Communications in Mathematics 26 (2018) 15–29

DOI: 10.2478/cm-2018-0003

Nonlinear $*$-Lie higher derivations of standard operator algebras

Mohammad Ashraf, Shakir Ali, Bilal Ahmad Wani

Abstract. Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\mathfrak{A}$ be a standard operator algebra on $\mathcal{H}$ which is closed under the adjoint operation. It is shown that every nonlinear $*$-Lie higher derivation $\mathcal{D} = \{\delta_n\}_{n\in\mathbb{N}}$ of $\mathfrak{A}$ is automatically an additive higher derivation on $\mathfrak{A}$. Moreover, $\mathcal{D} = \{\delta_n\}_{n\in\mathbb{N}}$ is an inner $*$-higher derivation.

1 Introduction

Let $\mathfrak{A}$ be an algebra over a commutative ring R. Recall that an R-linear mapping $d\colon \mathfrak{A} \to \mathfrak{A}$ is called a derivation if $d(AB) = d(A)B + Ad(B)$ for all $A, B \in \mathfrak{A}$; in particular, d is called an inner derivation if there exists some $X \in \mathfrak{A}$ such that $d(A) = AX - XA$ for all $A \in \mathfrak{A}$. An R-linear mapping $d\colon \mathfrak{A} \to \mathfrak{A}$ is called a Lie derivation if $d([A, B]) = [d(A), B] + [A, d(B)]$ for all $A, B \in \mathfrak{A}$, where $[A, B] = AB - BA$ is the usual Lie product. Furthermore, without linearity/additivity assumption, if d satisfies $d([A, B]) = [d(A), B] + [A, d(B)]$ for all $A, B \in \mathfrak{A}$, then d is called a nonlinear Lie derivation. The question of characterizing Lie derivations and revealing the relationship between derivations and Lie derivations have been studied by many authors (see [1], [2], [5], [6], [7], [8], [11], [12], [15], [18]).

2010 MSC: 47B47, 16W25, 46K15.

Key words: Nonlinear $*$-Lie derivation, nonlinear $*$-Lie higher derivation, additive $*$-higher derivation, standard operator algebra.

Affiliation:

Mohammad Ashraf – Department of Mathematics, Aligarh Muslim University, Aligarh-202002 India
E-mail: mashraf80@hotmail.com

Shakir Ali – Department of Mathematics, Aligarh Muslim University, Aligarh-202002 India
E-mail: shakir50@rediffmail.com

Bilal Ahmad Wani – Department of Mathematics, Aligarh Muslim University, Aligarh-202002 India
E-mail: bilalwanikmr@gmail.com

Let $\mathfrak{A}$ be an associative $*$-algebra over the complex field $\mathbb{C}$. A mapping $d\colon \mathfrak{A} \to \mathfrak{A}$ is said to be an additive $*$-derivation if it is an additive derivation and satisfies $d(A)^* = d(A^*)$ for all $A \in \mathfrak{A}$. Further, if $d\colon \mathfrak{A} \to \mathfrak{A}$ is a map (not necessarily linear) which satisfies $d([A,B]_*) = [d(A),B]_* + [A,d(B)]_*$ for all $A, B \in \mathfrak{A}$, where $[A,B]_* = AB - BA^*$, then d is known as a nonlinear $*$-Lie derivation of $\mathfrak{A}$.

In [16] Yu and Zhang showed that every nonlinear $*$-Lie derivation from a factor von Neumann algebra on an infinite-dimensional Hilbert space into itself is an additive $*$-derivation. It is to be noted that a factor von Neumann algebra is a von Neumann algebra whose centre is trivial. In [4] Wu Jing proved that every nonlinear $*$-Lie derivation on standard operator algebra is automatically linear. Moreover, it is an inner $*$-derivation .

Let us recall some basic facts related to Lie higher derivations and $*$-Lie higher derivations of an associative algebra. Many different kinds of higher derivations, which consist of a family of some additive mappings, have been widely studied in commutative and noncommutative rings. Let $\mathbb{N}$ be the set of non-negative integers and $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ be a family of linear mappings $d_n\colon \mathfrak{A} \to \mathfrak{A}$ such that $d_0 = \mathrm{id}_{\mathfrak{A}}$, the identity map on $\mathfrak{A}$. Then $\mathcal{D}$ is called

(i) a higher derivation on $\mathfrak{A}$ if for every $n \in \mathbb{N}$,

$$d_n(AB) = \sum_{i+j=n} d_i(A)d_j(B)$$

for all $A, B \in \mathfrak{A}$.

(ii) a Lie higher derivation on $\mathfrak{A}$ if for every $n \in \mathbb{N}$,

$$d_n([A,B]) = \sum_{i+j=n} \big[d_i(A), d_j(B)\big]$$

for all $A, B \in \mathfrak{A}$.

(iii) a $*$-Lie higher derivation on $\mathfrak{A}$ if for every $n \in \mathbb{N}$,

$$d_n([A,B]_*) = \sum_{i+j=n} \big[d_i(A), d_j(B)\big]_*$$

for all $A, B \in \mathfrak{A}$.

