

## The Existence of a Generalized Solution of an $m$ -Point Nonlocal Boundary Value Problem

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**Abstract.** An  $m$ -point nonlocal boundary value problem is posed for quasi-linear differential equations of first order on the plane. Nonlocal boundary value problems are investigated using the algorithm of reducing nonlocal boundary value problems to a sequence of Riemann-Hilbert problems for a generalized analytic function. The conditions for the existence and uniqueness of a generalized solution in the space are considered.

### Introduction

Nonlocal boundary value problems are quite an interesting generalization of classical problems and at the same time they are naturally obtained when constructing mathematical models in physics, engineering, sociology, ecology and so on [2], [7], [9], [10], [15], [25], [29], [33].

In the one-dimensional case, the known multipoint problems are actually problems with nonlocal conditions for ordinary differential equations and their research has a fairly long history. As for multidimensional nonlocal problems and related research, we can name the works of T. Carleman, A. Bels, F. E. Brovder and others [2], [8], [10]. It should be noted that the problems posed in these papers are problems with nonlocal conditions that are considered only on the boundary of the domain of definition of a differential operator.

In the work of J.R. Cannon [9], a nonlocal problem was posed, which initiated a new direction in the study of nonlocal boundary value problems and the problems of their numerical solution [1], [6], [25], [32]. In the work of A.V. Bitsadze

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and A.A. Samarski problem [7] was posed in a general form, but the uniqueness of the solution and its solvability was proved in the case of the Laplace equation and nonlocal boundary conditions. These works stimulated the appearance of interesting and original articles (see the papers of D.G. Gordeziani, Ya.A. Roitberg and Z.G. Sheftel, N.V. Zitarashu, S.D. Eidelman, T.Z. Dzhioeva and other works). Intensive studies of Bitsadze-Samarski nonlocal problems and their various generalizations began in the works of D.G. Gordeziani, A.L. Skubachevsky, V.P. Paneyakh, V.A. Il'in, I. Moiseev, G.V. Meladze, M.P. Sapagovas, I. Chegis, D.V. Kapanadze, V.L. Makarov, V.P. Mikhailov, A.G. Ghuschina, G. Avalishvili, L. Gurevich, and others (see [16], [17], [18], [19], [23], [24], [26], [28], [30], [34]). The papers of D.G. Sapagovas, G.K. Berikelashvili are certainly interesting from the standpoint of application and numerical methods (see e.g. [4], [30]). The results of research in this direction can be found on the works of V.L. Makarov, I.P. Gavriilyuk, D.G. Gordeziani, G.V. Meladze, D.O. Sittnik, B.V. Vasilik, S.V. Rao, V.V. Shelukhin, and others (see [21], [29], [33]).

In [5] an  $m$ -point nonlocal boundary value problem of Bitsadze-Samarskii type for an elliptic second-order equation in a rectangular domain is considered. V.V. Vasylyk [36] studied an  $m$ -point nonlocal problem for an elliptic differential equation with an operator coefficient in a Banach space. An exponentially convergent algorithm is proposed for the numerical solution of this problem. In [20], an  $m$ -point nonlocal initial-boundary value problem for a linear equation of parabolic type was investigated. To solve this problem, we propose an iterative process that allows us to reduce the non-local initial-boundary value problem to the classical Cauchy-Dirichlet problem. The properties of generalized analytic functions and the Riemann-Gilbert boundary value problems are studied in the monographs by I.N. Vekua [35] and in the works of G.F. Mandzhavidze and V. Tucke [27]. In [3], [12], [13], [14], [15] nonlocal boundary value problem is considered for quasilinear differential equations in the plane.

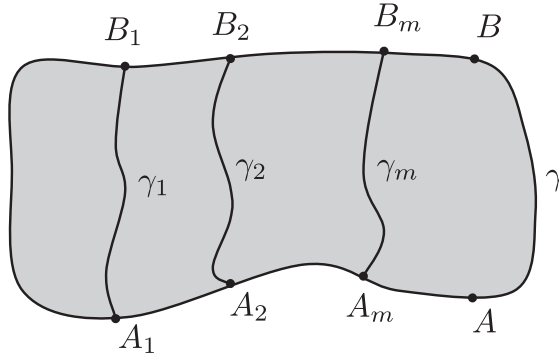
Obviously, the study of nonlocal boundary value problems, initial-boundary problems, the development and analysis of methods for their numerical solution is an actual, practically and theoretically very interesting, important area of mathematics.

