Communications in Mathematics 26 (2018) 1–10

DOI: 10.2478/cm-2018-0001

On self-similar subgroups in the sense of IFS

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Abstract. In this paper, we first give several properties with respect to subgroups of self-similar groups in the sense of iterated function system (IFS). We then prove that some subgroups of p-adic numbers $\mathbb{Q}_p$ are strong self-similar in the sense of IFS.

1 Introduction

The concept of fractal was introduced by B. Mandelbrot [6], referred to as the father of fractals, in the 1970s. Fractals are known as the geometry of nature and thus there are a vast range of applications including many branches of science like mathematics, physics, biology, medicine, finance and social sciences. Fractals have no precise definition but they have a common property called self-similarity. Namely, a fractal set is made up of smaller scale copies of itself. Iterated function systems (IFSs) are one of the most popular methods of obtaining self-similar sets. The theory of IFS was initiated by Hutchinson [5] and then it has developed by Barnsley [1]. An IFS on a complete metric space X is a finite family $\{f_1, f_2, \ldots, f_n\}$ of contracting maps $f_i \colon X \to X$ for $i = 1, 2, \ldots, n$. The well-known result of J. Hutchinson represents that if $\{f_1, f_2, \ldots, f_n\}$ is an IFS on X, then there is unique non-empty compact subset S of X such that

$$S = \bigcup_{i=1}^{n} f_i(S).$$

The set S is called the self-similar set or attractor associated with the IFS $\{f_1, f_2, \ldots, f_n\}$ (for details see [1], [3], [5]).

2010 MSC: 47H10, 28A80, 11E95
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In recent years, Saltan and Demir have investigated groups which is the attractor of an IFS. Saltan and Demir also show that a group satisfying Definition 1 or Definition 2 has proper subgroups being homomorphic or isomorphic to itself respectively. The following definitions, the proofs of propositions and theorem are elaborately given in [2], [8], [9].

Definition 1. ([2]) Let (G, d) be a compact topological group with a translation--invariant metric d. G is called a self-similar group in the sense of IFS, if there exists a proper subgroup H of finite index and a surjective homomorphism $\phi\colon G \to H$, which is a contraction with respect to d.

Definition 2. ([8]) Let (G, d) be a compact topological group with a translation--invariant metric d. G is called a strong self-similar group in the sense of IFS, if there exists a proper subgroup H of finite index and a group isomorphism $\phi\colon G \to H$, which is a contraction with respect to d.

Proposition 1. *([8]) A self-similar or a strong self-similar group is the attractor of an IFS.*

Proposition 1 explains the reason why we call these groups self-similar groups in the sense of IFS. In the literature, there are different definitions of self-similar groups. Since this paper only deals with self-similar group in the sense of IFS, in order to avoid unnecessary repetitions, we say self-similar groups instead of self-similar groups in the sense of IFS.

Proposition 2. *([8]) A self-similar group is a disconnected set.*

Theorem 1. *([8]) A strong self-similar group is a profinite group.*

A self-similar group can be considered as a fractal group since it is the attractor of an IFS. So, these groups have an important place for fractals and the structure of these groups is worth examining. In the literature, there is no known works giving the properties of subgroups of these groups. Actually, there is a natural relationship between self-similar (or strong self-similar) groups and their subgroups. When these relationships are considered, some small but important questions come to mind. To exemplify, when self-similar groups are finite, what features do their subgroups have? This case is stated in Proposition 3 and Corollary 1. The main goal of the present paper is to give an answer to the following question:

Let G be an infinite strong self-similar group and H be a subgroup of G. Which properties should H have to be a strong self-similar group?

In Proposition 4, the answer of above question is given using by Lemma 1. Moreover, several important results are obtained in Corollary 3, 4 and 5 and then these results are supported by an example. In Remark 1, it is given an another example the converse of Lemma 1 can not be true.

