Communications in Mathematics 26 (2018) 31–45

DOI: 10.2478/cm-2018-0004

The Properties of the Weighted Space $H^k_{2,\alpha}(\Omega)$ and Weighted Set $W^k_{2,\alpha}(\Omega, \delta)$

V.A. Rukavishnikov, E.V. Matveeva, E.I. Rukavishnikova

Abstract. We study the properties of the weighted space $H^k_{2,\alpha}(\Omega)$ and weighted set $W^k_{2,\alpha}(\Omega, \delta)$ for boundary value problem with singularity.

1 Introduction

A boundary value problem is said to possess a strong singularity if its solution u does not belong to the Sobolev space W^1_2 (H^1) or, in other words, the Dirichlet integral of the solution u diverges.

Boundary value problems with strong singularity caused by the singularity in the initial data or by the internal properties of the solution are found in the physics of plasma and gas discharge, electrodynamics, nuclear physics, nonlinear optics, and other branches of physics. In some particular cases, numerical methods for problems of electrodynamics and quantum mechanics with strong singularity were constructed, based on separation of singular and regular components, mesh refinement near singular points, multiplicative extraction of singularities, etc. (see, e.g., [2]–[3], [8], [9], [11]).

Boundary value problems with weak singularity of a solution ($u \in H^1$) is caused by the presence of corner points on the boundary of a domain and by a change in the type of boundary conditions are found in different mathematical models. By using special methods for extracting the singular part of the solution near corner points or applying grids refined toward the singularity point, it is possible

2010 MSC: 46E35
Key words: weighted functional spaces, weighted functional sets, weighted Sobolev spaces.
Affiliation:
V.A. Rukavishnikov, E.I. Rukavishnikova, Computing Center of Far-Eastern Branch Russian Academy of Sciences, Kim-Yu-Chen Str. 65, Khabarovsk 680000, Russia
E-mail: vark0102@mail.ru, rukavishnikova-55@mail.ru
E.V. Matveeva, Far Eastern State Transport University, Serisheva Str. 47, Khabarovsk 680021, Russia
E-mail: gabitus-ev@mail.ru

to construct high order accurate finite-element schemes (see, e.g., [1], [5]–[7], [10], [12]–[16], [27]).

In [18], it was suggested to define the solution of boundary value problem for second-order elliptic equation with singularity on a finite set of points belonging to boundary of a of a two-dimensional domain as an R_ν -generalized solutions in the weighted Sobolev space. Such a new concept of solution led to the distinction of two classes of boundary value problems: problems with coordinated and uncoordinated degeneracy of input data; it also made it possible to study the existence and uniqueness of solutions as well as its coercivity and differential properties in the weighted spaces and weighted sets (see [18]–[21]). In [22]–[26], we constructed and investigated the finite-element method for different boundary value problems.

For investigation of the weighted finite element methods with high-degree accuracy for singular boundary value problems with coordinated and uncoordinated degeneration of input data we need to know properties of the weighted space $H^k_{2,\alpha}(\Omega)$ and weighted set $W^k_{2,\alpha}(\Omega,\delta)$. In the present paper we study the properties of $H^k_{2,\alpha}(\Omega)$ and $W^k_{2,\alpha}(\Omega,\delta)$.

2 Basic notations

We denote the two-dimensional Euclidean space by $\mathbb{R}^2$ with $x=(x_1,x_2)$ and $dx=dx_1dx_2$. Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with piecewise smooth boundary $\partial\Omega$, and let $\overline{\Omega}$ be the closure of Ω, i.e. $\overline{\Omega}=\Omega\cup\partial\Omega$. We denote by $\bigcup\limits_{i=1}^{n}\tau_i$ a set of points τ_i, $i=1,\dots,n$, belonging to $\partial\Omega$, including the points of intersection of its smooth pieces.

Let O_i^δ be a disk of radius $\delta>0$ with its center in τ_i, $i=1,\dots,n$, i.e.

$$O_i^\delta=\{x:\|x-\tau_i\|\le\delta\},$$

and suppose that $O_i^\delta\cap O_j^\delta=\emptyset$, $i\ne j$. Let $\Omega'=\bigcup\limits_{i=1}^{n}\Omega_i$, where $\Omega_i=\Omega\cap O_i^\delta$, $i=1,\dots,n$.

Let $\rho(x)$ be a function that is positive everywhere, except in $\bigcup\limits_{i=1}^{n}\tau_i$, and satisfies the following conditions:

1. $\rho(x)=\delta$ for $x\in\Omega\setminus\bigcup\limits_{i=1}^{n}O_i^\delta$,

2. $\rho(x)=[(x_1-x_1^{(i)})^2+(x_2-x_2^{(i)})^2]^{1/2}$, $(x_1^{(i)},x_2^{(i)})=\tau_i$, for $x\in\Omega_i$, $i=1,\dots,n$.

Moreover, it is assumed that

$$\left|\frac{\partial^{|\lambda|}\rho^k(x)}{\partial x_1^{\lambda_1}\partial x_2^{\lambda_2}}\right|\le\sigma\cdot\rho^{k-|\lambda|}(x).$$

We introduce the weighted spaces $H^k_{2,\alpha}(\Omega)$ and $W^k_{2,\alpha}(\Omega)$ with norms:

$$\|u\|_{H^k_{2,\alpha}(\Omega)} = \left(\sum_{|\lambda|\le k}\int_\Omega \rho^{2(\alpha+|\lambda|-k)}|D^\lambda u|^2\,dx\right)^{1/2},$$

$$\|u\|_{W^k_{2,\alpha}(\Omega)} = \left(\sum_{|\lambda|\le k}\int_\Omega \rho^{2\alpha}|D^\lambda u|^2\,dx\right)^{1/2}.$$

Here $D^\lambda = \dfrac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1}\partial x_2^{\lambda_2}}$, $\lambda=(\lambda_1,\lambda_2)$ and $|\lambda|=\lambda_1+\lambda_2$; λ_1,λ_2 are integer nonnegative numbers, α, σ are some real nonnegative numbers, k is an integer nonnegative number. For $k=0$ we use the notation $H^0_{2,\alpha}(\Omega)=W^0_{2,\alpha}(\Omega)=L_{2,\alpha}(\Omega)$.

By $W^l_{2,\alpha+l-1}(\Omega,\delta)$ for $l\ge 1$ we denote a set of functions satisfying the following conditions:

(a) $|D^k u(x)| \le c_1\cdot\gamma^k\cdot k!\cdot(\rho^{\alpha+k}(x))^{-1}$ for $x\in\Omega'$, where $k=1,\dots,l$, the constants $c_1,\ \gamma\ge 1$ do not depend on k;

(b) $\|u\|_{L_{2,\alpha}(\Omega\setminus\Omega')}\ge c_2>0$, c_2=const; with the squared norm

$$\|u\|^2_{W^l_{2,\alpha+l-1}(\Omega,\delta)} = \sum_{|\lambda|\le l}\left\|\rho^{\alpha+l-1}|D^\lambda u|\right\|^2_{L_2(\Omega)}.$$

Let $L_{2,\alpha}(\Omega,\delta)$ be the set of functions satisfying conditions (a), (b) the norm

$$\|u\|_{L_{2,\alpha}(\Omega,\delta)} = \left(\int_\Omega \rho^{2\alpha}u^2dx\right)^{1/2}.$$

Denote by r and θ the polar coordinates in Ω_i, $i=1,\dots,n$.

