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Area Nevanlinna type classes of analytic functions in the unit disk and related spaces

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Abstract. The survey collects many recent advances on area Nevanlinna type classes and related spaces of analytic functions in the unit disk concerning zero sets and factorization representations of these classes and discusses approaches, used in proofs of these results.

1 Introduction

First theorems on zero sets of analytic functions from area Nevanlinna type classes appeared in works of Rolf Nevanlinna and Mkhitar Djrbashian many decades ago. These type results, as it was shown later, have various nice applications in theory of functions of a complex variable. The main purpose of this review article is to give a systematic survey of some recent sharp result on zero sets of weighted area Nevanlinna spaces in the unit disk as well as to present general methods for obtaining these type theorems. These methods and their modifications may work in more general situations also and it seems reasonable to illustrate them in this review paper. Almost all theorems we present were obtained in recent two decades by us or by our colleagues. The survey is important due to the fact that most deep results, included in it, have been published in the local Russian-language journals, and thus they are not available to the mathematical community.

The paper is organized as follows: second section presents results on zero sets in analytic area Nevanlinna spaces in the unit disk, the third section presents similar results in the half plane, finite complex plane and other domains; next it presents

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factorization representations of classes above, after it introduces the results concerning the classes with restrictions on α-characteristics; the sixth section presents the similar results for subharmonic functions and in final section we give remarks and discussions. We alert the reader that most results are given without proof, but precise references will be always provided to interested readers. Most sharp results on zeros have "modular" necessary and sufficient condition, like the classical Blaschke condition. The role of specific infinite product π_α, so called Djrbashian's product in the unit disc is very important in this paper. It helps to provide full characterization of zero sets of various analytic area Nevanlinna type spaces and also vital parametric representations. Proofs of such types theorems are largely based on a uniform assessment of the product π_α, obtained by F.A. Shamoyan in 1978. It's appeared in all proofs of such type results, which we present below in the unit disk.

The role of zero-counting $n(r)$ function and classical one-dimensional Jensen formula and various nice properties of counting $n(r)$ function is also very vital to us. At first authors considered the simplest case of Nevanlinna-Djrbashian spaces and described zero sets. Next, using modifications of the proof, based on classical estimates of complex function theory, various authors completely solved the same problem in various analytic area Nevanlinna type spaces. The case of upper half plane and some general domains will also be taken into consideration shortly.

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We denote by C, C_1, M with various lower or upper indices various positive constants in this paper.

We haven't discussed analytic Nevanlinna type spaces of several complex variables which is a new interesting and vital research area in this paper.

2 Zero sets of analytic area Nevanlinna type classes in the unit disk

This section is devoted to zero sets problem in the unit disk.

Let D be the unit disk in the complex plane $\mathbb{C}$,

$$D_r = \{z \in \mathbb{C} : |z| < r\}, \, 0 < r < 1 .$$

C^+ be an upper half plane on $\mathbb{C}$. If G is a certain simply connected domain in the complex plane, then we denote by $H(G)$ the set of all functions, analytic in G, by $M(G)$ we denote the set of all functions, meromorphic in G, and by $SH(G)$ we denote the set of all functions, subharmonic in G. By Z_f we denote zero set of function f; let $n(t, f)$ or shortly $n(t)$ be the number of zeros in a disk $|z| < t$, $t > 0$, i.e. if $\{z_k\} = Z_f$, then

$$n(t) = \mathrm{card}\{z_k : |z_k| < t\},$$

$n(t, \infty)$ be the number of poles of meromorphic function in $|z| < t$, $t > 0$; $n(0)$, $n(0, \infty)$ be the multiplicity of zero or a pole in the point $z = 0$ respectively, $a^+ = \max(a, 0)$, $a \in \mathbb{R}$.

