + (r^2+x^2)\lambda^{-c/2}}{1+\frac{1}{2}c}\right)^{1+c/2}\,.$$

From this, $K_c \supset \mathcal{E}_c(\lambda)$ is immediate. By (5), and an obvious linear substitution, (9) readily follows. Differentiating the right hand side of (9) with respect to λ and equating to zero, the choice $\lambda = \lambda_0$, as given in (8), turns out to be optimal. □

Lemma 2. *For any $c > 0$, define $r_0(c)$ as the unique*[4] *solution in $(0,1)$ of the equation*

$$2r_0^{c+2} - (c+2)r_0 + c = 0\,. \tag{10}$$

Put further

$$x_0(c) := \frac{\sqrt{r_0(c)^{-c} - r_0(c)^2}}{1-r_0(c)}\,. \tag{11}$$

[4] Let L denote the left-hand side of (10), then $\frac{\mathrm{d}L}{\mathrm{d}r_0} = 0$ iff $r_0 = r_* := 2^{-1/(c+1)}$. Hence L decreases on $[0, r_*]$ from c to $-r_*(c+1)$, and increases on $[r_*, 1]$ from $-r_*(c+1)$ to 0. Hence the uniqueness of $r_0(c)$.

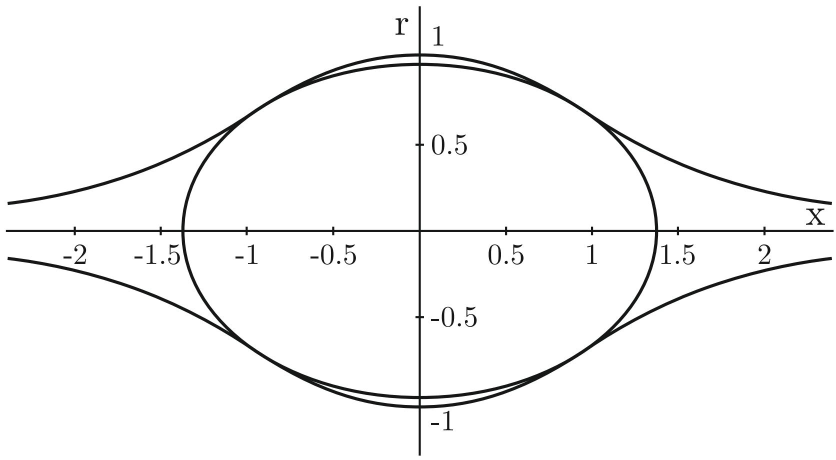

Figure 1: $c = 1$: The body K_1 and an optimal inscribed ellipsoid, in front view

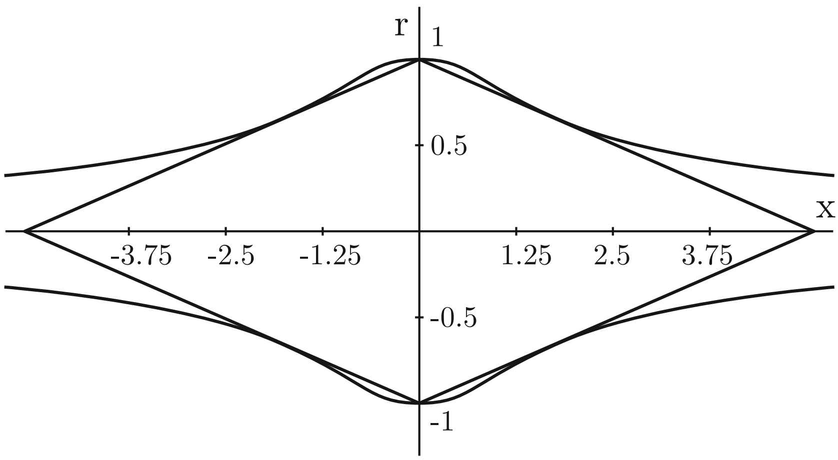

Figure 2: $c = 3$: The body K_3 and an optimal inscribed double cone, in front view

Then the double cone

$$\mathcal{C}_c : \quad \frac{|x|}{x_0(c)} + \sqrt{y^2 + z^2} \leq 1$$

is completely contained in K_c *and has critical determinant*

$$\Delta(\mathcal{C}_c) = \frac{\sqrt{6}}{8} x_0(c)\,. \tag{12}$$

Proof. Since both K_c and $\mathcal{C}_c$ are bodies of rotation, with respect to the x-axis, it suffices to discuss the situation in front view - in a (x, r)-plane, say, $r = \sqrt{y^2 + z^2}$.