Introduce the space $\mathcal{L}_{2,\alpha}(\Omega)$ with the squared norm

$$\|u(r,\theta)\|^2_{\mathcal{L}_{2,\alpha}(\Omega)} = \int_{\Omega'} r^{2\alpha}u^2(r,\theta)ds + \int_{\Omega\setminus\Omega'}\delta^{2\alpha}u^2dx,$$

where $ds=rdrd\theta$, α is a nonnegative real number.

By $\mathcal{W}^l_{2,\alpha+l-1}(\Omega,\delta)$, where α is a nonnegative number and l is a nonnegative integer we denote the set of functions satisfying the following conditions:

(a') $\left|D^\lambda u(r,\theta)\right| \le \tilde c_1\cdot\tilde\gamma^{|\lambda|}\cdot|\lambda|!\cdot\left(r^{\alpha+\lambda_1}\right)^{-1}$ for $(r,\theta)\in\Omega'$, where

$$D^\lambda u = \frac{\partial^{|\lambda|}u}{\partial r^{\lambda_1}\partial\theta^{\lambda_2}} = u_{r^{\lambda_1}\theta^{\lambda_2}},$$

$|\lambda|=1,\dots,l$, $\tilde c_1,\tilde\gamma\ge 1$ are constants independent of λ_1;

(b') $\|u\|_{\mathcal{L}_{2,\alpha}(\Omega\setminus\Omega')}\ge\tilde c_2>0$; with the squared norm

$$\|u\|^2_{\mathcal{W}^l_{2,\alpha+l-1}(\Omega,\delta)} = \sum_{|\lambda|\le l} \|\rho^{\alpha+l-1-\lambda_2}|D^\lambda u|\|^2_{\mathcal{L}_2(\Omega)}.$$

Let $\mathcal{L}_{2,\alpha}(\Omega,\delta)$ be the set of functions satisfying conditions (a'), (b') with the squared norm

$$\|u(r,\theta)\|^2_{\mathcal{L}_{2,\alpha}(\Omega,\delta)} = \int\limits_{\Omega'} r^{2\alpha} u^2(r,\theta) ds + \int\limits_{\Omega\setminus\Omega'} \delta^{2\alpha} u^2 dx.$$

3 The properties of the weighted space $H^k_{2,\alpha}(\Omega)$

Lemma 1. *a) Let $u \in H^1_{2,\alpha}(\Omega)$, then $\rho^\alpha u \in W^1_{2,0}(\Omega)$, $\rho^{\alpha-1} u \in L_{2,0}(\Omega)$ and*

$$|\rho^\alpha u|_{W^1_{2,0}(\Omega)} + \|\rho^{\alpha-1} u\|_{L_{2,0}(\Omega)} \le c_3 \|u\|_{H^1_{2,\alpha}(\Omega)}\,, \tag{1}$$

where c_3 is a positive constant independent of u.

b) Let $\rho^\alpha u \in W^1_{2,0}(\Omega)$ and $\rho^{\alpha-1} u \in L_{2,0}(\Omega)$, then $u \in H^1_{2,\alpha}(\Omega)$ and there exist positive constant c_4 and c_5 independent of u such that

$$c_4\,|\rho^\alpha u|_{W^1_{2,0}(\Omega)} + c_5 \|\rho^{\alpha-1} u\|_{L_{2,0}(\Omega)} \ge \|u\|_{H^1_{2,\alpha}(\Omega)}\,, \tag{2}$$

is valid.

Proof. a) We estimate the first term in the left-hand side of (1)

$$\begin{aligned} |\rho^\alpha u|^2_{W^1_{2,0}(\Omega)} &= \sum_{i=1}^2 \int\limits_\Omega \left(\frac{\partial}{\partial x_i}(\rho^\alpha u)\right)^2 \mathrm{d}x = \sum_{i=1}^2 \int\limits_\Omega \left(\frac{\partial \rho^\alpha}{\partial x_i} u + \rho^\alpha \frac{\partial u}{\partial x_i}\right)^2 \mathrm{d}x \\ &\le 2\sum_{i=1}^2 \int\limits_\Omega \left[\left(\frac{\partial \rho^\alpha}{\partial x_i} u\right)^2 + \rho^{2\alpha}\left(\frac{\partial u}{\partial x_i}\right)^2\right] \mathrm{d}x \\ &\le 4\delta^2\alpha^2 \|u\|^2_{L_{2,\alpha-1}(\Omega)} + 2|u|^2_{W^1_{2,\alpha}(\Omega)} \\ &\le \max\{4\delta^2\alpha^2, 2\}\|u\|^2_{H^1_{2,\alpha}(\Omega)}\,. \end{aligned} \tag{3}$$

By Lemma 1 a) u belongs to $H^1_{2,\alpha}(\Omega)$; therefore the second term in the left-hand side of (1) is estimated via the norm $\|u\|^2_{H^1_{2,\alpha}(\Omega)}$.

Then from this statement and estimate (3) we establish (1).

b) We write the obvious inequality

$$|\rho^\alpha u|^2_{W^1_{2,0}(\Omega)} \ge \sum_{i=1}^2 \int\limits_\Omega \left[\left(\frac{\partial \rho^\alpha}{\partial x_i} u\right)^2 - 2\left|\frac{\partial \rho^\alpha}{\partial x_i} u \rho^\alpha \frac{\partial u}{\partial x_i}\right| + \rho^{2\alpha}\left(\frac{\partial u}{\partial x_i}\right)^2\right] \mathrm{d}x\,. \tag{4}$$

We estimate the second term in the right-hand side of (4) using the Cauchy--Schwarz inequality, ε-inequality and we have

$$2\sum_{i=1}^{2}\int_{\Omega}\left|\left(\frac{\partial\rho^\alpha}{\partial x_i}u\right)\left(\rho^\alpha\frac{\partial u}{\partial x_i}\right)\right|\,\mathrm{d}x \le \varepsilon\sum_{i=1}^{2}\int_{\Omega}\rho^{2\alpha}\left(\frac{\partial u}{\partial x_i}\right)^2\mathrm{d}x+\frac{1}{\varepsilon}\sum_{i=1}^{2}\int_{\Omega}\left(\frac{\partial\rho^\alpha}{\partial x_i}u\right)^2\mathrm{d}x,$$

where $\varepsilon > 0$.