In the present paper we present an  $m$ -point nonlocal boundary value problem for first-order quasilinear differential equations in the plane. To investigate nonlocal boundary value problems, an algorithm is used to reduce nonlocal boundary value problems to a sequence of Riemann-Hilbert problems for generalized analytic functions. In this paper we prove a theorem on the existence and uniqueness of a generalized solution in the space  $C_\alpha(\overline{G})$ . We consider an  $m$ -point nonlocal boundary value problem for first-order linear differential equations in the plane. The existence of a generalized solution in the space  $C_\alpha^p(\overline{G})$  is proved.

## 1 $m$ -point nonlocal boundary value problem

Let  $G$  be the bounded domain on the complex plane  $E$  with the boundary  $\Gamma$  which is a closed simple Liapunov curve (i.e. the angle formed by the tangent to this curve

with the constant direction is continuous in the Hölder sense).



We take two simple points  $A, B$  on  $\Gamma$  and assume that at these points there exists the tangent to  $\Gamma$ . It is obvious that that these points divide the boundary  $\Gamma$  into two curves. One of these parts denoted by  $\gamma$  is an open Liapunov curve with the parametric equation  $z = z(s), 0 \leq s \leq \delta$ .

Let us choose simple points  $A_k, B_k, k = \overline{1, m}$ , on  $\Gamma \setminus \gamma$  and assume that at these points the tangent to  $\Gamma$  exists. Besides, we draw in  $G$  the simple smooth curves  $\gamma_k, k = \overline{1, m}$ , which connect  $A_k$  and  $B_k$ . The curves  $\gamma_k$  are assumed to have the tangents at  $A_k$  and  $B_k$  which do not coincide with the tangent to the contour  $\Gamma$  at the same points. It is assumed that  $\gamma_k$  is the image of  $\gamma$ , diffeomorphic to  $z_k = I(z)$  and with the parametric equation  $z_k = z_k(s), 0 \leq s \leq \delta, k = \overline{1, m}$ . Furthermore, it is assumed that  $\gamma_i \cap \gamma_j = \emptyset, i \neq j, \gamma_i \cap \gamma = \emptyset, i, j = \overline{1, m}$  and the distance between every two lines  $\gamma_1, \gamma_2, \dots, \gamma_m, \gamma$  is larger than some positive number  $\varepsilon = \text{const} > 0$ .

Suppose that  $z = x + iy \in G, w = w_1 + iw_2, \partial_{\bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$  is a generalized Sobolev derivative [35],  $C(\overline{G})$  is a Banach space consisting of all continuous functions on  $\overline{G}$ . The norm in  $C(\overline{G})$  is defined by the equality

$$\|f\|_{C(\overline{G})} = \text{Max}_{z \in \overline{G}} |f(z)|.$$

$C_\alpha(\overline{G})$  is the set of all bounded functions satisfying the Hölder condition with index  $\alpha$ . The norm in  $C_\alpha(\overline{G})$  is defined by the equality

$$\|f\|_{C_\alpha(\overline{G})} = \text{Max}_{z \in \overline{G}} |f(z)| + \text{Sup}_{z_1, z_2 \in \overline{G}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha}$$

Let us consider in  $\overline{G}$  the following  $m$ -point nonlocal boundary value problem

for quasilinear differential equations of first order

$$\partial_{\bar{z}}w = f(z, w, \bar{w}), \quad z \in G, \quad (1)$$

$$\begin{aligned} \operatorname{Re}[w(z)] &= \varphi(z), \quad z \in \Gamma \setminus \gamma, \\ \operatorname{Im}[w(z^*)] &= c, \quad z^* \in \Gamma \setminus \gamma, c = \text{const}, \end{aligned} \quad (2)$$

$$\begin{aligned} \operatorname{Re}[w(z(s))] &= \sum_{k=1}^m \sigma_k \operatorname{Re}[w(z_k(s))], \quad z(s) \in \gamma, z_k(s) \in \gamma_k, \\ 0 < \sigma_k &= \text{const}, k = \overline{1, m}. \end{aligned} \quad (3)$$