By symmetry, we may restrict the calculations to $x \geq 0, r \geq 0$. The curve k_c whose rotation generates ∂K_c is given by $r^c(x^2 + r^2) = 1$. Solving for x gives

$$x = \xi(r) := \sqrt{r^{-c} - r^2}\,.$$

$\partial\mathcal{C}_c$ is generated by the tangent

$$T: \quad x - \xi(r_0) = \xi'(r_0)(r - r_0) \tag{13}$$

which contains the point $(x, r) = (0, 1)$. Inserting this into (13) and carrying out some bulky analysis, we arrive at (10). Since $(x_0, 0)$ is the point of intersection of T with the x-axis, (11) follows by one more routine calculation. Finally, (6) readily implies (12).

By the way, the point of inflection of k_c is $\big(\xi(r_W(c)), r_W(c)\big)$ with

$$r_W(c) = \left(\frac{c}{2(c+1)}\right)^{1/(c+2)}.$$

It is easily checked that throughout $r_W(c) > r_0(c)$. □

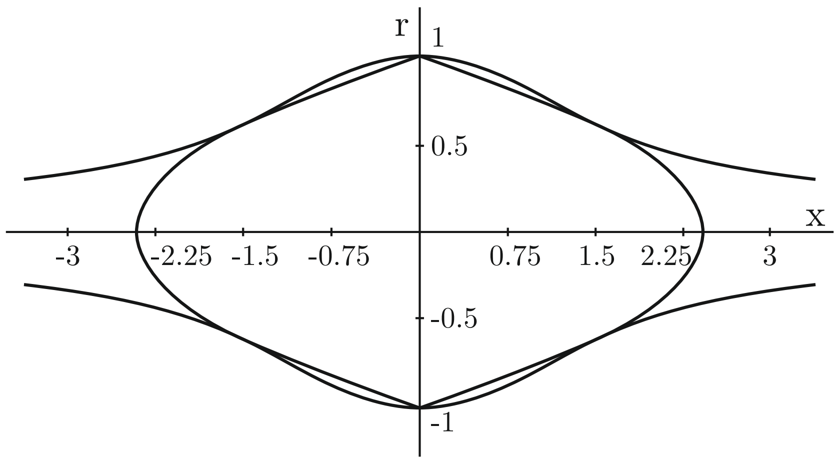

Figure 3: $c = 2$: The body K_2 and an optimal inscribed double paraboloid, in front view

Lemma 3. *For any $c > 0$, define $r_1(c)$ as the unique*[5] *solution in $(0, 1)$ of the equation*

$$2r_1^{c+2}(r_1^2 + 1) + c(1 - r_1^2) - 4r_1^2 = 0\,. \tag{14}$$

Put further

$$\alpha(c) := \frac{\sqrt{r_1(c)^{-c} - r_1(c)^2}}{1 - r_1(c)^2}\,. \tag{15}$$

[5]A similar argument applies as in footnote 4 in Lemma 2.

Then the double paraboloid

$$\mathcal{P}_c: \quad |x| \le \alpha(c)(1-y^2-z^2)$$

is completely contained in K_c *and has critical determinant*

$$\Delta(\mathcal{P}_c) = \frac{\alpha(c)}{2}\,. \tag{16}$$

Proof. Again we consider the situation in front view, in (x,r)-variables, $x,r \ge 0$. The aim is to choose $\alpha = \alpha(c)$ so that the parabola $p_c: \ x = \alpha(1-r^2)$ and the curve k_c have one point (x_1, r_1) in common (in the first quadrant), where also the derivative $x_1' = \frac{dx}{dr}\big|_{r=r_1}$ has the same value. In this way we get:

$$r_1^c(r_1^2 + x_1^2) = 1\,, \tag{17}$$

$$(c+2)r_1^{c+1} + cr_1^{c-1}x_1^2 + 2r_1^c x_1 x_1' = 0\,, \tag{18}$$

$$x_1 = \alpha(1-r_1^2)\,, \tag{19}$$

$$x_1' = -2\alpha r_1\,. \tag{20}$$

Dividing (20) by (19), we conclude that

$$x_1' = -\frac{2r_1}{1-r_1^2}\,x_1\,.$$

Using this in (18), we get

$$(c+2)r_1^{c+1} + x_1^2\left(cr_1^{c-1} - \frac{4r_1}{1-r_1^2}\right) = 0\,. \tag{21}$$

Solving (17) for x_1^2 and using this in (21), we obtain an equation in the single unknown r_1 which, after simplifying, is just (14). Further, (17) and (19) readily imply (15). Finally, (16) is immediate from (7).