(iv) an inner higher derivation on $\mathfrak{A}$ if there exist two sequences $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ in $\mathfrak{A}$ satisfying the conditions

$$X_0 = Y_0 = 1 \quad \text{and} \quad \sum_{i=0}^{n} X_i Y_{n-i} = \delta_{n0} = \sum_{i=0}^{n} Y_i X_{n-i}$$

such that $d_n(A) = \sum_{i=0}^{n} X_i A Y_{n-i}$, for all $A \in \mathfrak{A}$ and for every $n \in \mathbb{N}$, where δ_{n0} is the Kronecker sign.

If the linear assumption in the above definitions is dropped, then the corresponding higher derivation, Lie higher derivation and $*$-Lie higher derivation is said to be nonlinear higher derivation, nonlinear Lie higher derivation and nonlinear $*$-Lie higher derivation respectively. Moreover, if $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ is assumed to be the family of additive mappings, then in the above definition higher derivation, Lie higher derivation and $*$-Lie higher derivation is said to be additive higher derivation, additive Lie higher derivation and additive $*$-Lie higher derivation respectively. Note that d_1 is always a derivation, Lie derivation and $*$-Lie derivation if $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ is a higher derivation, Lie higher derivation and $*$-Lie higher derivation respectively.

The objective of this article is to investigate nonlinear $*$-Lie higher derivations on standard operator algebras which are closed under adjoint operation in infinite-dimensional complex Hilbert spaces. Many researchers have made important contributions to the related topics (see [3], [9], [13]). Xiao [14] proved that every nonlinear Lie higher derivation of triangular algebras is the sum of an additive higher derivation and a nonlinear functional vanishing on all commutators. Qi and Hou [10] gave a characterization of Lie higher derivations on nest algebras. Zhang et al., [17] showed that every nonlinear $*$-Lie higher derivation on factor von Neumann algebra is linear. Motivated by the above work in this article, we study nonlinear $*$-Lie higher derivations on standard operator algebras .

2 Nonlinear $*$-Lie higher derivations

Throughout this paper, $\mathbb{R}$ and $\mathbb{C}$ represents the set of real numbers and complex numbers respectively and $\mathcal{H}$ represents a complex Hilbert space. By $\mathcal{B}(\mathcal{H})$ we mean the algebra of all bounded linear operators on $\mathcal{H}$. Denote by $\mathcal{F}(\mathcal{H})$ the subalgebra of bounded finite rank operators. It is to be noted that $\mathcal{F}(\mathcal{H})$ forms a $*$-closed ideal in $\mathcal{B}(\mathcal{H})$. An algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ is said to be standard operator algebra in case $\mathcal{F}(\mathcal{H}) \subset \mathfrak{A}$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be a projection provided $P^* = P$ and $P^2 = P$. Note that, different from von Neumann algebras which are always weakly closed, a standard operator algebra is not necessarily closed. Recall that an algebra $\mathfrak{A}$ is prime if $A\mathfrak{A}B = 0$ implies either $A = 0$ or $B = 0$. It is to be noted that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem. Motivated by the work of Jing [4], we have obtained the following main result.

Theorem 1. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\mathfrak{A}$ be a standard operator algebra on $\mathcal{H}$ containing identity operator I. If $\mathfrak{A}$ is closed under the adjoint operation, then every nonlinear $*$-Lie higher derivation $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ from $\mathfrak{A}$ to $\mathcal{B}(\mathcal{H})$ is an additive $*$-higher derivation.*

Now take a projection $P_1 \in \mathfrak{A}$ and let $P_2 = I - P_1$. We write $\mathfrak{A}_{jk} = P_j\mathfrak{A}P_k$ for $j, k = 1, 2$. Then by Peirce decomposition of $\mathfrak{A}$ we have $\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{12} \oplus \mathfrak{A}_{21} \oplus \mathfrak{A}_{22}$. Note that any operator $A \in \mathfrak{A}$ can be expressed as $A = A_{11} + A_{12} + A_{21} + A_{22}$, and $A^*_{jk} \in \mathfrak{A}_{kj}$ for any $A_{jk} \in \mathfrak{A}_{jk}$.

We facilitate our discussion with the following known results.

Lemma 1. *[4, Lemma 2.1] Let $\mathfrak{A}$ be a standard operator algebra containing identity operator I in a complex Hilbert space which is closed under the adjoint operation. If $AB = BA^*$ holds true for all $B \in \mathfrak{A}$, then $A \in \mathbb{R}I$.*

Lemma 2. *[4, Proposition 2.7] Let $\mathfrak{A}$ be a standard operator algebra containing identity operator I in a complex Hilbert space which is closed under the adjoint operation. For any $A \in \mathfrak{A}$,*

(i) $[iP_1, A]_* = 0$ *implies* $A_{11} = A_{12} = A_{21} = 0$.

(ii) $[iP_2, A]_* = 0$ *implies* $A_{12} = A_{21} = A_{22} = 0$.

(iii) $[i(P_2 - P_1), A]_* = 0$ *implies* $A_{11} = A_{22} = 0$.

Now we shall use the hypothesis of Theorem 1 freely without any specific mention in proving the following lemmas.

Lemma 3. $d_n(0) = 0$ *for each* $n \in \mathbb{N}$.

Proof. We proceed by induction on $n \in \mathbb{N}$ with $n \geq 1$. If $n = 1$, by [4, Lemma 2.2], the result is true. Now assume that the result is true for $k < n$, i.e., $d_k(0) = 0$. Our aim is to show that d_n satisfies the similar property. Observe that

$$d_n(0) = d_n([0,0]_*) = \sum_{i+j=n} [d_i(0), d_j(0)]_* = [d_n(0), 0]_* + [0, d_n(0)]_* = 0.$$

□

Lemma 4. d_n *has the following properties:*

(i) *For any* $\lambda \in \mathbb{R}$, $d_n(\lambda I) \in \mathbb{R}I$.