The group of p-adic integer, $\mathbb{Z}_p$, constitutes an important model for strong self-similar groups. An IFS for $\mathbb{Z}_p$ and the self-similarity of the group of $\mathbb{Z}_p$ are stated without proof in [8]. In the last section, in Proposition 5 and Proposition 6 are given the proofs respectively. Finally, in Remark 2, a different way demonstrating self-similarity of subgroups of $\mathbb{Z}_p$ is presented.

2 Some properties of self-similar subgroups

We begin this section with the properties of finite self-similar group. Afterwards, we explain the general case.

Proposition 3. *If G is a finite group, then every subgroup of G is a self-similar group.*

Proof. Let G be a finite group and let e be the identity element of G. Then, G is a compact topological group with discrete metric. Similarly, so is every subgroup of G. Let G' be arbitrary subgroup of G. Thus,

$$\phi' \colon G' \to \{e\}$$

is both a contraction map and a group homomorphism. This concludes the proof. □

The following results are easily seen.

Corollary 1. *If G is a finite group and N is an arbitrary normal subgroup of G, then the factor group G/N is a self-similar group.*

Corollary 2. *If G is an infinite strong self-similar group and N is its normal subgroup providing Definition 2, then the finite factor group G/N is a self-similar group.*

In the following proposition, we mean the homomorphism ϕ given in Definition 2.

Lemma 1. *Let G be a strong self-similar group and let G_1 be a closed-proper subgroup of G. If $\phi(G_1)$ is a subgroup of G_1 of finite index, then G_1 is also a strong self-similar group.*

Proof. There exists a proper subgroup H of finite index and a group isomorphism $\phi \colon G \to H$, which is a contraction with respect to d, since G is a strong self-similar group. Moreover, G_1 is a compact topological group owing to the fact that every closed space of a compact space is compact and every subgroup of a topological group is a topological group. We now define the mapping

$$\phi|_{G_1} \colon G_1 \to \phi(G_1).$$

The mapping $\phi|_{G_1}$ is a group isomorphism and contraction with respect to d because of the definition of ϕ. We know that the index of $\phi(G_1)$ in G_1 is finite due to the hypothesis. Thus, the proof is completed. □

Proposition 4. *If ϕ, G and G_1 satisfy the condition of Lemma 1, then for every $n \in \mathbb{N}$*

$$\phi^n(G_1) = (\underbrace{\phi \circ \phi \circ \ldots \circ \phi}_{n \text{ times}})(G_1)$$

is also a strong self-similar group.

Proof. $\phi(G_1)$ is a closed subset of the group G_1 owing to the fact that G_1 is a closed subset of G and the mapping ϕ is a homeomorphism. Moreover, $\phi(G_1)$ is a subgroup of G_1 due to the hypothesis of Lemma 1. By using induction method, we show that $\phi^{n+1}(G_1)$ is a closed subgroup of $\phi^n(G_1)$ for every $n \in \mathbb{N}$.

We now prove that $\phi^2(G_1)$ is of finite index in $\phi(G_1)$. By Lemma 1, we know that $\phi(G_1)$ is a subgroup of G_1 of finite index. Let $[G_1 : \phi(G_1)] = m$ and let $x_0 = e$ be the identity element of G. Then, we can denote G_1 such that

$$G_1 = (\phi(G_1) * x_0) \cup (\phi(G_1) * x_1) \cup (\phi(G_1) * x_2) \cup \ldots \cup (\phi(G_1) * x_{m-1})$$

where $(\phi(G_1) * x_i) \cap (\phi(G_1) * x_j) = \emptyset$ for $i, j = 0, 1, 2, \ldots, (m-1)$ and $i \neq j$. Since ϕ is an isomorphism, we have

$$\begin{aligned}\phi(G_1) &= \phi\big(\phi(G_1) \cup (\phi(G_1) * x_1) \cup \cdots \cup (\phi(G_1) * x_{m-1})\big) \\ &= \phi^2(G_1) \cup \big(\phi^2(G_1) * \phi(x_1)\big) \cup \cdots \cup \big(\phi^2(G_1) * \phi(x_{m-1})\big)\end{aligned}$$

and

$$(\phi^2(G_1) * \phi(x_i)) \cap (\phi^2(G_1) * \phi(x_j)) = \emptyset$$

for $i, j = 0, 1, 2, \ldots, (m-1)$ and $i \neq j$. Hence, we obtain

$$[\phi(G_1) : \phi^2(G_1)] = m.$$

Proceeding in this manner, we get

$$[\phi^{n+1}(G_1) : \phi^n(G_1)] = m.$$

By Lemma 1, $\phi^n(G_1)$ is a strong self-similar group for each $n \in \mathbb{N}$.