Let Ω be the set of all measurable positive functions on $(0,1]$, for which there exist the numbers m_ω, q_ω from $(0,1]$, M_ω, such as

$$m_\omega \le \frac{\omega(\lambda r)}{\omega(r)} \le M_\omega,\ r \in (0,1],\ \lambda \in [q_\omega, 1]. \tag{1}$$

In particular, the examples of these features are functions of the form

$$t^\alpha \cdot \left(\ln\ln\ldots\ln\frac{a}{t}\right)^\beta, \quad a > 0$$

and all functions are from $C^1(0,1)$ class, for which the following estimate is valid:

$$\left|\frac{\omega'(t)(1-t)}{\omega(t)}\right| \le M_\omega.$$

We also introduce the following notation

$$\alpha_\omega = \frac{\ln m_\omega}{\ln q_\omega},\ \beta_\omega = \frac{\ln M_\omega}{\ln\frac{1}{q_\omega}}.$$

The Ω classes were studied in the monograph of E. Seneta (see [35]).

For any $0 < p, q \le \infty$, $s \in \mathbb{R}$ we denote by $B^s_{p,q}$ the O. Besov space on $\mathbb{T} = \{|z| = 1\}$ ([55, p. 141]).

New characteristics function $T(r,f)$ was introduced in the 1920s in classical papers of R. Nevanlinna and later used by many mathematicians in solving various problems of complex analysis. For any $f \in M(D)$ (see [23]):

$$T(r,f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^+ |f(re^{i\varphi})|\, d\varphi + N(r,f),$$

where

$$N(r,f) = \int_0^r \frac{n(t,\infty) - n(0,\infty)}{t}\, dt, \quad 0 < r < 1.$$

In addition we denote by N the Nevanlinna class or a class of functions with bounded characteristic, that is a class of functions $f \in M(D)$, for which

$$\sup_{0<r<1} T(r,f) < +\infty.$$

Nevanlinna constructed a new factorization of class N as a product of some factors which are easier to study (see ibid.):
Class N coincides with the set of functions $f \in M(D)$, allowing the following the representation

$$f(z) = e^{i\gamma} z^\lambda \frac{B(z,a_k)}{B(z,b_k)} \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\psi(\theta)\right\}, \tag{2}$$

where

$$B(z,a_k)=\prod_{k=1}^{+\infty}\frac{\bar a_k}{|a_k|}\frac{a_k-z}{1-\bar a_k z}$$

is a Blaschke product, $\{a_k\}$ *and* $\{b_k\}$ *are the sequences of points from* D*, satisfying the Blaschke condition:*

$$\sum_{k=1}^{+\infty}(1-|a_k|)<+\infty,\qquad \sum_{k=1}^{+\infty}(1-|b_k|)<+\infty, \tag{3}$$

and ψ *is a real-valued function with bounded variation on* $[0,2\pi]$, $\gamma\in\mathbb{R}$, $\lambda\in\mathbb{Z}$.

In the future, this result has found many applications in complex, harmonic and functional analysis.

In his well-known monograph [23] Nevanlinna also introduced the following class of meromorphic functions

$$S_\alpha(D):=\left\{f\in M(D):\int_0^1(1-r)^\alpha T(r,f)\,\mathrm{d}r<\infty\right\},\quad \alpha>-1,$$

and he found a simple necessary condition on zeros of functions from the this class.

That condition is the following classical one

$$\sum_{k=1}^{+\infty}(1-|z_k|)^{\alpha+2}<+\infty.$$

The complete canonic representation of S_α classes was later found by M. Djrbashian in his paper [9]. The sufficiency of this condition was proved by F.A. Shamoyan in 1978 in paper [37].