Again, it is easily checked numerically that throughout $r_W(c) > r_1(c)$. □

We are now in a position to summarize the results obtained.

Theorem 1. *For* $0 < c < 4$, *the critical determinant of the starbody*

$$K_c: \quad (y^2+z^2)^{c/2}(x^2+y^2+z^2) \le 1$$

can be estimated from below by

$$\Delta(K_c) \ge \max\big(\Delta(\mathcal{E}_c), \Delta(\mathcal{C}_c), \Delta(\mathcal{P}_c)\big)\,.$$

Here, $\Delta(\mathcal{E}_c), \Delta(\mathcal{C}_c), \Delta(\mathcal{P}_c)$ *are given in Lemmas 1–3.*

Further, for $c \ge 4$,

$$\Delta(K_c) \ge \Delta(\mathcal{C}_c)\,.$$

Remark 1. As can be seen from the table below, for $c \in \{1, 1.2\}$, the sharpest lower bound for $\Delta(K_c)$ can be obtained by inscribing an ellipsoid. For $c \in \{1.4, 1.6, 1.8, 2, 2.2\}$, inscribing a double paraboloid yields the best result, while for $c \in \{2.4, 2.6, 2.8, 3\}$, an inscribed double cone is the best choice.

c	$\Delta(\mathcal{E}_c)$	$\Delta(\mathcal{P}_c)$	$\Delta(\mathcal{C}_c)$
1	**0.9186**	0.8810	0.7785
1.2	**0.9612**	0.9473	0.8601
1.4	1.0045	**1.0130**	0.9408
1.6	1.0485	**1.0780**	1.0207
1.8	1.0935	**1.1428**	1.1000
2	1.1398	**1.2071**	1.1790
2.2	1.1875	**1.2712**	1.2576
2.4	1.2371	1.3351	**1.3358**
2.6	1.2890	1.3988	**1.4139**
2.8	1.3437	1.4623	**1.4917**
3	1.4019	1.5257	**1.5693**

Table 2: The critical determinants of $\mathcal{E}_c, \mathcal{P}_c, \mathcal{C}_c$, for $1 \leq c \leq 3$, in step lengths of 0.2.

References

[1] J.V. Armitage: On a method of Mordell in the geometry of numbers. *Mathematika* 2 (2) (1955) 132–140.

[2] H. Davenport, K. Mahler: Simultaneous Diophantine approximation. *Duke Math. J.* 13 (1946) 105–111.

[3] H. Davenport: On a theorem of Furtwängler. *J. London Math.Soc.* 30 (1955) 185–195.

[4] P.M. Gruber, C.G. Lekkerkerker: *Geometry of numbers.* North Holland, Amsterdam (1987).

[5] H. Minkowski: Dichteste gitterförmige Lagerung kongruenter Körper. *Nachr. Kön. Ges. Wiss. Göttingen* (1904) 311–355.

[6] W.G. Nowak: The critical determinant of the double paraboloid and Diophantine approximation in $\mathbb{R}^3$ and $\mathbb{R}^4$. *Math. Pannonica* 10 (1999) 111–122.

[7] W.G. Nowak: Diophantine approximation in $\mathbb{R}^s$: On a method of Mordell and Armitage. In: *Algebraic number theory and Diophantine analysis. Proceedings of the conference held in Graz, Austria, August 30 to September 5, 1998, W. de Gruyter, Berlin.* (2000) 339–349.

[8] W.G. Nowak: Lower bounds for simultaneous Diophantine approximation constants. *Comm. Math.* 22 (1) (2014) 71–76.

[9] W.G. Nowak: Simultaneous Diophantine approximation: Searching for analogues of Hurwitz's theorem. In: *T.M. Rassias and P.M. Pardalos (eds.), Essays in mathematics and its applications.* Springer, Switzerland (2016) 181–197.

[10] W.G. Nowak: On the critical determinants of certain star bodies. *Comm. Math.* 25 (1) (2017) 5–11.

[11] K. Ollerenshaw: The critical lattices of a sphere. *J. London Math. Soc.* 23 (1949) 297–299.

[12] J.V. Whitworth: The critical lattices of the double cone. *Proc. London Math. Soc.* 2 (1) (1951) 422–443.

Author's address:
INSTITUTE OF MATHEMATICS, DEPARTMENT OF INTEGRATIVE BIOLOGY, BOKU WIEN, 1180 VIENNA, AUSTRIA

E-mail: nowak@boku.ac.at

Received: 21 February, 2017
Accepted for publication: 6 July, 2017
Communicated by: Karl Dilcher