It means that there are self-similar subgroups of G such that

$$G_1, \phi(G_1), \phi^2(G_1), \phi^3(G_1), \ldots, \phi^n(G_1), \ldots$$

where $\phi^{n+1}(G_1) \leq \phi^n(G_1)$ for $n = 0, 1, 2, \ldots$. □

Corollary 3. *Let G be a strong self-similar group and H be its subgroup given in Definition 2. By Proposition 4, G naturally has strong self-similar subgroups such that*

$$H, \phi(H), \phi^2(H), \phi^3(H), \ldots, \phi^n(H), \ldots$$

where $\phi^{n+1}(H) \leq \phi^n(H)$ for $n = 0, 1, 2, \ldots$. Hence, if there is a strong self-similar group, this means that we actually have a lot of strong self-similar groups.

It is well-known that every profinite group is a residually finite group. In Theorem 1, we prove that a strong self-similar group is a profinite group and thus it is a residually finite group. By using Corollary 3, we arrive at the same result from a different way as follows:

Corollary 4. *Let G be a strong self-similar group. G is compact and thus it is complete. By Corollary 3, G has closed nested subgroups whose diameters tend to zero such that*

$$H, \phi(H), \phi^2(H), \phi^3(H), \ldots, \phi^n(H), \ldots$$

where $\phi^{n+1}(H) \leq \phi^n(H)$ *for* $n = 0, 1, 2, \ldots$. *So, we have*

$$\bigcap_{n \in \mathbb{N}} \phi^n(H) = \{e\}$$

due to the Cantor intersection theorem. This implies that G *is a residually finite group.*

Corollary 5. *If we consider that the group* H *given in Definition 2 is a normal subgroup of* G*, then* $\phi^n(H)$ *is a normal subgroup of* G *for every* $n \in \mathbb{N}$*. Moreover,* $\phi^n(H)$ *is an open subgroup for every* $n \in \mathbb{N}$ *due to the fact that* $\phi^n(H)$ *is a closed subgroup for every* $n \in \mathbb{N}$ *and it is of finite index in* G*. Thus, we can explicitly express that*

$$G \cong \varprojlim_{n \in \mathbb{N}} G/\phi^n(H).$$

Due to Proposition 4, we know that $[G : \phi^n(G)] = p^n$ *if* $[G : H] = p$ *where* p *is a prime number. In such a case,* G *is a pro-*p *group.*

In the following example, we give an application of Corollary 3, 4 and 5.

Example 1. Let us consider the infinite product group

$$G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots .$$

Let $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$ be arbitrary elements of G. If we equip the group G with the metric

$$d(x, y) = 2\left|\sum_{i=1}^{\infty} \frac{x_i - y_i}{3^i}\right|,$$

then we obtain the classical Cantor set. Thus, G is a compact topological group. On the group G, we define a contraction mapping as follows:

$$\begin{aligned} \phi \colon \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots &\to \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \\ (x_1, x_2, x_3, \ldots) &\mapsto (0, x_1, x_2, x_3, \ldots) \end{aligned}$$

It is clear that $\phi(G) = \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z} \times \cdots$ is a subgroup of G and ϕ is an isomorphism. It follows that $[G : \phi(G)]$=2 owing to the fact that

$$G = \phi(G) \cup [\phi(G) + (1, 0, 0, \ldots)].$$

This shows that G is a strong self-similar group.