Using this inequality we strengthen (4)

$$|\rho^\alpha u|^2_{W^1_{2,0}(\Omega)} \ge (1-\varepsilon)|u|^2_{H^1_{2,\alpha}(\Omega)} - \left(\frac{1}{\varepsilon}-1\right)\sum_{i=1}^{2}\int_{\Omega}\left(\frac{\partial\rho^\alpha}{\partial x_i}u\right)^2\mathrm{d}x\,. \tag{5}$$

Then, taking into account the inequality

$$\sum_{i=1}^{2}\int_{\Omega}\left(\frac{\partial\rho^\alpha}{\partial x_i}u\right)^2\mathrm{d}x \le 4\delta^2\alpha^2\,\|u\|^2_{L_{2,\alpha-1}(\Omega)}\,,$$

choosing $\varepsilon < 1$, from (5) we get the estimate

$$\frac{1}{1-\varepsilon}\,|\rho^\alpha u|^2_{W^1_{2,0}(\Omega)} + \frac{4\delta^2\alpha^2}{\varepsilon}\|\rho^{\alpha-1}u\|^2_{L_{2,0}(\Omega)} \ge |u|^2_{H^1_{2,\alpha}(\Omega)}. \tag{6}$$

We add to both sides of (6) $\|u\|^2_{L_{2,\alpha-1}(\Omega)}$ and apply the inequality $(|a|+|b|)^2 \ge a^2+b^2$ to its right-hand side. Then we extract the square root from both sides of this relation and obtain (2). □

Theorem 1. *a) If $u \in H^k_{2,\alpha}(\Omega)$, then $\rho^{\alpha-(k-s)}u \in W^s_{2,0}(\Omega)$, $s = 1,\dots,k$ and*

$$|\rho^\alpha u|_{W^k_{2,0}(\Omega)} + \ldots + \left|\rho^{\alpha-k}u\right|_{L_{2,0}(\Omega)} \le c_6\|u\|_{H^k_{2,\alpha}(\Omega)}, \tag{7}$$

where c_6 is a positive constant independent of u.

b) If $\rho^{\alpha-(k-s)}u \in W^s_{2,0}(\Omega)$, $s = 1,\dots,k$, then $u \in H^k_{2,\alpha}(\Omega)$ and there exist positive constants $c^*_0,\dots,c^*_k$ independent of u such that the inequality

$$c^*_k\,|\rho^\alpha u|_{W^k_{2,0}(\Omega)} + \ldots + c^*_0\left|\rho^{\alpha-k}u\right|_{L_{2,0}(\Omega)} \ge \|u\|_{H^k_{2,\alpha}(\Omega)} \tag{8}$$

is valid.

Proof. The theorem will be proved by induction on k_1 $(0 \le k_1 < k)$.

a) By the condition of the theorem $u \in H^k_{2,\alpha}(\Omega)$. It is obvious, that $\rho^{\alpha-k}u \in L_{2,0}(\Omega)$, i.e. for $k_1 = 0$ the statement of theorem a) is proved.

   Assume that for some number k_1 $(0 \le k_1 < k)$ the functions $\rho^{\alpha-(k-s)}u$ belong to the space $W^s_{2,0}(\Omega)$ for all s $(0 \le s \le k_1)$ and the inequality

$$\left|\rho^{\alpha-(k-k_1)}u\right|_{W^{k_1}_{2,0}(\Omega)} + \ldots + \left|\rho^{\alpha-k}u\right|_{L_{2,0}(\Omega)} \le c_7\|u\|_{H^{k_1}_{2,\alpha-(k-k_1)}(\Omega)} \qquad (9)$$

   holds.

   We show that $\rho^{\alpha-(k-k_1-1)}u \in W^{k_1+1}_{2,0}(\Omega)$. To this end, we estimate the seminorm $\left|\rho^{\alpha-(k-k_1-1)}u\right|_{W^{k_1+1}_{2,0}(\Omega)}$. We have

$$\left|\rho^{\alpha-(k-k_1-1)}u\right|^2_{W^{k_1+1}_{2,0}(\Omega)} = \sum_{|\lambda|=k_1+1}\int\limits_\Omega\left[\sum_{i=0}^{k_1+1} C^i_{k_1+1}\left(D^i\rho^{\alpha-(k-k_1-1)}\right)D^{\lambda-i}u\right]^2 \mathrm{d}x. \qquad (10)$$

   For any $\lambda = (\lambda_1, \lambda_2)$ and $|\lambda| = k_1 + 1$ the inequality

$$\int\limits_\Omega\left[\sum_{i=0}^{k_1+1} C^i_{k_1+1}\left(D^i\rho^{\alpha-(k-k_1-1)}\right)D^{\lambda-i}u\right]^2 \mathrm{d}x \le (k_1+2)\int\limits_\Omega\left[\sum_{i=0}^{k_1+1} C^i_{k_1+1}\left(D^i\rho^{\alpha-(k-k_1-1)}\right)D^{\lambda-i}u\right]^2 \mathrm{d}x \qquad (11)$$

   is valid. Here $C^i_{k_1+1}$ is a combination of $k_1 + 1$ things i at a time.

   It is easy to note that

$$\left|D^i\rho^{\alpha-(k-k_1-1)}(x)\right| \le c_8\rho^{\alpha-(k-k_1-1)-i}(x), \quad x \in \overline{\Omega}. \qquad (12)$$

   From (10) and (11), (12) we have

$$\left|\rho^{\alpha-(k-k_1-1)}u\right|_{W^{k_1+1}_{2,0}(\Omega)} \le c_9\|u\|_{H^{k_1+1}_{2,\alpha-(k-k_1-1)}(\Omega)}$$

   and $\rho^{\alpha-(k-k_1-1)}u \in W^{k_1+1}_{2,0}(\Omega)$. By virtue of induction the statement of Theorem 1 a) is proved.

b) If $0 \le s \le k$ then according to conditions of the Theorem 1 b) the functions $\rho^{\alpha-(k-s)}u$ belong to the spaces $W^s_{2,0}(\Omega)$.

   For $s = 0$ $\rho^{\alpha-k}u \in W^0_{2,0}(\Omega)$ or $u \in L_{2,\alpha-k}(\Omega)$, i.e. for $k_1 = 0$ the statement of the Theorem 1 b) is valid.

We suppose that $u \in H^{k_1}_{2,\alpha-(k-k_1)}(\Omega)$ for some number k_1 $(0 \le k_1 < k)$ and the inequality

$$\tilde{c}^*_{k_1} \left|\rho^{\alpha-(k-k_1)}u\right|_{W^{k_1}_{2,0}(\Omega)} + \ldots + \tilde{c}^*_0 \left|\rho^{\alpha-k}u\right|_{L_{2,0}(\Omega)} \ge \|u\|_{H^{k_1}_{2,\alpha-(k-k_1)}(\Omega)} \tag{13}$$

holds. Here $\tilde{c}^*_{k_1}, \ldots, \tilde{c}^*_0$ are positive constants independent of u.

