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On $x^n + y^n = n!z^n$

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Abstract. In p. 219 of R.K. Guy's *Unsolved Problems in Number Theory*, 3rd edn., Springer, New York, 2004, we are asked to prove that the Diophantine equation $x^n + y^n = n!z^n$ has no integer solutions with $n \in \mathbb{N}_+$ and $n > 2$. But, contrary to this expectation, we show that for $n = 3$, this equation has infinitely many primitive integer solutions, i.e. the solutions satisfying the condition $\gcd(x, y, z) = 1$.

1 Introduction

In [4], Ribet showed that the equation $x^n + y^n = 2z^n$ has no solutions for $n > 2$ apart from the trivial $x = y = z$. Elkies [1] has also examined 'twists' of the Fermat equation, $x^3 + y^3 = nz^3$. In [2], Erdős and Obláh showed that $x^p \pm y^p = n!$ has no solution with $p > 2$. In p. 219 of [3], we are asked to prove that $x^n + y^n = n!z^n$ has no integer solutions with $n \in \mathbb{N}_+$ and $n > 2$. But, contrary to this anticipation, we prove the following Theorem.

Theorem 1. *The Diophantine equation*

$$x^n + y^n = n!z^n \tag{1}$$

has infinitely many primitive integer solutions when $n = 3$, *i.e. the solutions satisfying the condition* $\gcd(x, y, z) = 1$.

In fact, we show a technique to generate these solutions and give four numerical results. For other values of $n \geq 4$, we do not know if (1) has solutions in integers.

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2 Diophantine equation $x^3 + y^3 = 3!z^3$

The following Lemma will be used to prove Theorem 1.

Lemma 1. *For any given $a, b, c \in \mathbb{Z} \setminus \{0\}$ the Diophantine equation*

$$aA^3 + bB^3 = cC^3 \tag{2}$$

has infinitely many primitive integer solutions $(A_i, B_i, C_i)_{i \geq 0}$, i.e. the solutions satisfying the condition $\gcd(aA_i, bB_i, cC_i) = 1$ if (A_0, B_0, C_0) is a solution of this equation such that $\gcd(aA_0, bB_0, cC_0) = 1$.

Proof. The well known identity

$$m(m+2n)^3 - n(2m+n)^3 = (m+n)(m-n)^3, \tag{3}$$

which is true for any real values of m and n can be easily verified by direct expansion of its LHS and RHS terms. In (3), if we choose m and n to be coprime, *i.e.*, $\gcd(m, n) = 1$, then it is easy to show that

$$\begin{aligned} \gcd(m,(m+n)) &= \gcd(m,(m-n)) = \gcd\big((m+2n),(m+n)\big) \\ &= \gcd\big((m+2n),(m-n)\big) = \gcd(n,(m+n)) = \gcd(n,(m-n)) \\ &= \gcd\big((2m+n),(m+n)\big) = \gcd\big((2m+n),(m-n)\big) = 1. \end{aligned}$$

Hence, with coprime integers m and n, the three terms: $m(m+2n)^3$, $n(2m+n)^3$ and $(m+n)(m-n)^3$ of (3) are pairwise coprime.
Since (A_0, B_0, C_0) is a primitive solution of (2), we get

$$aA_0^3 + bB_0^3 = cC_0^3. \tag{4}$$

Taking $m = aA_0^3$, $n = bB_0^3$ in (3) and using (4), we have

$$a\big(A_0(aA_0^3 + 2bB_0^3)\big)^3 + b\big(-B_0(2aA_0^3 + bB_0^3)\big)^3 = c\big(C_0(aA_0^3 - bB_0^3)\big)^3. \tag{5}$$

Since the three terms of (4) are pairwise coprime, the three terms of resulting equation (5), after substitution in (3), will also be pairwise coprime.

Comparing (5) with (2), we see that an initial primitive integer solution (A_0, B_0, C_0) of (2) will lead to the next primitive integer solution:

$$(A_1, B_1, C_1) = \big(A_0(aA_0^3 + 2bB_0^3), -B_0(2aA_0^3 + bB_0^3), C_0(aA_0^3 - bB_0^3)\big). \tag{6}$$

As (A_1, B_1, C_1) is a primitive solution of (2), we get

$$aA_1^3 + bB_1^3 = cC_1^3. \tag{7}$$

Taking $m = aA_1^3$, $n = bB_1^3$ in (3) and using (7), we have

$$a\big(A_1(aA_1^3 + 2bB_1^3)\big)^3 + b\big(-B_1(2aA_1^3 + bB_1^3)\big)^3 = c\big(C_1(aA_1^3 - bB_1^3)\big)^3. \tag{8}$$