By Corollary 3, $H = \phi(G) = \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z} \times \cdots$ is a strong self-similar subgroup of G. In addition that, for every $n \in \mathbb{N}$

$$\phi^n(G) = \underbrace{\{0\} \times \{0\} \times \cdots \times \{0\}}_{n \text{ times}} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$$

is a strong self-similar group.

By Corollary 4, G is a residually finite group in that

$$\bigcap_{n\in\mathbb{N}} \phi^n(H) = \{0\} \times \{0\} \times \{0\} \times \cdots .$$

By Corollary 5, G is a pro-2 group by reason of the fact that H is a normal subgroup of G and $[G : \phi(G)]$=2.

Remark 1. The converse of Lemma 1 is not true. To show this case, we first consider the group G defined in Example 1. We now take the subgroup

$$G_1 = \mathbb{Z}/2\mathbb{Z} \times \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$$

of G. It is clear that G_1 is a compact topological group by the subspace metric of d with respect to G_1. On the group G_1, we define a contraction mapping as follows:

$$\begin{aligned} \varphi\colon G_1 &\to \{0\} \times \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \\ (x_1, 0, x_3, x_4, \dots) &\mapsto (0, 0, x_1, 0, x_3, x_4, \dots). \end{aligned}$$

It is obvious that

$$\varphi(G_1) = \{0\} \times \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$$

is a subgroup of G_1 and φ is a group isomorphism. Furthermore,

$$[G_1 : \varphi(G_1)] = 4$$

owing to the fact that

$$G_1 = \varphi(G_1) \cup [\varphi(G_1) + x] \cup [\varphi(G_1) + y] \cup [\varphi(G_1) + z]$$

where $x = (1, 0, 0, 0, \dots)$, $y = (0, 0, 0, 1, 0, 0, 0, \dots)$ and $z = (1, 0, 0, 1, 0, 0, 0, \dots)$ are the elements of G_1. It follows that G_1 is a strong self-similar group.

However, it is obvious that

$$\phi(G_1) = \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \{0\} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z} \times \cdots$$

is not a subgroup of G_1.

3 Some subgroups of $\mathbb{Q}_p$ and their self-similarity

The group of p-adic integers is one of the most important examples of topological and profinite groups. Due to this reason, it will be a remarkable example for self-similar groups. In the following, we only give the properties of group structure.

We now recall p-adic numbers and their properties (for details see [4], [7], [10]). The p-adic integers, $\mathbb{Z}_p$, is the set

$$\left\{ \sum_{i=0}^{\infty} a_i p^i \;\middle|\; a_i \in \{0, 1, 2, \dots, (p-1)\} \right\}$$

and each elements of this set is called a p-adic integer. Let $x = \sum_{i\geq 0} x_i p^i$ and $y = \sum_{i\geq 0} y_i p^i$ be elements of $\mathbb{Z}_p$. The sum of x and y, $z = \sum_{i\geq 0} z_i p^i$, is determined by

$$\sum_{i=0}^{m} z_i p^i \equiv \sum_{i=0}^{m} (x_i + y_i) p^i \pmod{p^{m+1}} \tag{1}$$

for each $m \in \{0, 1, 2, \ldots\}$ where $z_i \in \{0, 1, \ldots, (p-1)\}$. It is well-know that $(\mathbb{Z}_p, +)$ is a group called the group of p-adic integers.

The p-adic numbers, $\mathbb{Q}_p$, is the set

$$\left\{ \sum_{i=-\infty}^{\infty} a_i p^i \;\middle|\; a_i \in \{0, 1, 2, \ldots, (p-1)\} \text{ and } a_{-i} = 0 \text{ for large } i \right\}.$$