We show that the function u belongs to the space $H^{k_1+1}_{2,\alpha-(k-k_1-1)}(\Omega)$. To this end we estimate the seminorm $|u|_{H^{k_1+1}_{2,\alpha-(k-k_1-1)}(\Omega)}$. The proof of this estimate is similar to the proof of Lemma 1 b). Thus

$$\begin{aligned}
&\left|\rho^{\alpha-(k-k_1-1)}u\right|^2_{W^{k_1+1}_{2,0}(\Omega)} \\
&= \sum_{|\lambda|=k_1+1} \int_\Omega \left[\sum_{i=0}^{k_1+1} C^i_{k_1+1}\left(D^i\rho^{\alpha-(k-k_1-1)}\right)D^{\lambda-i}u\right]^2 \mathrm{d}x \\
&\ge |u|^2_{H^{k_1+1}_{2,\alpha-(k-k_1-1)}(\Omega)} - 2\sum_{|\lambda|=k_1+1}\left\{\left|\rho^{\alpha-(k-k_1-1)}D^\lambda u\right|_{L_{2,0}(\Omega)} \times\right. \\
&\qquad \left.\times \left[\int_\Omega \left(\sum_{i=1}^{k_1+1} C^i_{k_1+1}\left(D^i\rho^{\alpha-(k-k_1-1)}\right)D^{\lambda-i}u\right)^2 \mathrm{d}x\right]^{1/2}\right\} \\
&\quad + \sum_{|\lambda|=k_1+1}\int_\Omega \left(\sum_{i=1}^{k_1+1} C^i_{k_1+1}\left(D^i\rho^{\alpha-(k-k_1-1)}\right)D^{\lambda-i}u\right)^2 \mathrm{d}x.
\end{aligned} \tag{14}$$

By means of the Cauchy-Schwarz inequality, ε-inequality we estimate the second term in the right-hand side of (14)

$$\begin{aligned}
&2\sum_{|\lambda|=k_1+1}\left\{\left|\rho^{\alpha-(k-k_1-1)}D^\lambda u\right|_{L_{2,0}(\Omega)} \times\right. \\
&\qquad \left.\times \left[\int_\Omega \left(\sum_{i=1}^{k_1+1} C^i_{k_1+1}\left(D^i\rho^{\alpha-(k-k_1-1)}\right)D^{\lambda-i}u\right)^2 dx\right]^{1/2}\right\} \\
&\le \varepsilon|u|^2_{H^{k_1+1}_{2,\alpha-(k-k_1-1)}(\Omega)} \\
&\qquad + \frac{1}{\varepsilon}\sum_{|\lambda|=k_1+1}\int_\Omega \left(\sum_{i=1}^{k_1+1} C^i_{k_1+1}\left(D^i\rho^{\alpha-(k-k_1-1)}\right)D^{k_1+1-i}u\right)^2 \mathrm{d}x\,,
\end{aligned}$$

where $\varepsilon > 0$.

Using the last inequality and choosing ε such that $\varepsilon < 1$ we write (14) in the form

$$|u|^2_{H^{k_1+1}_{2,\alpha-(k-k_1-1)}(\Omega)} \leq \frac{1}{1-\varepsilon}\left|\rho^{\alpha-(k-k_1-1)}u\right|^2_{W^{k_1+1}_{2,0}(\Omega)} + \frac{1}{\varepsilon}\sum_{|\lambda|=k_1+1}\int_\Omega\left(\sum_{i=1}^{k_1+1}C^i_{k_1+1}\left(D^i\rho^{\alpha-(k-k_1-1)}\right)D^{\lambda-i}u\right)^2 \mathrm{d}x\,. \quad (15)$$

Using inequalities $\left(\sum\limits_{i=1}^{n} a_i\right)^2 \leq n\sum\limits_{i=1}^{n} a_i^2$ and (12) we estimate the second term in the right-hand side (15). As a result we have

$$\frac{1}{\varepsilon}\sum_{|\lambda|=k_1+1}\int_\Omega\left(\sum_{i=1}^{k_1+1}C^i_{k_1+1}\left(D^i\rho^{\alpha-(k-k_1-1)}\right)D^{\lambda-i}u\right)^2 \mathrm{d}x \leq c_{10}\|u\|^2_{H^{k_1}_{2,\alpha-(k-k_1)}(\Omega)}\,. \quad (16)$$

Taking into account that u belongs to $H^{k_1}_{2,\alpha-(k-k_1)}(\Omega)$, the conditions of Theorem 1 b) and (16), (15), (13) we obtain the estimate

$$|u|_{H^{k_1+1}_{2,\alpha-(k-k_1-1)}(\Omega)} \leq \tilde{c}^*_{k_1+1}\left|\rho^{\alpha-(k-k_1-1)}u\right|_{W^{k_1+1}_{2,0}(\Omega)} + \ldots + \tilde{c}^*_0\left|\rho^{\alpha-k}u\right|_{L_{2,0}(\Omega)}$$

and $u \in H^{k_1+1}_{2,\alpha-(k-k_1-1)}(\Omega)$. Thus the statement of theorem 1 b) is proved. □

4 Properties of the functions of the set of $W^k_{2,\alpha}(\Omega,\delta)$

Lemma 2. *For any function u in the set $W^1_{2,\alpha}(\Omega,\delta)$ there exist a parameter α^* that*

$$\|u\|_{L_{2,\alpha^*-1}(\Omega',\delta)} \leq c_{11}\|u\|_{L_{2,\alpha^*}(\Omega,\delta)}, \quad (17)$$

where $0 < c_{11} < 1$.

Proof. Taking into account condition (a), one can show that for $\alpha_1 > \alpha+1$ we have

$$\|u\|^2_{L_{2,\alpha_1-1}(\Omega_i,\delta)} = \int_{\Omega_i}\rho^{2(\alpha_1-1)}u^2\,\mathrm{d}x \leq c_1^2\int_{\Omega_i}\rho^{-2\alpha}\rho^{2(\alpha_1-1)}\,\mathrm{d}x \leq \frac{c_1^2c_{12}^{(i)}\delta^{2(\alpha_1-\alpha)}}{2(\alpha_1-\alpha)},$$

where $c_{12}^{(i)}$ is a constant. Here, the index i is determined in Ω_i. Then

$$\|u\|^2_{L_{2,\alpha_1-1}(\Omega',\delta)} \leq \frac{c_1^2\cdot c_{12}\delta^{2(\alpha_1-\alpha)}}{2(\alpha_1-\alpha)}\,. \quad (18)$$

From the second condition (b), for $u \in W^1_{2,\alpha}(\Omega,\delta)$ we write the inequality

$$\|u\|^2_{L_{2,\alpha_1}(\Omega,\delta)} \ge \|u\|^2_{L_{2,\alpha_1}(\Omega\setminus\Omega',\delta)} = \int\limits_{\Omega\setminus\Omega'} \rho^{2\alpha_1} u^2 \,\mathrm{d}x$$
$$= \int\limits_{\Omega\setminus\Omega'} \rho^{2(\alpha_1-\alpha)}\rho^{2\alpha}u^2\,\mathrm{d}x = \delta^{2(\alpha_1-\alpha)}\|u\|^2_{L_{2,\alpha}(\Omega\setminus\Omega',\delta)} \ge c_2^2\delta^{2(\alpha_1-\alpha)}. \tag{19}$$

Obviously there is α_1, which is denoted by α^*, such that from (18), (19) estimate (17) follows with constant c_{11} $(0 < c_{11} < 1)$. □

Remark 1. For any function u in the set $W^{k+1}_{2,\alpha}(\Omega,\delta)$ there exists a parameter α^*, that

$$\|u\|_{W^k_{2,\alpha^*-1}(\Omega',\delta)} \le c_{13}\|u\|_{W^k_{2,\alpha^*}(\Omega,\delta)},$$

where $0 < c_{13} < 1$.