**Theorem 1.** *Let the following conditions be fulfilled:*

1. the function  $f(z, w, \bar{w})$  is defined for  $z \in G$ ,  $|w| < R$ ,  $f(z, 0, 0) \in L_p(\overline{G})$ ,  $p > 2$  and

$$|f(z, w, \bar{w}) - f(z, w_0, \bar{w}_0)| \leq L(|w - w_0| + |w - \bar{w}_0|);$$

2.  $\varphi(z) \in C_\alpha(\Gamma \setminus \gamma)$ ,  $\alpha > \frac{1}{2}$ ,  $0 < \sum_{k=1}^m \sigma_k < 1$ ,  $0 < \sigma_k = \text{const}$ ,  $k = \overline{1, m}$ ;

3. there exists a number  $R_1 > 0$ ,  $R_1 \leq R$  such that the inequality

$$\|\psi_n\|_{C_\alpha(\overline{G})} + (C_1 + \|T_G\|_{L_p(\overline{G}), C_\alpha(\overline{G})})(2L|G|^{1/p}R_1) \leq R_1,$$

where  $|G| = \text{mes } G$ ;

$$2|G|^{1/p}L(C_1 + \|T_G\|_{L_p(\overline{G}), C_\alpha(\overline{G})}) < 1,$$

is fulfilled.

Then the solution of problem (1)–(3) exists in the space  $C_\alpha(\overline{G})$  and is unique.

*Proof.* To investigate the existence of a generalized solution of problem (1)–(3) we consider the following iteration process

$$\partial_{\bar{z}}w_n = f(z, w_n, \bar{w}_n), \quad z \in G, \quad (4)$$

$$\begin{aligned} \operatorname{Re}[w_n(z)] &= \varphi(z), \quad z \in \Gamma \setminus \gamma, \\ \operatorname{Im}[w_n(z^*)] &= c, \quad z^* \in \Gamma \setminus \gamma, \end{aligned} \quad (5)$$

$$\begin{aligned} \operatorname{Re}[w_n(z(s))] &= \sum_{k=1}^m \sigma_k \operatorname{Re}[w_{n-1}(z_k(s))], \quad z(s) \in \gamma, z_k(s) \in \gamma_k, \\ k &= \overline{1, m}, n = 1, 2, 3, \dots, \end{aligned} \quad (6)$$

where  $\operatorname{Re}[w_0(z_k)]$  is any function from  $C_\alpha(\gamma_k)$ ,  $\alpha > \frac{1}{2}$ ,  $k = \overline{1, m}$ , that continuously adjoins the values of  $\varphi(z)$  at the ends of the contour  $\gamma_k$ .

For every  $n \in \mathbb{N}$ , problem (4)–(6) is a Riemann-Hilbert type problem and its regular generalized solution belongs to the space  $C_\alpha(\overline{G})$  [27], [35].

Consider the function  $v_n = w_{n+1} - w_n$ . Then from (4)–(6) it follows that the function  $v_n$  is a solution of the problem

$$\partial_{\bar{z}}v_n = f(z, w_{n+1}, \bar{w}_{n+1}) - f(z, w_n, \bar{w}_n) \equiv F(z, w_n, \bar{w}_n, w_{n+1}, \bar{w}_{n+1}), \quad z \in G, \tag{7}$$

$$\begin{aligned} \operatorname{Re}[v_n(z)] &= 0, & z \in \Gamma \setminus \gamma, \\ \operatorname{Im}[v_n(z^*)] &= 0, & z^* \in \Gamma \setminus \gamma, \end{aligned} \tag{8}$$

$$\begin{aligned} \operatorname{Re}[v_n(z(s))] &= \sum_{k=1}^m \sigma_k \operatorname{Re}[v_{n-1}(z_k(s))], & z(s) \in \gamma, z_k(s) \in \gamma_k, \\ & & k = \overline{1, m}, n = 1, 2, 3, \dots \end{aligned} \tag{9}$$