(ii) *For any* $A \in \mathfrak{A}$ *with* $A = A^*$, $d_n(A) = d_n(A^*) = d_n(A)^*$.

(iii) *For any* $\lambda \in \mathbb{C}$, $d_n(\lambda I) \in \mathbb{C}I$.

Proof. We proceed by induction on $n \in \mathbb{N}$ with $n \geq 1$. By Lemmas 2.3, 2.4 & 2.5 of [4] the result is true for $n = 1$.

Assume that the result is true for $k < n$, i.e.,

$$d_k(\lambda I) \in \mathbb{R}I,\ d_k(A) = d_k(A^*) = d_k(A)^*,\ d_k(\lambda I) \in \mathbb{C}I.$$

Our aim is to show that d_n satisfies the similar property. By the induction hypothesis;

(i) For any $\lambda \in \mathbb{R}$, since $d_k(\lambda I) \in \mathbb{R}I$, i.e., $d_k(\lambda I) = d_k(\lambda I)^* \in \mathbb{R}I$

$$\begin{aligned} 0 &= d_n([\lambda I, A]_*) = [d_n(\lambda I), A]_* + [\lambda I, d_n(A)]_* + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [d_i(\lambda I), d_j(A)]_* \\ &= d_n(\lambda I)A - Ad_n(\lambda I)^* . \end{aligned}$$

This gives us that $d_n(\lambda I)A = Ad_n(\lambda I)^*$. By Lemma 1, we have $d_n(\lambda I) \in \mathbb{R}I$.

(ii) Using (i), we have for $A = A^*$

$$0 = d_n([A, I]_*) = [d_n(A), I]_* + [A, d_n(I)]_* + \sum_{\substack{i+j=n \\ 0<i,j\le n-1}} [d_i(A), d_j(I)]_*$$
$$= d_n(A) - d_n(A)^* .$$

(iii) For any $\lambda \in \mathbb{C}$ and $A \in \mathfrak{A}$ with $A = A^*$, applying (ii), we see that

$$0 = d_n([A, \lambda I]_*) = [d_n(A), \lambda I]_* + [A, d_n(\lambda I)]_* + \sum_{\substack{i+j=n \\ 0<i,j\le n-1}} [d_i(A), d_j(\lambda I)]_*$$
$$= Ad_n(\lambda I) - d_n(\lambda I)A .$$

This yields that $d_n(\lambda I)A = Ad_n(\lambda I)$ for all $A \in \mathfrak{A}$ with $A = A^*$, and hence $d_n(\lambda I) \in \mathbb{C}I$.

□

Lemma 5. *$d_n(\frac{1}{2}iI) = 0$ for each $n \in \mathbb{N}$ with $n \ge 1$ and $d_n(iA) = id_n(A)$ for all $A \in \mathfrak{A}$.*

Proof. The result is true for $n = 1$ by [4, Lemma 2.6]. Assume that the result is true for $k < n$, i.e., $d_k(\frac{1}{2}iI) = 0$. Now we compute

$$d_n\Big(-\frac{1}{2}I\Big) = d_n\Big(\Big[\frac{1}{2}iI, \frac{1}{2}iI\Big]_*\Big)$$
$$= \Big[d_n\Big(\frac{1}{2}iI\Big), \frac{1}{2}iI\Big]_* + \Big[\frac{1}{2}iI, d_n\Big(\frac{1}{2}iI\Big)\Big] + \sum_{\substack{p+q=n \\ 0<p,q\le n-1}} \Big[d_p\Big(\frac{1}{2}iI\Big), d_q\Big(\frac{1}{2}iI\Big)\Big]_*$$
$$= id_n\Big(\frac{1}{2}iI\Big) + \frac{1}{2}i\Big\{d_n\Big(\frac{1}{2}iI\Big) - d_n\Big(\frac{1}{2}iI\Big)\Big\}^*.$$

Since both $d_n\big(-\frac{1}{2}I\big)$ and $\frac{1}{2}i\big\{d_n\big(\frac{1}{2}iI\big) - d_n\big(\frac{1}{2}iI\big)\big\}^*$ are self-adjoint, $id_n\big(\frac{1}{2}iI\big)$ is also self-adjoint, and hence it follows that

$$d_n\Big(\frac{1}{2}iI\Big) = -d_n\Big(\frac{1}{2}iI\Big)^*.$$

Thus, the above computation gives that

$$d_n\Big(-\frac{1}{2}I\Big) = 2id_n\Big(\frac{1}{2}iI\Big) . \tag{1}$$

Similarly, we can obtain from the fact $\big[-\frac{1}{2}iI, -\frac{1}{2}iI\big] = \frac{1}{2}I$ that $d_n\big(-\frac{1}{2}iI\big)^* = -d_n\big(-\frac{1}{2}iI\big)$ and $d_n\big(-\frac{1}{2}I\big) = -2id_n\big(-\frac{1}{2}iI\big)$. Thus $d_n\big(-\frac{1}{2}iI\big) = -d_n\big(\frac{1}{2}iI\big)$. Now we compute