Remark 2. If $u \in W^1_{2,\alpha^*}(\Omega,\delta)$, then $\rho^{\alpha^*}u \in W^1_{2,0}(\Omega,\delta)$ and

$$|\rho^{\alpha^*}u|_{W^1_{2,0}(\Omega,\delta)} \le c_{14}\|u\|_{W^1_{2,\alpha^*}(\Omega,\delta)},$$

where c_{14} is a positive constant independent of u.

Lemma 3. *a) Let $\rho^{\alpha^*+1}u \in W^2_{2,0}(\Omega,\delta)$ and $u \in W^1_{2,\alpha^*}(\Omega,\delta)$. Then $u \in W^2_{2,\alpha^*+1}(\Omega,\delta)$ and there exist positive constants c_{15} and c_{16} independent of u such that*

$$|u|^2_{W^2_{2,\alpha^*+1}(\Omega,\delta)} \le c_{15}|\rho^{\alpha^*+1}u|^2_{W^2_{2,0}(\Omega,\delta)} + c_{16}\left(\|u\|^2_{W^1_{2,\alpha^*}(\Omega',\delta)} + \|u\|^2_{L_{2,\alpha^*}(\Omega,\delta)}\right).$$

b) Let $u \in W^1_{2,\alpha^*}(\Omega,\delta)$ and $|u|_{W^2_{2,\alpha^*+1}(\Omega,\delta)}$ is a finite seminorm. Then $\rho^{\alpha^*+1}u \in W^2_{2,0}(\Omega,\delta)$ and there exist positive constants c_{17} and c_{18} independent of u such that

$$\left|\rho^{\alpha^*+1}u\right|^2_{W^2_{2,0}(\Omega,\delta)} \le c_{17}\,|u|^2_{W^2_{2,\alpha^*+1}(\Omega,\delta)} + c_{18}\left(|u|^2_{W^1_{2,\alpha^*}(\Omega',\delta)} + \|u\|^2_{L_{2,\alpha^*}(\Omega,\delta)}\right). \tag{20}$$

Proof. a) We write the inequality

$$|\rho^{\alpha^*+1}u|^2_{W^2_{2,0}(\Omega,\delta)} \ge \int\limits_\Omega \sum_{l=1}^2\Bigg\{\left(\frac{\partial^2 u}{\partial x_l^2}\right)^2\rho^{2(\alpha^*+1)} - 2\left|\frac{\partial^2\rho^{\alpha^*+1}}{\partial x_l^2}u + 2\frac{\partial\rho^{\alpha^*+1}}{\partial x_l}\frac{\partial u}{\partial x_l}\right| \times$$
$$\times\left|\frac{\partial^2 u}{\partial x_l^2}\rho^{\alpha^*+1}\right| + \left(\frac{\partial^2\rho^{\alpha^*+1}}{\partial x_l^2}u + 2\frac{\partial\rho^{\alpha^*+1}}{\partial x_l}\frac{\partial u}{\partial x_l}\right)^2 + \left(\frac{\partial^2 u}{\partial x_1\partial x_2}\rho^{\alpha^*+1}\right)^2$$
$$- 2\left|\frac{\partial^2 u}{\partial x_1\partial x_2}\rho^{\alpha^*+1}\right|\left|\frac{\partial u}{\partial x_1}\frac{\partial\rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2}\frac{\partial\rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2\rho^{\alpha^*+1}}{\partial x_1\partial x_2}u\right|$$
$$+ \left(\frac{\partial u}{\partial x_1}\frac{\partial\rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2}\frac{\partial\rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2\rho^{\alpha^*+1}}{\partial x_1\partial x_2}u\right)^2\Bigg\}\,\mathrm{d}x. \tag{21}$$

We estimate the second and fifth terms in the right-hand side of (21) using ε-inequality and we have

$$2\sum_{l=1}^{2}\int\limits_{\Omega}\left|\frac{\partial^2\rho^{\alpha^*+1}}{\partial x_l^2}u+2\frac{\partial\rho^{\alpha^*+1}}{\partial x_l}\frac{\partial u}{\partial x_l}\right|\left|\frac{\partial^2 u}{\partial x_l^2}\rho^{\alpha^*+1}\right|\,\mathrm{d}x\le$$
$$\le\varepsilon\sum_{l=1}^{2}\int\limits_{\Omega}\left(\frac{\partial^2 u}{\partial x_l^2}\rho^{\alpha^*+1}\right)^2 dx+\frac{1}{\varepsilon}\sum_{l=1}^{2}\int\limits_{\Omega}\left(\frac{\partial^2\rho^{\alpha^*+1}}{\partial x_l^2}u+2\frac{\partial\rho^{\alpha^*+1}}{\partial x_l}\frac{\partial u}{\partial x_l}\right)^2\mathrm{d}x\,, \tag{22}$$

$$2\sum_{l=1}^{2}\int\limits_{\Omega}\left|\frac{\partial^2 u}{\partial x_1\partial x_2}\rho^{\alpha^*+1}\right|\left|\frac{\partial u}{\partial x_1}\frac{\partial\rho^{\alpha^*+1}}{\partial x_2}+\frac{\partial u}{\partial x_2}\frac{\partial\rho^{\alpha^*+1}}{\partial x_1}+\frac{\partial^2\rho^{\alpha^*+1}}{\partial x_1\partial x_2}u\right|\,\mathrm{d}x$$
$$\le\varepsilon\sum_{l=1}^{2}\int\limits_{\Omega}\left(\frac{\partial^2 u}{\partial x_1\partial x_2}\rho^{\alpha^*+1}\right)^2\mathrm{d}x$$
$$+\frac{1}{\varepsilon}\sum_{l=1}^{2}\int\limits_{\Omega}\left(\frac{\partial u}{\partial x_1}\frac{\partial\rho^{\alpha^*+1}}{\partial x_2}+\frac{\partial u}{\partial x_2}\frac{\partial\rho^{\alpha^*+1}}{\partial x_1}+\frac{\partial^2\rho^{\alpha^*+1}}{\partial x_1\partial x_2}u\right)^2\mathrm{d}x \tag{23}$$

where $\varepsilon>0$.

Using right-hand sides of these inequalities we strengthen (21)

$$|\rho^{\alpha^*+1}u|^2_{W^2_{2,0}(\Omega,\delta)}\ge(1-\varepsilon)\int\limits_{\Omega}\left\{\sum_{l=1}^{2}\left(\frac{\partial^2 u}{\partial x_l^2}\rho^{\alpha^*+1}\right)^2+\left(\frac{\partial^2 u}{\partial x_1\partial x_2}\rho^{\alpha^*+1}\right)^2\right\}\mathrm{d}x$$
$$+\left(1-\frac{1}{\varepsilon}\right)\sum_{l=1}^{2}\int\limits_{\Omega}\left(\frac{\partial^2\rho^{\alpha^*+1}}{\partial x_l^2}u+2\frac{\partial\rho^{\alpha^*+1}}{\partial x_l}\frac{\partial u}{\partial x_l}\right)^2\mathrm{d}x$$
$$+\left(1-\frac{1}{\varepsilon}\right)\int\limits_{\Omega}\left(\frac{\partial u}{\partial x_1}\frac{\partial\rho^{\alpha^*+1}}{\partial x_2}+\frac{\partial u}{\partial x_2}\frac{\partial\rho^{\alpha^*+1}}{\partial x_1}+\frac{\partial^2\rho^{\alpha^*+1}}{\partial x_1\partial x_2}u\right)^2\mathrm{d}x. \tag{24}$$