We can reduce the solution of problem (7)–(9) to the non-linear integral equation [27]:

$$v_n(z) = \psi_n(z) + \phi_n(z) - \frac{1}{\pi} \iint_G \frac{F(\zeta, w_n(\zeta), \bar{w}_n(\zeta), w_{n+1}(\zeta), \bar{w}_{n+1}(\zeta))}{\zeta - z} d\xi d\eta, \tag{10}$$

where  $\zeta = \xi + i\eta$ ,  $\psi(z)$  is a holomorphic function satisfying conditions (8)–(9) and is a holomorphic function such that the difference

$$\phi_n(z) - \frac{1}{\pi} \iint_G \frac{F(\zeta, w_n, \bar{w}_n, w_{n+1}, \bar{w}_{n+1})}{\zeta - z} d\xi d\eta$$

satisfies homogeneous boundary conditions and an a priori estimate has the form [27]

$$\|\phi_n\|_{C_\alpha(\bar{G})} \leq C_1 \|F\|_{L_p(\bar{G})}, \quad C_1 = \text{const} > 0.$$

The integral operator in the right-hand part of equation (10) is denoted by

$$T_G[z, F] = -\frac{1}{\pi} \iint_G \frac{F(\zeta, w_n(\zeta), \bar{w}_n(\zeta), w_{n+1}(\zeta), \bar{w}_{n+1}(\zeta))}{\zeta - z} d\xi d\eta, \quad \zeta = \xi + i\eta.$$

The operator  $T_G$  maps the space  $L_p(\bar{G})$  into  $C_\beta(\bar{G})$ ,  $\beta = \frac{p-2}{p} < \alpha$  [35].

Assume that conditions 1–3 are fulfilled, then there exists a unique solution of problem (7)–(9) in a ball  $\|v_n\|_{C_\alpha(\bar{G})} \leq R_1$  [27].

Let us estimate the function  $v_n(z)$  from equality (10) in the metric of the space  $C(\bar{G})$ :

$$\|v_n\|_{C(\bar{G})} \leq \|\psi_n\|_{C(\bar{G})} + \|\phi_n\|_{C(\bar{G})} + \|T_G[F]\|_{C(\bar{G})} \tag{11}$$

Using the previous estimates, from inequality (11) we obtain:

$$\|v_n\|_{C(\bar{G})} \leq \|\psi_n\|_{C(\bar{G})} + \left( C_1 + \|T_G\|_{L_p(\bar{G}), C_\alpha(\bar{G})} \right) \|F\|_{L_p(\bar{G})}. \tag{12}$$

By virtue of 1 we have

$$\begin{aligned} |F(z, w_n, \bar{w}_n, w_{n+1}, \bar{w}_{n+1})| &= |f(z, w_{n+1}, \bar{w}_{n+1}) - f(z, w_n, \bar{w}_n)| \\ &\leq 2L|w_{n+1} - w_n| = 2L|v_n|. \end{aligned}$$

With 1 taken into account, the latter inequality implies that the complex function  $F(z, w_n, \bar{w}_n, w_{n+1}, \bar{w}_{n+1})$ , belongs to the space  $L_p(\bar{G})$ . Then

$$\|F\|_{L_p(\bar{G})} \leq 2L\|v_n\|_{L_p(\bar{G})} \leq 2L|G|^{1/p}\|v_n\|_{C(\bar{G})}.$$

Thus, from inequality (12) we can write that

$$\|v_n\|_{C(\bar{G})} \leq \|\psi_n\|_{C(\bar{G})} + 2L|G|^{1/p}\left(C_1 + \|T_G\|_{C_\alpha(\bar{G})}\right)\|v_n\|_{C(\bar{G})},$$

i.e., taking 3 into account, we finally obtain

$$\|v_n\|_{C(\bar{G})} \leq \frac{\|\psi_n\|_{C(\bar{G})}}{1 - 2L|G|^{1/p}\left(C_1 + \|T_G\|_{C_\alpha(\bar{G})}\right)}. \tag{13}$$

Note that the function  $\psi_n(z)$  is the solution of the following problem:

$$\begin{aligned} \partial_{\bar{z}}\psi_n(z) &= 0, & z \in G, \\ \operatorname{Re}[\psi_n(z)] &= 0, & z \in \Gamma \setminus \gamma, \\ \operatorname{Im}[\psi_n(z^*)] &= 0, \end{aligned}$$

$$\operatorname{Re}[\psi_n(z)] = \sum_{k=1}^m \sigma_k \operatorname{Re}[\psi_{n-1}(z_k(s))], \quad z \in \gamma, z_k \in \gamma_k, k = \overline{1, m}, n = 1, 2, 3, \dots,$$

$$\psi_0(z) = w_1(z) - w_0(z).$$

Since  $\operatorname{Re}[\psi_n(z)]$  is a harmonic function,  $0 < \sum_{k=1}^m \sigma_k < 1$ , all the conditions of Schwartz' lemma [11] are fulfilled for it and there exists  $0 < q < 1$  which is independent of  $\psi_n$  and for which the following inequality [5] is fulfilled:

$$\|\psi_n\|_{C(\bar{G})} \leq Mq^n,$$

where the constant  $M > 0$  depend only on  $\varphi(z)$ .