For each $\beta>-1$ we denote as $\pi_\beta(z,z_k)$ an infinite product of M. Djrbashian with zeros at $\{z_k\}_1^\infty\subset D$ points (see [9]):

$$\pi_\beta(z,z_k)=\prod_{k=1}^{+\infty}\left(1-\frac{z}{z_k}\right)\exp(-U_\beta(z,z_k)),\quad z\in D,$$

where

$$U_\beta(z,z_k)=\frac{2(\beta+1)}{\pi}\int_{-\pi}^{\pi}\int_0^1\frac{(1-\rho^2)^\beta\ln|1-\frac{\rho e^{i\theta}}{z_k}|}{(1-z\rho e^{-i\theta})^{\beta+2}}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta,\quad z\in\mathbb{C}.$$

Note that π_β products are converging absolutely and uniformly on compact subsets of D if and only if the following is valid

$$\sum_{k=1}^{+\infty}(1-|z_k|)^{\beta+2}<+\infty. \tag{4}$$

Let us remark that the product $\pi_\beta(z, \alpha_k)$ has appeared naturally in the integral representations of the holomorphic functions by the kernel

$$K_\alpha(\zeta, z) = \frac{\alpha+1}{\pi}\frac{(1-|\zeta|^2)^\alpha}{(1-\overline{\zeta}z)^{\alpha+2}}, \qquad \zeta, z \in D, \quad \alpha > -1,$$

as there is the Blaschke product in the integral representation of the bounded type functions by the Poisson-Jensen formula (see [23], [30]).

Note also that uniform estimate of π_β products, that was found by F. Shamoyan in his paper [37], has many applications in the solution of problems related to the description of zero sets of Nevanlinna type spaces in the disk.

To his earlier investigations, related with the Nevanlinna classes, F. Shamoyan turned back again in 1992 in his paper [38]. In this important work the following complete analogue of representation (2) for S_α classes was given.

Theorem 1. *The following assertions are equivalent:*

a) $f \in S_\alpha$*;*

b) f *admits the following representation in* D*:*

$$f(z) = c_\lambda z^\lambda \frac{\pi_\beta(z, a_k)}{\pi_\beta(z, b_k)} \exp\left(\int_{-\pi}^{\pi} \frac{\psi(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta}{(1-\mathrm{e}^{-\mathrm{i}\theta}z)^{\beta+1}}\right), \quad \beta > \alpha + 1,$$

where $\{a_k\}$, $\{b_k\}$ *are arbitrary sequences from* D*, satisfying condition* (4)*,* $\psi \in C^{(k)}(T)$*, such as*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|\psi^{(k)}(\mathrm{e}^{\mathrm{i}\theta+t}) - 2\psi^{(k)}(\mathrm{e}^{\mathrm{i}\theta}) + \psi^{(k)}(\mathrm{e}^{\mathrm{i}\theta-t})|}{|t|^{1+\beta-\alpha}}\,\mathrm{d}t\,\mathrm{d}\theta < +\infty,$$

where $k = [\beta - \alpha - 1]$*.*

In 1999 in his another important work [39] F. Shamoyan completely generalizes so called Nevanlinna-Djrbashian spaces of meromorphic functions in the following natural direction: he considered S^p_ω ($\omega \in \Omega$, $0 < p < +\infty$) spaces of functions. We define them as follows:

$$S^p_\omega(D) := \left\{ f \in M(D) : \|T(f)\|_{L^p(\omega)} = \left(\int_0^1 \omega(1-r)T^p(r,f)\,\mathrm{d}r\right)^{\frac{1}{p}} < +\infty \right\}.$$

For $\omega(t) \equiv 1$, $p = +\infty$, we have $S^\infty_1 = N$, where N is a class of functions with bounded characteristic. For $p = 1$, $\omega(t) = t^\alpha$, $\alpha > -1$, we have $S^1_\alpha = S_\alpha$, where S_α is a standard Nevanlinna-Djrbashian class.

In particular, in that paper F. Shamoyan obtained the full description of zero sets of S^p_ω classes and provided factorization of that class. Namely, the following statement was provided.