By means of the inequality $\left(\sum\limits_{i=1}^{n}a_i\right)^2\le n\sum\limits_{i=1}^{n}a_i^2$ we estimate the second and third terms in the right-hand side of 24

$$\sum_{l=1}^{2}\int\limits_{\Omega}\left(\frac{\partial^2\rho^{\alpha^*+1}}{\partial x_l^2}u+2\frac{\partial\rho^{\alpha^*+1}}{\partial x_l}\frac{\partial u}{\partial x_l}\right)^2\mathrm{d}x$$
$$\le 8(\alpha^*+1)^2\sum_{l=1}^{2}\int\limits_{\Omega'}\rho^{2\alpha^*}\left(\frac{\partial u}{\partial x_l}\right)^2\mathrm{d}x+4(\alpha^*+1)^2(\alpha^*)^2\int\limits_{\Omega'}\rho^{2(\alpha^*-1)}u^2\mathrm{d}x$$
$$\le 8(\alpha^*+1)^2|u|^2_{W^1_{2,\alpha^*}(\Omega',\delta)}+4(\alpha^*+1)^2(\alpha^*)^2c_{11}^2\|u\|^2_{L_{2,\alpha^*}(\Omega,\delta)}, \tag{25}$$

$$\int\limits_\Omega \left(\frac{\partial u}{\partial x_1}\frac{\partial \rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2}\frac{\partial \rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_1 \partial x_2} u\right)^2 \mathrm{d}x$$
$$\leq 3(\alpha^*+1)^2 |u|^2_{W^1_{2,\alpha^*}(\Omega',\delta)} + 3(\alpha^*+1)^2(\alpha^*)^2 c_{11}^2 \|u\|^2_{L_{2,\alpha^*}(\Omega,\delta)}. \quad (26)$$

Using inequalities (25) and (26) we write (24) in the form

$$|\rho^{\alpha^*+1}u|^2_{W^2_{2,0}(\Omega,\delta)} \geq (1-\varepsilon)|u|^2_{W^2_{2,\alpha^*+1}(\Omega,\delta)}$$
$$- \frac{1-\varepsilon}{\varepsilon}(\alpha^*+1)^2\left(11|u|^2_{W^1_{2,\alpha^*}(\Omega',\delta)} + 7(\alpha^*)^2 c_{11}^2 \|u\|^2_{L_{2,\alpha^*}(\Omega,\delta)}\right).$$

Choosing $\varepsilon < 1$ we obtain

$$|u|^2_{W^2_{2,\alpha^*+1}(\Omega,\delta)} \leq \frac{1}{1-\varepsilon}|\rho^{\alpha^*+1}u|^2_{W^2_{2,0}(\Omega,\delta)}$$
$$+ \frac{1}{\varepsilon}(\alpha^*+1)^2 \max\{11, 7(\alpha^*)^2 c_{11}^2\}\left(|u|^2_{W^1_{2,\alpha^*}(\Omega',\delta)} + \|u\|^2_{L_{2,\alpha^*}(\Omega,\delta)}\right).$$

Thus the statement of Lemma 1 a) is proved.

b) We write the inequality

$$|\rho^{\alpha^*+1}u|^2_{W^2_{2,0}(\Omega,\delta)} \leq \sum_{l=1}^{2}\int\limits_\Omega \left\{\left(\frac{\partial^2 u}{\partial x_l^2}\right)^2 \rho^{2(\alpha^*+1)}\right.$$
$$+2\left|\frac{\partial^2\rho^{\alpha^*+1}}{\partial x_l^2}u + 2\frac{\partial\rho^{\alpha^*+1}}{\partial x_l}\frac{\partial u}{\partial x_l}\right|\left|\frac{\partial^2 u}{\partial x_l^2}\rho^{\alpha^*+1}\right| + \left.\left(\frac{\partial^2\rho^{\alpha^*+1}}{\partial x_l^2}u + 2\frac{\partial\rho^{\alpha^*+1}}{\partial x_l}\frac{\partial u}{\partial x_l}\right)^2\right\}\mathrm{d}x$$
$$+\int\limits_\Omega\left\{\left(\frac{\partial^2 u}{\partial x_1\partial x_2}\rho^{\alpha^*+1}\right)^2 + \right.$$
$$+2\left|\frac{\partial^2 u}{\partial x_1\partial x_2}\rho^{\alpha^*+1}\right|\left|\frac{\partial u}{\partial x_1}\frac{\partial \rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2}\frac{\partial \rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_1 \partial x_2} u\right|$$
$$+\left.\left(\frac{\partial u}{\partial x_1}\frac{\partial \rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2}\frac{\partial \rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_1 \partial x_2} u\right)^2\right\}\mathrm{d}x.$$

Then from this statement and estimates (22), (23), (25) and (26) we establish (20).

□

Theorem 2. *Let* $\rho^p u_{r^k} \in \mathcal{W}^2_{2,\alpha^*+1}(\Omega,\delta)$ *and* $u \in \mathcal{W}^m_{2,\alpha^*+p+m-(k+1)}(\Omega,\delta)$ *for* $m = 2,\dots,k+1$. *Then* $u \in \mathcal{W}^{k+2}_{2,\alpha^*+p+1}(\Omega,\delta)$ *and there exist positive constants* c_{19} *and* c_{20} *independent of* u *such that the estimate*

$$|u|^2_{\mathcal{W}^{k+2}_{2,\alpha^*+p+1}(\Omega,\delta)} \leq c_{19}|\rho^p u_{r^k}|^2_{\mathcal{W}^2_{2,\alpha^*+1}(\Omega,\delta)} + c_{20}\sum_{\substack{\lambda_1=0\\|\lambda|=k+2}}^{k+1}\|\rho^{\alpha^*+p+1-\lambda_2}|D^\lambda u|\|^2_{\mathcal{L}_2(\Omega,\delta)}$$

is valid.

Proof. We lower estimate seminorm function $\rho^p u_{r^k}$ in the set $\mathcal{W}^2_{2,\alpha^*+1}(\Omega,\delta)$. We obtain