$$\begin{aligned} d_n\Big(\frac{1}{2}iI\Big) &= d_n\Big(\Big[-\frac{1}{2}iI, -\frac{1}{2}I\Big]_*\Big) \\ &= \Big[d_n\Big(-\frac{1}{2}iI\Big), -\frac{1}{2}I\Big]_* + \Big[-\frac{1}{2}iI, d_n\Big(-\frac{1}{2}I\Big)\Big]_* \\ &\quad + \sum_{\substack{p+q=n \\ 0<p,q\leq n-1}} \Big[d_p\Big(-\frac{1}{2}iI\Big), d_q\Big(\frac{1}{2}I\Big)\Big]_* \\ &= -id_n\Big(-\frac{1}{2}iI\Big) - id_n\Big(-\frac{1}{2}I\Big) = d_n\Big(\frac{1}{2}iI\Big) - id_n\Big(-\frac{1}{2}I\Big). \end{aligned}$$

It follows that $d_n\big(-\frac{1}{2}I\big) = 0$, and so, by the equality (1), we have $d_n\big(\frac{1}{2}iI\big) = 0$. Now, for any $A \in \mathfrak{A}$, we have by induction hypothesis

$$\begin{aligned} d_n(iA) &= d_n\Big(\Big[\frac{1}{2}iI, A\Big]_*\Big) \\ &= \Big[d_n\Big(\frac{1}{2}iI\Big), -A\Big]_* + \Big[\frac{1}{2}iI, d_n(A)\Big]_* + \sum_{\substack{p+q=n \\ 0<p,q\leq n-1}} \Big[d_p\Big(\frac{1}{2}iI\Big), d_q(A)\Big]_* \\ &= id_n(A). \end{aligned}$$

□

Lemma 6. *For any $A_{12} \in \mathfrak{A}_{12}$ and $B_{21} \in \mathfrak{A}_{21}$,*

$$d_n(A_{12} + B_{21}) = d_n(A_{12}) + d_n(B_{21}).$$

Proof. We proceed by induction on $n \in \mathbb{N}$ with $n \geq 1$. By [4, Lemma 2.8] the result is true for $n = 1$.

Assume that the result is true for $k < n$, i.e., $d_k(A_{12}+B_{21}) = d_k(A_{12})+d_k(B_{21})$. Let $M = d_n(A_{12} + B_{21}) - d_n(A_{12}) - d_n(B_{21})$. We now show that $M = 0$.

By the induction hypothesis, we have

$$\begin{aligned} 0 &= d_n\big([i(P_2 - P_1), A_{12} + B_{21}]_*\big) \\ &= \big[d_n\big(i(P_2 - P_1)\big), A_{12} + B_{21}\big]_* + \big[i(P_2 - P_1), d_n(A_{12} + B_{21})\big]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r\big(i(P_2 - P_1)\big), d_s(A_{12} + B_{21})\big]_* \\ &= \big[d_n\big(i(P_2 - P_1)\big), A_{12} + B_{21}\big]_* + \big[i(P_2 - P_1), d_n(A_{12} + B_{21})\big]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r\big(i(P_2 - P_1)\big), d_s(A_{12}) + d_s(B_{21})\big]_*. \end{aligned}$$

On the other hand,

$$\begin{aligned}
0 &= d_n\big(\big[i(P_2-P_1),A_{12}\big]_*\big)+d_n\big(\big[i(P_2-P_1),B_{21}\big]_*\big)\\
&= \big[d_n\big(i(P_2-P_1)\big),A_{12}\big]_*+\big[i(P_2-P_1),d_n(A_{12})\big]_*\\
&\quad+\sum_{\substack{r+s=n\\0<r,s\leq n-1}}\big[d_r\big(i(P_2-P_1)\big),d_s(A_{12})\big]_*+\big[d_n\big(i(P_2-P_1)\big),B_{21}\big]_*\\
&\quad+\big[i(P_2-P_1),d_n(B_{21})\big]_*+\sum_{\substack{r+s=n\\0<r,s\leq n-1}}\big[d_r\big(i(P_2-P_1)\big),d_s(B_{21})\big]_*\\
&= \big[d_n\big(i(P_2-P_1)\big),A_{12}+B_{21}\big]_*+\big[i(P_2-P_1),d_n(A_{12})+d_n(B_{21})\big]_*\\
&\quad+\sum_{\substack{r+s=n\\0<r,s\leq n-1}}\big[d_r\big(i(P_2-P_1)\big),d_s(A_{12})+d_s(B_{21})\big]_*.
\end{aligned}$$

Comparing the above two equations, we arrive at $[i(P_2-P_1),M]_*=0$. It follows from Lemma 2 that $M_{11}=M_{22}=0$. Now we calculate $d_n(A_{12}-A_{12}^*)$ in two ways