Using this estimate, from (13) we can write

$$\|v_n\|_{C(\bar{G})} \leq \frac{M}{1 - 2L|G|^{1/p}\left(C_1 + \|T_G\|_{C_\alpha(\bar{G})}\right)}q^n. \tag{14}$$

Now from (14) we can conclude that the series  $\sum_{k=1}^\infty v_k$  converges uniformly to zero in the domain  $\bar{G}$ . Hence it follows that the sequence  $\{w_n(z)\}$  is fundamental in  $C(\bar{G})$  and has the limit  $w(z) \in C(\bar{G})$ .

Let us consider the integral representation for the function  $w_n(z)$ :

$$w_n(z) = \psi'_n(z) + \phi'_n(z) - \frac{1}{\pi} \iint_G \frac{f(\zeta, w_n, \bar{w}_n)}{\zeta - z} d\xi d\eta, \tag{15}$$

where  $\psi'_n(z)$  is a holomorphic function that satisfies conditions (5)–(6), and  $\phi'_n(z)$  is a holomorphic function such that the difference

$$\phi'_n(z) - \frac{1}{\pi} \iint_G \frac{f(\zeta, w_n, \bar{w}_n)}{\zeta - z} d\xi d\eta$$

satisfies the homogeneous boundary conditions.

From representation (15) we can conclude that  $w(z)$  is the solution of problem (1)–(3) and  $w(z) \in C_\alpha(\overline{G})$ . By the uniqueness of the holomorphic solution and the integral representation (15) we conclude that this solution is unique in the class  $C_\alpha(\overline{G})$ .  $\square$

## 2 Linear problem

Consider in the domain  $\overline{G}$  the following  $m$ -point nonlocal boundary value problem for a linear differential equation of first order

$$\begin{aligned} \partial_{\bar{z}}w &= A(z)w + B(z)\bar{w} + d(z) \quad z \in G, \\ \operatorname{Re}[w(z)] &= 0, \quad z \in \Gamma \setminus \gamma, \\ \operatorname{Im}[w(z^*)] &= 0, \quad z^* \in \Gamma \setminus \gamma, \\ \operatorname{Re}[w(z(s))] &= \sum_{k=1}^m \sigma_k \operatorname{Re}[w(z_k(s))], \quad z(s) \in \gamma, z_k(s) \in \gamma_k, \\ &0 < \sigma_k = \operatorname{const}, k = \overline{1, m}. \end{aligned} \tag{16}$$

Assume that  $A(z), B(z), d(z) \in L_p(\overline{G}), p > 2, |A|, |B| \leq N$ . Denote by  $C_\alpha^p(\overline{G})$  the set of functions  $w(z) \in C_\alpha(\overline{G})$  such that

$$\begin{aligned} \operatorname{Re}[w(z)] &= 0, \quad z \in \Gamma \setminus \gamma, \quad \operatorname{Im}[w(z^*)] = 0, \quad z^* \in \Gamma \setminus \gamma, \\ \operatorname{Re}[w(z(s))] &= \sum_{k=1}^m \sigma_k \operatorname{Re}[w(z_k(s))], \quad z(s) \in \gamma, z_k(s) \in \gamma_k, k = \overline{1, m} \end{aligned} \tag{17}$$

and having the finite norm

$$\|w\|_{C_\alpha^p(\overline{G})} = \|w\|_{C_\alpha(\overline{G})} + \|\partial_{\bar{z}}w\|_{L_p(\overline{G})} < +\infty. \tag{18}$$

The set  $C_\alpha^p(\overline{G})$  is a linear normed space over the real field with the norm defined by means of equality (18). If  $p > q > 2$ , then  $C_\alpha^p(\overline{G}) \subset C_\alpha^q(\overline{G})$  and

$$\|w\|_{C_\alpha^q(\overline{G})} \leq \ell \|w\|_{C_\alpha^p(\overline{G})},$$

where  $\ell$  is a positive constant and  $w$  is any element from  $C_\alpha^p(\overline{G})$ . The following theorem holds true.