So,

$$(A_2, B_2, C_2) = \big(A_1(aA_1^3 + 2bB_1^3), -B_1(2aA_1^3 + bB_1^3), C_1(aA_1^3 - bB_1^3)\big). \tag{9}$$

Since the three terms of (7) are pairwise coprime, the three terms of resulting equation (8), after substitution in (3), will also be pairwise coprime. Thus, any primitive integer solution (A_i, B_i, C_i) of (2) is related to its next primitive integer solution $(A_{i+1}, B_{i+1}, C_{i+1})$ as

$$(A_{i+1}, B_{i+1}, C_{i+1}) = \big(A_i(aA_i^3 + 2bB_i^3), -B_i(2aA_i^3 + bB_i^3), C_i(aA_i^3 - bB_i^3)\big). \tag{10}$$

Thus, by iteration, (2) can have infinitely many primitive integer solutions, once a primitive integer solution of this equation is known. □

Proof. (Theorem 1) By putting $n = 3$ in (1), we get

$$x^3 + y^3 = 3!z^3. \tag{11}$$

Taking $a = b = 1$, $c = 3!$ and replacing (A, B, C) by (x, y, z), the equation (2) transforms into equation (11). Now, to show that (11) has infinitely many primitive integer solutions, the first step is to find three integers $x_0, y_0, z_0 \in \mathbb{Z} \setminus \{0\}$ such that

$$x_0^3 + y_0^3 = 3!z_0^3, \tag{12}$$

and $\gcd(x_0, y_0, 3!z_0) = 1$.

For any real values of p and q the identity

$$(3p + q)^3 + (3p - q)^3 = 3!p(9p^2 + 3q^2) \tag{13}$$

can be easily proved to be true.
Putting $p = 9$ and $q = 10$ in (13), we get

$$37^3 + 17^3 = 3! \times 21^3. \tag{14}$$

Comparing (14) with (12), we get

$$x_0, y_0, z_0 = 37, 17, 21; \tag{15}$$

and

$$\gcd(x_0, y_0, 3!z_0) = \gcd(37, 17, 3!21) = 1. \tag{16}$$

Now, (11) has an initial solution given by (15) and this solution is primitive because of the condition (16). To find the solutions of (11) explicitly, take $a = b = 1$, and replace (A_i, B_i, C_i) by (x_i, y_i, z_i) in (6), (9) and (10). Thus,

$$(x_1, y_1, z_1) = \big(x_0(x_0^3 + 2y_0^3), -y_0(2x_0^3 + y_0^3), z_0(x_0^3 - y_0^3)\big). \tag{17}$$

$$(x_2, y_2, z_2) = \big(x_1(x_1^3 + 2y_1^3), -y_1(2x_1^3 + y_1^3), z_1(x_1^3 - y_1^3)\big). \tag{18}$$

$$(x_{i+1}, y_{i+1}, z_{i+1}) = \big(x_i(x_i^3 + 2y_i^3), -y_i(2x_i^3 + y_i^3), z_i(x_i^3 - y_i^3)\big). \tag{19}$$

Hence, in accordance with Lemma 1, (11) has infinitely many primitive integer solutions. □

3 Some solutions of $x^3 + y^3 = 3!z^3$

Using (15), (17), (18) and (19), the first four primitive integer solutions of $x^3+y^3 = 3!z^3$ are calculated starting with $(x_0, y_0, z_0) = (37, 17, 21)$, and they are given in the Table 1. Since all of x_3, y_3 and z_3 are calculated to be 'negative', we took all of them to be 'positive', because both (x_3, y_3, z_3) and $(-x_3, -y_3, -z_3)$ are the solutions of the given equation. In Table 1, interchange $x_i \leftrightarrow y_i$ for solutions starting with $(y_0, x_0, z_0) = (37, 17, 21)$.

i	$x_i(y_i)$	$y_i(x_i)$	z_i
0	37	17	21
1	2237723	−1805723	960540
2	−1276454530530789	2983517129811443	1641849890114429
	553459441	3945011441	4337512360
3	6779598051038214	7922205726625496	4360668418820711
	2472326399266506	0819025292611212	1709500245932408
	1838773573375138	1617686087939438	5167366543342937
	7073793470619938	2456658060516086	4773448186461962
	6093375292356829	2111364183033645	7938530544150686
	7473185577965857	0448115419524772	1017701946929489
	67361	568639	111120

Table 1: First four solutions of $x^3 + y^3 = 3!z^3$

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