Addition in $\mathbb{Z}_p$ is determined by (1) can be naturally extended to $\mathbb{Q}_p$. Hence, $\mathbb{Q}_p$ is a group with this operation. Let $\sum_{i=-\infty}^{\infty} x_i p^i$ be a nonzero element of $\mathbb{Q}_p$. Thus, there is a first index $v(x) \geq 0$ such that $x_v \neq 0$. This index is called the order of x and is denoted by $\mathrm{ord}_p(x)$. In other words, $\mathrm{ord}_p(x) = \min\{v \mid x_v \neq 0\}$. If $x_i = 0$ for $i = 0, 1, 2, \ldots$, then $\mathrm{ord}_p(x) = \infty$. On the other hand, the p-adic value of x is denoted by

$$|x|_p = \begin{cases} 0 & \text{if } x_i = 0 \text{ for } i = 0, 1, 2, \ldots, \\ p^{-\mathrm{ord}_p(x)} & \text{otherwise} \end{cases}$$

and induces the p-adic metric $d_p(x, y) = |x - y|_p$ for $x, y \in \mathbb{Q}_p$. Moreover, $\mathbb{Q}_p$ is the metric completion of $\mathbb{Q}$ with respect to the p-adic metric. It is easily seen that the group of p-adic numbers is a topological group. Moreover, the group of p-adic integers can be defined as

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid \; |x|_p \leq 1\}$$

and thus, it is an important subgroup of $\mathbb{Q}_p$. It is easily shown that $\mathbb{Z}_p$ is compact and thus, it is complete. Furthermore, $\mathbb{Q}_p$ is complete and locally compact.

3.1 A family of self-similar subgroups of $\mathbb{Q}_p$

In this section, we first give an IFS for a family of subgroups of $\mathbb{Q}_p$. Then, we show that the group of p-adic integers is a strong self-similar group. Finally, in Remark 2, we also obtain some self-similar subgroups of $\mathbb{Z}_p$ by using Corollary 3.

In the following proposition, we take $\phi = \phi_0$ to compatible with Definition 1.

Proposition 5. *Let $n \in \mathbb{Z}$ and*

$$\phi_i : \mathbb{Q}_p \to \mathbb{Q}_p$$
$$\sum_{k=m}^{\infty} a_k p^k \mapsto i p^n + \sum_{k=m}^{\infty} a_k p^{k+1}$$

where $i \in \{0, 1, \ldots, p-1\}$. Then, the attractor of the iterated function system $\{\mathbb{Q}_p, \{\phi_i\}_{i\in\{0,1,\ldots,p-1\}}\}$ is

$$G_n = \left\{ a_n p^n + a_{n+1} p^{n+1} + a_{n+2} p^{n+2} + \ldots \;\middle|\; a_i \in \{0, 1, \ldots, p-1\} \right\}.$$

Proof. It is well-known that $\mathbb{Q}_p$ is complete and $\mathbb{Z}_p$ is compact. Similarly, it is easily shown that G_n is compact for each $n \in \mathbb{Z}$. First, we show that ϕ_i is a contraction mapping for $i = 0, 1, 2, \ldots, p-1$. Let a and b be arbitrary elements of $\mathbb{Q}_p$. Then, there are $m, s \in \mathbb{Z}$ such that $a = \sum_{k=m}^{\infty} a_k p^k$ and $b = \sum_{k=s}^{\infty} b_k p^k$. If we define

$$d_p(a, b) = |a - b|_p = \frac{1}{p^t},$$

then we compute

$$d_p(\phi_i(a), \psi_i(b)) = |\phi_i(a) - \phi_i(b)|_p = \frac{1}{p^{t+1}}.$$

So, ϕ_i is a contraction mapping with contraction constant $\frac{1}{p}$ for $i = 0, 1, 2, \ldots, p-1$. Finally, we prove

$$G_n = \bigcup_{i=0}^{p-1} \phi_i(G_n).$$

Given an arbitrary $a = a_n p^n + a_{n+1} p^{n+1} + \ldots \in G_n$. If we take $b = a_{n+1} p^n + a_{n+2} p^{n+1} + \ldots \in G_n$, then we obtain $\phi_i(b) = a$ where $i = a_n$. It follows that