Theorem 2. *Let $0 < p < +\infty$, $\omega \in \Omega$, $\beta_\omega \in [0,1)$. The following assertions are equivalent:*

1. *$Z_f = \{z_k\}$, $f \in S^p_\omega$;*
2. *the series*
$$\sum_{k=1}^{+\infty} \frac{n_k^p \omega(\frac{1}{2^k})}{2^{k(p+1)}} \tag{5}$$
is convergent, where $n_k = n\left(1 - \frac{1}{2^k}\right)$, $k = 1, 2, \ldots$.

We present a factorization theorem for the important special case

$$S^p_{\omega,a} = S^p_\omega \cap H(D), \quad \omega(t) = t^\alpha, \quad \alpha > -1\,.$$

In this case we will denote $S^p_{\omega,a}$ as $S^p_{\alpha,a}$.

Theorem 3. *Let $\alpha > -1$, $\beta > \frac{\alpha+1}{p}$. The following assertions are equivalent:*

1. *$f \in S^p_{\alpha,a}$;*
2. *function f allows the following representation in D*
$$f(z) = c_\lambda z^\lambda \pi_\beta(z, z_k) \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\psi(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta}{(1-\mathrm{e}^{-\mathrm{i}\theta}z)^{\beta+1}}\right), \quad z \in D,$$
where $\{z_k\}_1^\infty$ is an arbitrary sequence from D, satisfying the condition
$$\sum_{k=1}^{+\infty} \frac{n_k^p}{2^{k(p+\alpha+1)}} < +\infty,$$
$\psi \in B^s_{1,p}$, $s = \beta - \frac{\alpha+1}{p}$, $\lambda \in \mathbb{Z}$, $c_\lambda \in \mathbb{C}$.

In 2000 in the work of F.A. Shamoyan and E.N. Shubabko [44] a complete investigation of some new classes of meromorphic functions in the unit disk was provided. Namely they introduced and studied the following spaces of meromorphic functions with a fixed “weighted” growth of the Nevanlinna characteristic:

$$S^\infty_\omega(\alpha)(D) := \left\{f \in M(D) : T(r,f) \le C_f \frac{\omega(1-r)}{(1-r)^\alpha}\right\}, \quad \alpha > 0,$$

where $C_f > 0$ is a positive constant, whose values depends only on f, $r \in [0,1)$, $T(r,f)$ is the Nevanlinna characteristics of function f.

In case of $\omega(t) = 1$, the corresponding class of function we denote by

$$S^\infty_\alpha(D) := \left\{f \in M(D) : T(r,f) \le \frac{C_f}{(1-r)^\alpha}\right\}, \quad \alpha > 0,$$

Analytic parts of these classes we denote by $S^\infty_{\omega,a}(\alpha)$ and $S^\infty_{\alpha,a}$ respectively.

The S_α^∞ classes appeared in start of XX century in classical works of Nevanlinna (see [23]). He tried to extend some earlier results of J. Hadamard and E. Borel (see [14]) to the case of all meromorphic functions in a disk. In particular, R. Nevanlinna found that if $\{a_k\}$, $\{b_k\}$ are the set of zeros and poles of a certain function from the class S_α^∞, then

$$\sum_{k=1}^{+\infty}(1-|a_k|)^{\alpha+2+\varepsilon}<+\infty, \qquad \sum_{k=1}^{+\infty}(1-|b_k|)^{\alpha+2+\varepsilon}<+\infty,$$

for each $\varepsilon > 0$. But he fails to provide a complete description of root sets of functions from above spaces. Attempts to obtain a complete description of these sets have also been made by Japanese mathematician M. Tsuji (see [56], [57]). This problem was finally solved by F.A. Shamoyan and E.N. Shubabko (see [44] and also [46]). Namely the following result was provided:

Theorem 4. *In order to $a=\{a_k\}_1^\infty, b=\{b_k\}_1^\infty \subset D$ are the set of zero and poles certain function $f \in S_\alpha^\infty$, it is necessary and sufficient, that*

$$n(r,a)=\operatorname{card}\{a_k : |a_k|<r\} \le \frac{c_1}{(1-r)^{\alpha+1}}\,, \tag{6}$$

$$n(r,b)=\operatorname{card}\{b_k : |b_k|<r\} \le \frac{c_2}{(1-r)^{\alpha+1}}\,,$$

$0<r<1$, $c_1,c_2>0$.