$$\begin{aligned}
d_n(A_{12}-A_{12}^*) &= d_n\big([A_{12}+B_{21},P_2]_*\big)\\
&= \big[d_n(A_{12}+B_{21}),P_2\big]_*+\big[A_{12}+B_{21},d_n(P_2)\big]_*\\
&\quad+\sum_{\substack{r+s=n\\0<r,s\leq n-1}}\big[d_r(A_{12}+B_{21}),d_s(P_2)\big]_*\\
&= \big[d_n(A_{12}+B_{21}),P_2\big]_*+\big[A_{12}+B_{21},d_n(P_2)\big]_*\\
&\quad+\sum_{\substack{r+s=n\\0<r,s\leq n-1}}\big[d_r(A_{12})+d_r(B_{21}),d_s(P_2)\big]_*.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d_n(A_{12}-A_{12}^*) &= d_n\big([A_{12},P_2]\big)_*+d_n\big([B_{21},P_2]\big)_*\\
&= \big[d_n(A_{12}),P_2\big]_*+\big[A_{12},d_n(P_2)\big]_*\\
&\quad+\sum_{\substack{r+s=n\\0<r,s\leq n-1}}\big[d_r(A_{12}),d_s(P_2)\big]_*\\
&\quad+\big[d_n(B_{21}),P_2\big]_*+\big[B_{21},d_n(P_2)\big]_*\\
&\quad+\sum_{\substack{r+s=n\\0<r,s\leq n-1}}\big[d_r(B_{21}),d_s(P_2)\big]_*\\
&= \big[d_n(A_{12})+d_n(B_{21}),P_2\big]_*+\big[A_{12}+B_{21},d_n(P_2)\big]_*\\
&\quad+\sum_{\substack{r+s=n\\0<r,s\leq n-1}}\big[d_r(A_{12})+d_r(B_{21}),d_s(P_2)\big]_*.
\end{aligned}$$

The above two identities give us that $[M, P_2]_* = 0$. But

$$[M, P_2]_* = MP_2 - P_2M^* = (M_{12} + M_{21})P_2 - P_2(M_{12}^* + M_{21}^*) = M_{12} - M_{12}^*.$$

Hence it follows that $M_{12} = 0$.

Similarly, using the fact that

$$\begin{aligned} d_n(B_{21} - B_{21}^*) &= d_n\big([A_{12} + B_{21}, P_1]_*\big) \\ &= d_n\big([A_{12}, P_1]\big)_* + d_n\big([B_{21}, P_1]\big)_*, \end{aligned}$$

one can show that $M_{21} = 0$. □

Lemma 7. *For any $A_{11} \in \mathfrak{A}_{11}$, $B_{12} \in \mathfrak{A}_{12}$, $C_{21} \in \mathfrak{A}_{21}$ and $D_{22} \in \mathfrak{A}_{22}$;*

(i) $d_n(A_{11} + B_{12} + C_{21}) = d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})$.

(ii) $d_n(B_{12} + C_{21} + D_{22}) = d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22})$.

Proof. (i) We proceed by induction on $n \in \mathbb{N}$ with $n \geq 1$. By [4, Lemma 2.9] the result is true for $n = 1$.

Assume that the result is true for $k < n$, that is,

$$d_k(A_{11} + B_{12} + C_{21}) = d_k(A_{11}) + d_k(B_{12}) + d_k(C_{21}).$$

Let

$$M = d_n(A_{11} + B_{12} + C_{21}) - d_n(A_{11}) - d_n(B_{12}) - d_n(C_{21}).$$

We now show that $M = 0$.

By the induction hypothesis, we have by Lemma 6,

$$\begin{aligned} d_n(iB_{12}) + d_n(iC_{21}) &= d_n(iB_{12} + iC_{21}) \\ &= d_n\big([iP_2, A_{11} + B_{12} + C_{21}]_*\big) \\ &= \big[d_n(iP_2), A_{11} + B_{12} + C_{21}\big]_* \\ &\quad + \big[iP_2, d_n(A_{11} + B_{12} + C_{21})\big]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(iP_2), d_s(A_{11} + B_{12} + C_{21})\big]_* \\ &= \big[d_n(iP_2), A_{11} + B_{12} + C_{21}\big]_* \\ &\quad + \big[iP_2, d_n(A_{11} + B_{12} + C_{21})\big]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(iP_2), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})\big]_*. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} d_n(iB_{12}) + d_n(iC_{21}) &= d_n\big([iP_2, A_{11}]_*\big) + d_n\big([iP_2, B_{21}]_*\big) + d_n\big([iP_2, C_{21}]_*\big) \\ &= \big[d_n(iP_2), A_{11}\big]_* + \big[iP_2, d_n(A_{11})\big]_* + \sum_{\substack{r+s=n\\0<r,s\leq n-1}} \big[d_r(iP_2), d_s(A_{11})\big]_* \\ &\quad + \big[d_n(iP_2), B_{12}\big]_* + \big[iP_2, d_n(B_{12})\big]_* + \sum_{\substack{r+s=n\\0<r,s\leq n-1}} \big[d_r(iP_2), d_s(B_{12})\big]_* \\ &\quad + \big[d_n(iP_2), C_{21}\big]_* + \big[iP_2, d_n(C_{21})\big]_* + \sum_{\substack{r+s=n\\0<r,s\leq n-1}} \big[d_r(iP_2), d_s(C_{21})\big]_* \\ &= \big[d_n(iP_2), A_{11} + B_{12} + C_{21}\big]_* + \big[iP_2, d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})\big]_* \\ &\quad + \sum_{\substack{r+s=n\\0<r,s\leq n-1}} \big[d_r(iP_2), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})\big]_*. \end{aligned}$$

Comparing the above two equalities, we have $[iP_2, M]_* = 0$ and hence it follows from Lemma 2 (ii), that $M_{12} = M_{21} = M_{22} = 0$.