$$
\begin{aligned}
|\rho^p u_{r^k}|^2_{\mathcal{W}^2_{2,\alpha^*+1}(\Omega,\delta)} &= \int\limits_{\Omega'} r^{2(\alpha^*+1)}\Bigg\{\left(\frac{\partial^2}{\partial r^2}(r^p u_{r^k})\right)^2 \\
&+ \left(\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(r^p u_{r^k})\right)^2 + \left(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(r^p u_{r^k})\right)^2\Bigg\}\mathrm{d}s + \int\limits_{\Omega\setminus\Omega'} \delta^{2(\alpha^*+1)}\left|D^2 u\right|^2 \mathrm{d}x \\
&= \int\limits_{\Omega'} r^{2(\alpha^*+1)}\Bigg\{\left(r^p u_{r^{k+2}} + 2pr^{p-1}u_{r^{k+1}} + p(p-1)r^{p-2}u_{r^k}\right)^2 \\
&+ \left(r^{p-1}u_{r^{k+1}\theta} + pr^{p-2}u_{r^k\theta}\right)^2 + \left(r^{p-2}u_{r^k\theta^2}\right)^2\Bigg\}\mathrm{d}s + \int\limits_{\Omega\setminus\Omega'} \delta^{2(\alpha^*+1)}\left|D^2 u\right|^2 \mathrm{d}x \\
&\geq \int\limits_{\Omega'} r^{2(\alpha^*+1)}\Bigg\{r^{2p}u^2_{r^{k+2}} - 2|r^p u_{r^{k+2}}||2pr^{p-1}u_{r^{k+1}} + p(p-1)r^{p-2}u_{r^k}| + \\
&+ \left(2pr^{p-1}u_{r^{k+1}} + p(p-1)r^{p-2}u_{r^k}\right)^2\Bigg\}ds + \int\limits_{\Omega'} r^{2(\alpha^*+1)}\Big\{r^{2(p-1)}u^2_{r^{k+1}\theta} \\
&- 2|r^{p-1}u_{r^{k+1}\theta}||pr^{p-2}u_{r^k\theta}| + p^2 r^{2(p-2)}u^2_{r^k\theta} + r^{2(p-2)}u^2_{r^k\theta^2}\Big\}\,\mathrm{d}s. \qquad (27)
\end{aligned}
$$

Using ε-inequality we estimate the second and sixth terms in the right-hand side of (27)

$$
\begin{aligned}
&2\int\limits_{\Omega'} r^{2(\alpha^*+1)}|r^p u_{r^{k+2}}||2pr^{p-1}u_{r^{k+1}} + p(p-1)r^{p-2}u_{r^k}|\,\mathrm{d}s \\
&\leq \varepsilon\int\limits_{\Omega'} r^{2(\alpha^*+p+1)}u^2_{r^{k+2}}\,\mathrm{d}s + \frac{1}{\varepsilon}\int\limits_{\Omega'} r^{2(\alpha^*+1)}\left(2pr^{p-1}u_{r^{k+1}} + p(p-1)r^{p-2}u_{r^k}\right)^2\mathrm{d}s\,,
\end{aligned}
$$

$$
\begin{aligned}
&2\int\limits_{\Omega'} r^{2(\alpha^*+1)}|r^{p-1}u_{r^{k+1}\theta}||pr^{p-2}u_{r^k\theta}| \\
&\qquad\qquad \leq \varepsilon\int\limits_{\Omega'} r^{2(\alpha^*+p)}u^2_{r^{k+1}\theta}\mathrm{d}s + \frac{p^2}{\varepsilon}\int\limits_{\Omega'} r^{2(\alpha^*+p-1)}u^2_{r^k\theta}\mathrm{d}s\,.
\end{aligned}
$$

Substituting the right-hand side of this inequality with a minus sign in (27), we obtain

$$|\rho^p u_{r^k}|^2_{\mathcal{W}^2_{2,\alpha^*+1}(\Omega,\delta)} \geq (1-\varepsilon)\int\limits_{\Omega'} r^{2(\alpha^*+p+1)} u^2_{r^{k+2}}\,\mathrm{d}s + (1-\varepsilon)\int\limits_{\Omega'} r^{2(\alpha^*+p)} u^2_{r^{k+1}\theta}\,\mathrm{d}s$$
$$+\left(1-\frac{1}{\varepsilon}\right)\int\limits_{\Omega'} r^{2(\alpha^*+1)}\Big(2pr^{p-1}u_{r^{k+1}} + +p(p-1)r^{p-2}u_{r^k}\Big)^2\mathrm{d}s$$
$$+\,p^2\left(1-\frac{1}{\varepsilon}\right)\int\limits_{\Omega'} r^{2(\alpha^*+p-1)} u^2_{r^k\theta}\mathrm{d}s + \int\limits_{\Omega'} r^{2(\alpha^*+p-1)} u^2_{r^k\theta^2}\,\mathrm{d}s. \tag{28}$$

Taking into account that

$$\int\limits_{\Omega'} r^{2(\alpha^*+1)}\left(2pr^{p-1}u_{r^{k+1}} + p(p-1)r^{p-2}u_{r^k}\right)^2\mathrm{d}s$$
$$\leq 8p^2\int\limits_{\Omega'} r^{2(\alpha^*+p)} u^2_{r^{k+1}}\mathrm{d}s + 2p^2(p-1)^2\int\limits_{\Omega'} r^{2(\alpha^*+p-1)} u^2_{r^k}\mathrm{d}s\,,$$

from inequality (28) we get

$$|\rho^p u_{r^k}|^2_{\mathcal{W}^2_{2,\alpha^*+1}(\Omega,\delta)} \geq (1-\varepsilon)\int\limits_{\Omega'} r^{2(\alpha^*+p+1)}\left[u^2_{r^{k+2}} + \left(\frac{1}{r}u^2_{r^{k+1}\theta}\right)^2 + \left(\frac{1}{r^2}u^2_{r^k\theta^2}\right)^2\right]\mathrm{d}s$$
$$-\left(1-\frac{1}{\varepsilon}\right)\left[8p^2\int\limits_{\Omega'} r^{2(\alpha^*+p)} u^2_{r^{k+1}}\,\mathrm{d}s + 2p^2(p-1)^2\int\limits_{\Omega'} r^{2(\alpha^*+p-1)} u^2_{r^k}\,\mathrm{d}s\right.$$
$$\left. +\,p^2\int\limits_{\Omega'} r^{2(\alpha^*+p-1)} u^2_{r^k\theta}\,\mathrm{d}s\right].$$

Choosing $\varepsilon < 1$ we obtain

$$\int\limits_{\Omega'} r^{2(\alpha^*+p+1)}\left[u^2_{r^{k+2}} + \left(\frac{1}{r}u^2_{r^{k+1}\theta}\right)^2 + \left(\frac{1}{r^2}u^2_{r^k\theta^2}\right)^2\right]\mathrm{d}s$$
$$\leq \frac{1}{1-\varepsilon}\left|\rho^p u_{r^k}\right|^2_{\mathcal{W}^2_{2,\alpha^*+p}(\Omega,\delta)} + \frac{p^2}{\varepsilon}\int\limits_{\Omega'} r^{2(\alpha^*+p)}\left[8u^2_{r^{k+1}} + 2(p-1)^2\frac{u^2_{r^k}}{r^2} + \frac{u^2_{r^k\theta}}{r^2}\right]\mathrm{d}s\,.$$

Add to both sides of the last inequality the sum of the form

$$\sum_{\substack{\lambda_1=0\\|\lambda|=k+2}}^{k-1}\left\|\rho^{\alpha^*+p+1-\lambda_2}|D^\lambda u|\right\|^2_{\mathcal{L}_2(\Omega,\delta)} + \int\limits_{\Omega\setminus\Omega'}\delta^{2(\alpha^*+p+1)}\left|D^2u\right|^2\mathrm{d}x\,.$$