$$G_n \subseteq \bigcup_{i=0}^{p-1} \phi_i(G_n).$$

We now take an arbitrary $a \in \bigcup_{i=0}^{p-1} \phi_i(G_n)$. Therefore, there is $b = \sum_{j=n}^{\infty} b_j p^j \in G_n$ such that $a = \phi_i(b)$ for $i \in \{0, 1, \ldots, p-1\}$. This shows that $\phi_i(G_n) \subseteq G_n$ owing to the fact that

$$a = \phi_i(b) = ip^n + \sum_{j=n}^{\infty} b_j p^{j+1} \in G_n.$$

This concludes the proof. □

Corollary 6. *In the case of $n = 0$, it is $G_0 = \mathbb{Z}_p$ and it is well-known that its IFS is $\{\mathbb{Q}_p, \{\phi_i\}_{i \in \{0,1,\ldots,p-1\}}\}$ where*

$$\phi_i \colon \mathbb{Q}_p \to \mathbb{Q}_p$$
$$\sum_{k=m}^{\infty} a_k p^k \mapsto i + \sum_{k=m}^{\infty} a_k p^{k+1}$$

for $i = 0, 1, \ldots, p-1$.

Proposition 6. *For each $n \in \mathbb{Z}$, G_n is a strong self-similar group.*

Proof. It is well-known that $\mathbb{Z}_p$ is compact topological group. Similarly, one can show that G_n is compact topological group for each $n \in \mathbb{Z}$. It is obvious that d_p is a translation-invariant metric. In Proposition 5, we show that ϕ_0 is a contraction mapping. Moreover,

$$\phi_0(G_n) = \big\{a_n p^{n+1} + a_{n+1}p^{n+2} + a_{n+2}p^{n+3} + \dots \;\big|\; a_n \in \{0,1,2,\dots,p-1\}\big\}$$

is a subgroup of G_n of the index p. So, it is enough to show that ϕ_0 is a group homomorphism. Let $a = \sum_{k=n}^{\infty} a_k p^k$ and $b = \sum_{k=m}^{\infty} b_k p^k$ be arbitrary elements of $\mathbb{Q}_p$ where $n, m \in \mathbb{Z}$. By using (1), it is obtained that

$$a + b = \sum_{k=t}^{\infty} c_k p^k$$

where $t = \min\{n, m\}$ and $c_k \in \{0,1,2,\dots,p-1\}$. Thus, we compute

$$\phi_0(a+b) = \sum_{k=t}^{\infty} c_k p^{k+1}.$$

Since $\phi_0(a) = \sum_{k=n}^{\infty} a_k p^{k+1}$ and $\phi_0(b) = \sum_{k=m}^{\infty} b_k p^{k+1}$, it is easily seen that

$$\phi_0(a) + \phi_0(b) = \sum_{k=t}^{\infty} c_k p^{k+1}.$$

That is, ϕ_0 is a group homomorphism. Moreover, it is obvious that ϕ_0 is one-to--one. This shows that G_n is a strong self-similar group for each $n \in \mathbb{Z}$.

□

Remark 2. We also obtain some self-similar subgroups of $\mathbb{Z}_p$ by using Corollary 3. It is well-known that $p\mathbb{Z}_p$ is a closed proper subgroup of $\mathbb{Z}_p$. If we consider the mapping

$$\phi\colon \mathbb{Z}_p \to p\mathbb{Z}_p$$
$$\sum_{k=0}^{\infty} a_k p^k \mapsto \sum_{k=0}^{\infty} a_k p^{k+1}$$

then the subgroups of $\mathbb{Z}_p$

$$p\mathbb{Z}_p, p^2\mathbb{Z}_p, p^3\mathbb{Z}_p, \dots, p^n\mathbb{Z}_p, \dots$$

are strong self-similar subgroups.

Acknowledgements

The author thanks editor Stephen Glasby and the anonymous reviewers for their constructive suggestions. This paper is supported by the Anadolu University Research Fund under Contract 1306F169.

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Received: 21 December, 2016
Accepted for publication: 7 September, 2017
Communicated by: Stephen Glasby