We now show that $M_{11} = 0$. Note that

$$\big[i(P_2 - P_1), B_{12}\big]_* = \big[i(P_2 - P_1), C_{21}\big]_* = 0.$$

We have

$$\begin{aligned} d_n\big([i(P_2 - P_1), A_{11} + B_{12} + C_{21}]_*\big) &= d_n\big([i(P_2 - P_1), A_{11}]_*\big) \\ &\quad + d_n\big([i(P_2 - P_1), B_{12}]_*\big) + d_n\big([i(P_2 - P_1), C_{21}]_*\big). \end{aligned}$$

Using the similar arguments as used above, we get $[i(P_2 - P_1), M]_* = 0$. Therefore by Lemma 2, $M_{11} = 0$. Hence we are done.

(ii) Considering $d_n([iP_1, B_{12}+C_{21}+D_{22}]_*)$ and $d_n([i(P_2-P_1), B_{12}+C_{21}+D_{22}]_*)$, with the similar argument as in (i), one can obtain

$$d_n(B_{12} + C_{21} + D_{22}) = d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22}).$$

□

Lemma 8. *For any* $A_{11} \in \mathfrak{A}_{11}$, $B_{12} \in \mathfrak{A}_{12}$, $C_{21} \in \mathfrak{A}_{21}$ *and* $D_{22} \in \mathfrak{A}_{22}$*;*

$$d_n(A_{11} + B_{12} + C_{21} + D_{22}) = d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22}).$$

Proof. By [4, Lemma 2.10], the result is true for $n = 1$. Assume that the result is true for $k < n$, i.e.,

$$d_k(A_{11} + B_{12} + C_{21} + D_{22}) = d_k(A_{11}) + d_k(B_{12}) + d_k(C_{21}) + d_k(D_{22}).$$

Our aim is to show that the result is true for every $n \in \mathbb{N}$. Let

$$M = d_n(A_{11} + B_{12} + C_{21} + D_{22}) - d_n(A_{11}) - d_n(B_{12}) - d_n(C_{21}) - d_n(D_{22}).$$

Note that $[iP_1, D_{22}]_* = 0$, by induction hypothesis, we have

$$\begin{aligned}
d_n\big([iP_1, A_{11} + B_{12} + C_{21} + D_{22}]_*\big) &= [d_n(iP_1), A_{11} + B_{12} + C_{21} + D_{22}]_* \\
&\quad + \big[iP_1, d_n(A_{11} + B_{12} + C_{21} + D_{22})\big]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(iP_1), d_s(A_{11} + B_{12} + C_{21} + D_{22})\big]_* \\
&= \big[d_n(iP_1), A_{11} + B_{12} + C_{21} + D_{22}\big]_* \\
&\quad + \big[iP_1, d_n(A_{11} + B_{12} + C_{21} + D_{22})\big]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(iP_1), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21}) + d_s(D_{22})\big]_*.
\end{aligned}$$

On the other hand, we have by (i) of Lemma 7,

$$\begin{aligned}
d_n\big([iP_1, A_{11} + B_{12} + C_{21} + D_{22}]_*\big) \\
&= d_n\big([iP_1, A_{11} + B_{12} + C_{21}]_*\big) + d_n\big([iP_1, D_{22}]_*\big) \\
&= [d_n(iP_1), A_{11} + B_{12} + C_{21}]_* \\
&\quad + \big[iP_1, d_n(A_{11} + B_{12} + C_{21})\big]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(iP_1), d_s(A_{11} + B_{12} + C_{21})\big]_*. \\
&\quad + \big[d_n(iP_1), D_{22}\big]_* + \big[iP_1, d_n(D_{22})\big]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(iP_1), d_s(D_{22})\big]_* \\
&= \big[d_n(iP_1), A_{11} + B_{12} + C_{21}\big]_* \\
&\quad + \big[iP_1, d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})\big]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(iP_1), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})\big]_* \\
&\quad + \big[d_n(iP_1), D_{22}\big]_* + \big[iP_1, d_n(D_{22})\big]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(iP_1), d_s(D_{22})\big]_*.
\end{aligned}$$

Comparing the above two equalities, it follows that $[iP_1, M] = 0$, and hence by Lemma 2, $M_{11} = M_{12} = M_{21} = 0$. Using the fact that $[iP_2, A_{11}] = 0$ and the above similar arguments, we obtain $[iP_2, M]_* = 0$ which leads to $M_{22} = 0$. This completes the proof. □

Lemma 9. *For any $A_{jk}, B_{jk} \in \mathfrak{A}_{jk}$, where $j, k \in 1, 2$, we have*

$$d_n(A_{jk} + B_{jk}) = d_n(A_{jk}) + d_n(B_{jk})$$

Proof. We separate the proof in two distinct cases.