Then

$$
\begin{aligned}
|u|^2_{\mathcal{W}^{k+2}_{2,\alpha^*+p+1}(\Omega,\delta)} \le & \frac{1}{1-\varepsilon}|\rho^p u_{r^k}|^2_{\mathcal{W}^2_{2,\alpha^*+p}(\Omega,\delta)} \\
& + \frac{p^2}{\varepsilon}\int\limits_{\Omega'} r^{2(\alpha^*+p)}\left[8u^2_{r^{k+1}} + 2(p-1)^2\frac{u^2_{r^k}}{r^2} + \frac{u^2_{r^k\theta}}{r^2}\right]\mathrm{d}s \\
& + \sum_{\substack{\lambda_1=0\\|\lambda|=k+2}}^{k-1}\|\rho^{\alpha^*+p+1-\lambda_2}|D^\lambda u|\|^2_{\mathcal{L}_2(\Omega,\delta)} + \int\limits_{\Omega\setminus\Omega'}\delta^{2(\alpha^*+p+1)}\left|D^2u\right|^2\mathrm{d}x \\
\le & \frac{1}{1-\varepsilon}|\rho^p u_{r^k}|^2_{\mathcal{W}^2_{2,\alpha^*+p}(\Omega,\delta)} + c_{20}\sum_{\substack{\lambda_1=0\\|\lambda|=k+2}}^{k+1}\|\rho^{\alpha^*+p+1-\lambda_2}|D^\lambda u|\|^2_{\mathcal{L}_2(\Omega,\delta)},
\end{aligned}
$$

where $c_{20} = 2\max\left\{\dfrac{8p^2}{\varepsilon}, \dfrac{(p-1)^2p^2}{\varepsilon}, 1\right\}$. □

References

[1] T. Apel, A-M. Sändig, J.R. Whiteman: Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains. *Math. Methods Appl. Sci.* 19 (1) (1996) 63–85.

[2] D. Arroyo, A. Bespalov, N. Heuer: On the finite element method for elliptic problems with degenerate and singular coefficients. *Math. Comp.* 76 (258) (2007) 509–537.

[3] F. Assous, J. Ciarlet, E. Garcia, J. Segré: Time-dependent Maxwell's equations with charges in singular geometries. *Comput. Methods Appl. Mech. Engrg.* 196 (1–3) (2006) 665–681.

[4] F. Assous, P. Ciarlet Jr, J. Segré: Numerical solution to the time-dependent Maxwell equations in two-dimensional singular domains: the singular complement method. *J. Comput. Phys.* 161 (1) (2000) 218–249.

[5] T. Belytschko, R. Gracie, G. Ventura: A review of extended generalized finite element methods for material modeling. *Model. Simul. Sci. Eng.* 17 (4) (2009) 043001.

[6] S. Bordas, M. Duflot, P. Le: A simple error estimator for extended finite elements. *Int. J. Numer. Methods Eng.* 24 (2008) 961–971.

[7] A. Byfut, A. Schrödinger: hp-Adaptive extended finite element method. *Int. J. Numer. Methods Eng.* 89 (2012) 1392–1418.

[8] M. Costabel, M. Dauge: Weighted regularization of Maxwell equations in polyhedral domains. *Numer. Math.* 93 (2) (2002) 239–277.

[9] M. Costabel, M. Dauge, C. Schwab: Exponential convergence of hp-FEM for Maxwell equations with weighted regularization in polygonal domains. *Math. Models Methods Appl. Sci.* 15 (4) (2005) 575–622.

[10] V. Ivannikov, C. Tiago, J.P. Moitinho de Almeida, P. Díez: Meshless methods in dual analysis: theoretical and implementation issues. In: *Proceedings of the V International Conference on Adaptive Modeling and Simulation (ADMOS 2011), Paris, France.* (2011) 291–308.

[11] H. Li, V. Nistor: Analysis of a modified Schrödinger operator in 2D: regularity, index, and FEM. *J. Comput. Appl. Math.* 224 (1) (2009) 320–338.

[12] G.R. Liu, T. Nguyen-Thoi: *Smoothed finite elements methods*. CRC Press/Taylor & Francis Group (2010).

[13] P. Morin, R.H. Hochetto, K.G. Siebert: Convergence of adaptive finite element methods. *SIAM Rev.* 44 (5) (2002) 631–658.

[14] H. Nguyen-Xuana, G.R. Liuc, S. Bordasd, S. Natarajane, T. Rabczukf: An adaptive singular ES-FEM for mechanics problems with singular field of arbitrary order. *Comput. Methods Appl .Mech. Engrg.* 253 (2013) 252–273.

[15] T. Nguyen-Thoi, H. Vu-Do, T. Rabczuk, H. Nguyen-Xuan: A node-based smoothed finite element method (NS-FEM) for upper bound solution to visco-elastoplastic analyses of solids using triangular and tetrahedral meshes. *Comput. Methods Appl. Mech. Engrg.* 199 (2010) 3005–3027.

[16] V.P. Nguyen, T. Rabczuk, S. Bordas, M. Dufolt: Meshless methods: A review and computer implementation aspects. *Math. Comput. Simulation.* 79 (2008) 763–813.

[17] V.A. Rukavishnikov: On the differential properties of R_ν-generalized solution of Dirichlet problem. *Dokl. Akad. Nauk.* 309 (6) (1989) 1318–1320.

[18] V.A. Rukavishnikov: On the existence and uniqueness of an R_ν-generalized solution of a boundary value problem with uncoordinated degeneration of the input data. *Dokl. Math.* 90 (2) (2014) 562–564.

[19] V.A. Rukavishnikov, A.O. Mosolapov: New numerical method for solving time-harmonic Maxwell equations with strong singularity. *J. Comput. Phys.* 231 (6) (2012) 2438–2448.

[20] V.A Rukavishnikov, A.O Mosolapov: Weighted edge finite element method for Maxwell's equations with strong singularity. *Dokl. Math.* 87 (2) (2013) 156–159.

[21] V.A. Rukavishnikov, S.G. Nikolaev: On the R_ν-generalized solution of the Lamé system with corner singularity. *Dokl. Math.* 92 (1) (2015) 421–423.

[22] V.A. Rukavishnikov, S.G. Nikolaev: Weighted finite element method for an elasticity problem with singularity. *Dokl. Math.* 88 (3) (2013) 705–709.

[23] V. Rukavishnikov, E. Rukavishnikova: On the existence and uniqueness of R_ν-generalized solution for Dirichlet problem with singularity on all boundary. *Abstr. Appl. Anal. 2014* (2014) 568726.

[24] V.A. Rukavishnikov, E.I. Rukavishnikova: Dirichlet problem with degeneration of the input data on the boundary of the domain. *Differ. Equ.* 52 (5) (2016) 681–685.

[25] V.A. Rukavishnikov, H.I. Rukavishnikova: The finite element method for boundary value problem with strong singularity. *J. Comput. Appl. Math.* 234 (9) (2010) 2870–2882.

[26] V.A. Rukavishnikov, H.I. Rukavishnikova: On the error estimation of the finite element method for the boundary value problems with singularity in the Lebesgue weighted space. *Numer. Funct. Anal. Optim.* 34 (12) (2013) 1328–1347.

[27] O.C. Zienkiewicz, R.L. Taylor, J.Z. Zhu: *The finite element method: its basis and fundamentals. Sixth edition.* Elsevier (2005).

Received: 12 February, 2018
Accepted for publication: 2 April, 2018
Communicated by: Ari Laptev