**Theorem 2.** For any function  $d(z) \in L_p(\overline{G}), p > 2$ , a solution  $w(z)$  of problem (16) exists, belongs to the space  $C_\alpha^p(\overline{G})$  and the following a priori estimate holds for it

$$\|w\|_{C_\alpha^p(\overline{G})} \leq \lambda \|d\|_{L_p(\overline{G})}, \tag{19}$$

where  $\lambda$  is the positive constant depending only on  $p, N$  and  $|G| = \operatorname{mes} G$ .

*Proof.* The existence and uniqueness of the solution of problem (16) immediately follows from Theorem 1. It remains to prove the validity of the a priori estimate

(19). We have to reduce problem (16) to an integral equation. For this we introduce the operator

$$T_G[z, f] = -\frac{1}{\pi} \iint_G \frac{f(t)}{t-z} d\xi d\eta, \quad t = \xi + i\eta$$

and the operator  $S_G[z, f]$  from  $L_p(\overline{G})$  into a subset of analytic functions, which satisfies the conditions

$$\begin{aligned} \operatorname{Re}\{T_G[z, f] + S_G[z, f]\} &= 0, \quad z \in \Gamma \setminus \gamma, \\ \operatorname{Re}\{T_G[z, f] + S_G[z, f]\} &= \sum_{k=1}^m \sigma_k \operatorname{Re}\{T_G[z_k, f] + S_G[z_k, f]\}, \quad z \in \gamma, \quad z_k \in \gamma_k, \\ \operatorname{Im}\{T_G[z^*, f] + S_G[z^*, f]\} &= 0, \end{aligned} \tag{20}$$

where  $z^* \in \Gamma \setminus \gamma$  is a fixed point.

Due to conditions (20) we define the operator uniquely. Let us define the operators

$$\begin{aligned} P(f) &= T_G[z, f] + S_G[z, f], \\ P_{AB}(f) &= P(Af) + P(B\bar{f}), \end{aligned} \tag{21}$$

where the functions  $A(z)$  and  $B(z)$  are from the right-hand part of equation (16).

Taking now into account that  $\partial_{\bar{z}}P(f) = f(z)$ , it can be easily proved that the solution of problem (16) satisfies the following integral equation

$$w(z) = P_{AB}(w) + P(d). \tag{22}$$

It is likewise easy to show that problems (16) and (22) are equivalent. Using the properties of the operators  $T_G[z, f]$  and  $P(f)$  [35], it can be shown that these operators are completely continuous over the field of real numbers. It is obvious that the operator  $P_{AB}(f)$ , too, is completely continuous.

Since for  $d(z) = 0$  equation (16) has only the trivial solution, the equation  $w(z) = P_{AB}(w)$  will also have only the trivial solution. Hence, because the operator  $P_{AB}(f)$  is completely continuous, we obtain the existence and boundedness of the operator  $(I - P_{AB})^{-1}$ , where  $I$  is the identity operator.

We introduce the notation

$$\|I - P_A\|_{C_\alpha(\overline{G}), L_p(\overline{G})}^{-1} = M, \quad \|P\|_{L_p(\overline{G}), C_\alpha(\overline{G})} = M_p,$$

where  $M$  and  $M_p$  - are positive constants. From equation (22) we immediately obtain

$$\|w(z)\|_{C_\alpha(\overline{G})} \leq MM_p \|d\|_{L_p(\overline{G})}. \tag{23}$$

Equation (16) immediately implies

$$\|\partial_{\bar{z}}w\|_{L_p(\overline{G})} \leq 2N \|w\|_{C_\alpha(\overline{G})} + \|d\|_{L_p(\overline{G})}. \tag{24}$$

From inequalities (23), (24) we obtain the estimate

$$\|w\|_{C_\alpha^p(\overline{G})} = \|w\|_{C_\alpha(\overline{G})} + \|\partial_{\bar{z}}w\|_{L_p(\overline{G})} \leq \lambda \|d\|_{L_p(\overline{G})},$$

where  $\lambda = MM_p(2N + 1) + 1$ . □



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