Case I: $j \neq k$

On one side, by Lemma 8, we have

$$d_n(iA_{jk} + iB_{jk} + iA^*_{jk} + iB_{jk}A^*_{jk}) \\ = d_n(iA_{jk} + iB_{jk}) + d_n(iA^*_{jk}) + d_n(iB_{jk}A^*_{jk})\,.$$

On the other hand, using Lemmas 6 and 8, by induction, we have

$$\begin{aligned} d_n(iA_{jk} + iB_{jk} + iA^*_{jk} + iB_{jk}A^*_{jk}) &= d_n\big([iP_j + iA_{jk}, P_k + B_{jk}]_*\big) \\ &= \big[d_n(iP_j + iA_{jk}), P_k + B_{jk}\big]_* + \big[iP_j + iA_{jk}, d_(P_k + B_{jk})\big]_* \\ &\quad + \sum_{\substack{r+s=n\\0<r,s\leq n-1}} \big[d_r(iP_j + iA_{jk}), d_s(P_k + B_{jk})\big]_* \\ &= \big[d_n(iP_j) + d_n(iA_{jk}), P_k + B_{jk}\big]_* \\ &\quad + \big[iP_j + iA_{jk}, d_(P_k) + d_n(B_{jk})\big]_* \\ &\quad + \sum_{\substack{r+s=n\\0<r,s\leq n-1}} \big[d_r(iP_j) + d_r(iA_{jk}), d_s(P_k) + d_s(B_{jk})\big]_* \\ &= d_n\big([iP_j, P_k]_*\big) + d_n\big([iP_j, B_{jk}]_*\big) \\ &\quad + d_n\big([iA_{jk}, P_k]_*\big) + d_n\big([iA_{jk}, B_{jk}]_*\big) \\ &= d_n(iB_{jk}) + d_n(iA_{jk} + iA^*_{jk}) + d_n(iB_{jk}A^*_{jk}) \\ &= d_n(iB_{jk}) + d_n(iA_{jk}) + d_n(iA^*_{jk}) + d_n(iB_{jk}A^*_{jk}). \end{aligned}$$

Comparing the above two equalities, we can conclude that

$$d_n(A_{jk} + B_{jk}) = d_n(A_{jk}) + d_n(A^*_{jk})\,.$$

Case II: $j = k$.

Let $A_{jj}, B_{jj} \in \mathfrak{A}_{jj}$ and $n \in \{1,2\}$ with $n \neq j$. We have

$$\begin{aligned} 0 &= d_n\big([iP_n, A_{jj} + B_{jj}]_*\big) \\ &= \big[d_n(iP_n), A_{jj} + B_{jj}\big]_* + \big[iP_n, d_n(A_{jj} + B_{jj})\big]_* \\ &\quad + \sum_{\substack{r+s=n\\0<r,s\leq n-1}} \big[d_r(iP_n), d_s(A_{jj} + B_{jj})\big]_* \\ &= \big[d_n(iP_n), A_{jj} + B_{jj}\big]_* + \big[iP_n, d_n(A_{jj} + B_{jj})\big]_* \\ &\quad + \sum_{\substack{r+s=n\\0<r,s\leq n-1}} \big[d_r(iP_n), d_s(A_{jj}) + d_s(B_{jj})\big]_*. \end{aligned}$$

On the other hand we have,

$$\begin{aligned}
0 &= d_n\big([iP_n, A_{jj}]_*\big) + d_n\big([iP_n, B_{jj}]_*\big) \\
&= \big[d_n(iP_n), A_{jj}\big]_* + \big[iP_n, d_n(A_{jj})\big]_* + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(iP_n), d_s(A_{jj})\big]_*. \\
&\quad + \big[d_n(iP_n), B_{jj}\big]_* + \big[iP_n, d_n(B_{jj})\big]_* + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(iP_n), d_s(B_{jj})\big]_* \\
&= \big[d_n(iP_n), A_{jj} + B_{jj}\big]_* + \big[iP_n, d_n(A_{jj}) + d_n(B_{jj})\big]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(iP_n), d_s(A_{jj}) + d_s(B_{jj})\big]_*.
\end{aligned}$$

Take $M = d_n(A_{jj} + B_{jj}) - d_n(A_{jj}) - d_n(B_{jj})$. The above computation yields that $[iP_n, M]_* = 0$. By Lemma 2, we have $M_{nj} = M_{jn} = M_{nn} = 0$. We now show that $M_{jj} = 0$. For any $C_{jn} \in \mathfrak{A}_{jn}$, using Case I, we compute

$$\begin{aligned}
d_n\big([A_{jj} + B_{jj}, C_{jn}]_*\big) &= \big[d_n(A_{jj} + B_{jj}), C_{jn}\big]_* + \big[A_{jj} + B_{jj}, d_n(C_{jn})\big]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(A_{jj} + B_{jj}), d_s(C_{jn})\big]_* \\
&= \big[d_n(A_{jj} + B_{jj}), C_{jn}\big]_* + \big[A_{jj} + B_{jj}, d_n(C_{jn})\big]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(A_{jj}) + d_r(B_{jj}), d_s(C_{jn})\big]_*.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
d_n\big([A_{jj} + B_{jj}, C_{jn}]_*\big) &= d_n\big(A_{jj}C_{jn} + B_{jj}C_{jn}\big) \\
&= d_n(A_{jj}C_{jn}) + d_n(B_{jj}C_{jn}) \\
&= d_n\big([A_{jj}, C_{jn}]_*\big) + d_n\big([B_{jj}, C_{jn}]_*\big) \\
&= \big[d_n(A_{jj}), C_{jn}\big]_* + \big[A_{jj}, d_n(C_{jn})\big]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(A_{jj}), d_s(C_{jn})\big]_* \\
&\quad + \big[d_n(B_{jj}), C_{jn}\big]_* + \big[B_{jj}, d_n(C_{jn})\big]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0<r,s\leq n-1}} \big[d_r(B_{jj}), d_s(C_{jn})\big]_*.
\end{aligned}$$

Comparing the above two equalities, we obtain $[M, C_{jn}]_* = 0$ which leads to $M_{jj}C_{jn} = 0$. Since $\mathfrak{A}$ is prime, we see that $M_{jj} = 0$, which completes the proof.

□

Lemma 10. *d_n is an additive $*$-higher derivation on $\mathfrak{A}$.*

Proof. We first show that d_n is additive. For arbitrary $A, B \in \mathfrak{A}$, we write $A = \sum_{j,k=1}^{2} A_{jk}$ and $B = \sum_{j,k=1}^{2} B_{jk}$. It follows from Lemmas 8 and 9 that

$$\begin{aligned} d_n(A+B) &= d_n\left\{\sum_{j,k=1}^{2}(A_{jk}+B_{jk})\right\} \\ &= \sum_{j,k=1}^{2} d_n(A_{jk}+B_{jk}) \\ &= \sum_{j,k=1}^{2} \big(d_n(A_{jk})+d_n(B_{jk})\big) \\ &= d_n\left(\sum_{j,k=1}^{2} A_{jk}\right) + d_n\left(\sum_{j,k=1}^{2} B_{jk}\right) \\ &= d_n(A)+d_n(B). \end{aligned}$$

We now show that $d_n(A^*) = d_n(A)^*$.

For any $A \in \mathfrak{A}$, it follows from Lemmas 4 and 5 that

$$\begin{aligned} d_n(A^*) &= d_n(\Re A - i\Im A) = d_n(\Re A) - d_n(i\Im A) \\ &= d_n(\Re A) - i d_n(\Im A) = d_n(\Re A)^* - i d_n(\Im A)^* \\ &= d_n(\Re A)^* + (i d_n(\Im A))^* = d_n(\Re A)^* + d_n(i\Im A)^* \\ &= (d_n(\Re A + i\Im A))^* = d_n(A)^*. \end{aligned}$$

To complete the proof, we need to show that d_n is a higher derivation on $\mathfrak{A}$.

Since d_n is additive, it follows from Lemma 5, that $d_n(iI) = 0$. It is to be noted that $[iI + A, B]_* = 2iB + AB - BA^*$.

$$\begin{aligned} d_n(2iB) + d_n(AB) - d_n(BA^*) &= d_n\big([iI+A,B]_*\big) \\ &= \big[d_n(iI+A),B\big]_* + \big[iI+A,d_n(B)\big]_* + \sum_{\substack{r+s=n\\0<r,s\le n-1}} \big[d_r(iI+A),d_s(B)\big]_* \\ &= \big[d_n(iI)+d_n(A),B\big]_* + \big[iI+A,d_n(B)\big]_* \\ &\quad + \sum_{\substack{r+s=n\\0<r,s\le n-1}} \big[d_r(iI)+d_r(A),d_s(B)\big]_* \\ &= \big[d_n(A),B\big]_* + \big[iI+A,d_n(B)\big]_* + \sum_{\substack{r+s=n\\0<r,s\le n-1}} \big[d_r(A),d_s(B)\big]_* \\ &= d_n(A)B - Bd_n(A)^* + 2id_n(B) + Ad_n(B) - d_n(B)A^* \\ &\quad + \sum_{\substack{r+s=n\\0<r,s\le n-1}} \big(d_r(A)d_s(B) - d_s(B)d_r(A)^*\big). \end{aligned}$$

It follows that

$$d_n(AB) - d_n(BA^*) = d_n(A)B - Bd_n(A)^* + Ad_n(B) - d_n(B)A^* + \sum_{\substack{r+s=n \\ 0<r,s\le n-1}} \big(d_r(A)d_s(B) - d_s(B)d_r(A)^*\big).$$

Replacing A by iA in the above equality, we get

$$d_n(AB) + d_n(BA^*) = d_n(A)B + Bd_n(A)^* + Ad_n(B) + d_n(B)A^* + \sum_{\substack{r+s=n \\ 0<r,s\le n-1}} \big(d_r(A)d_s(B) + d_s(B)d_r(A)^*\big).$$

Thus we have,

$$\begin{aligned} d_n(AB) &= d_n(A)B + Ad_n(B) + \sum_{\substack{r+s=n \\ 0<r,s\le n-1}} d_r(A)d_s(B) \\ &= \sum_{r+s=n} d_r(A)d_s(B). \end{aligned}$$

This shows that d_n is an additive higher derivation with $d_n(A^*) = d_n(A)^*$. Hence d_n is an additive $*$-higher derivation on $\mathfrak{A}$, which completes the proof. □

Note that every additive derivation $d\colon \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is an inner derivation (see [12]). Nowicki [9] proved that if every additive (linear) derivation of $\mathfrak{A}$ is inner, then every additive (linear) higher derivation of $\mathfrak{A}$ is inner (see also [13]). So by Theorem 1, the following corollary is immediate.

Corollary 1. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\mathfrak{A}$ be a standard operator algebra on $\mathcal{H}$ containing identity operator I. If $\mathfrak{A}$ is closed under the adjoint operation, then every nonlinear $*$-Lie higher derivation $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ is inner with $d_n(A^*) = d_n(A)^*$ for each $A \in \mathfrak{A}$ and every $n \in \mathbb{N}$.*

Acknowledgement

The authors are highly indebted to the referee for his/her valuable remarks which have improved the paper immensely.

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Received: 13 July, 2017
Accepted for publication: 2 February, 2018
Communicated by: Stephen Glasby