This result acted as cornerstone for further investigations of F.A. Shamoyan and his pupils and in particular it immediately provides factorization of S_α^∞ class.

In the following theorem we give a complete characterization of zero sets in more general case.

Theorem 5. *(see [46]) Let $\omega \in \Omega$, $\alpha > \alpha_\omega$. The following statements are valid:*

1. *If $f \in S_{\omega,a}^\infty(\alpha)$, then*

$$n(r,f) \le c_f \frac{\omega(1-r)}{(1-r)^{\alpha+1}}, \quad 0<r<1. \tag{7}$$

2. *If $\{z_k\}$ is an arbitrary sequence from D, satisfying the condition* (7) *and $\beta > \alpha+\beta_\omega-1$, then the π_β product converges uniformly in D and belongs to $S_{\omega,a}^\infty(\alpha)$ class.*

Remark 1. In our theorems we always assume that f is nontrivial function from considering class, that is $f \not\equiv 0, f \not\equiv \infty$.

In recent work of first author and H. Li the following general classes of analytic functions of the Nevanlinna type were investigated (see [48])

$$(NA)_{p,\beta,\alpha}(D) := \left\{ f \in H(D) : \int_0^1 \left[\sup_{0<r<R} T(r,f)(1-r)^{\beta} \right]^p (1-R)^{\alpha}\, \mathrm{d}R < \infty \right\},$$

$$N^{\infty,p}_{\alpha,\beta}(D) := \left\{ f \in H(D) : \sup_{0<R\le 1} \int_0^R \left[\int_{\mathbb{T}} \ln^+ |f(|z|\xi)|\, \mathrm{d}\xi \right]^p (1-|z|)^{\alpha}\, \mathrm{d}|z|\, (1-R)^{\beta} < \infty \right\},$$

$$N_{\alpha,\beta_1,p}(D) := \left\{ f \in H(D) : \int_0^1 \left[\int_{|z|\le R} \ln^+ |f(z)|(1-|z|)^{\alpha}\, \mathrm{d}m_2(z) \right]^p (1-R)^{\beta_1}\, \mathrm{d}R < \infty \right\},$$

for all $0 < p < \infty$, $\alpha > -1$, $\beta \ge 0$, $\beta_1 > -1$.

The following statements are valid.

Theorem 6. *Let $0 < p < \infty$, $\alpha > -1$, $\beta \ge 0$. The following assertions are equivalent:*

1. $\exists f \in (NA)_{p,\beta,\alpha} : \{z_k\} = Z_f$,
2. $\displaystyle\sum_{k=1}^{+\infty} \frac{n_k^p}{2^{k((\beta p+1)+\alpha+1)}} < +\infty.$

Theorem 7. *Let $0 < p < \infty$, $\alpha > -1$, $\beta \ge 0$. The following assertions are equivalent:*

1. $\exists f \in N^{\infty,p}_{\alpha,\beta} : \{z_k\} = Z_f$,
2. $\displaystyle n(r) \le \frac{C}{(1-r)^{\frac{(\alpha+\beta+p+1)}{p}}}, \quad 0 < r < 1.$

Theorem 8. *Let $0 < p < \infty$, $\alpha > -1$, $\beta_1 > -1$. The following assertions are equivalent:*

1. $\exists f \in N_{\alpha,\beta_1,p} : \{z_k\} = Z_f$,
2. $\displaystyle\sum_{k=1}^{+\infty} \frac{n_k^p}{2^{k(\alpha p+\beta_1+2p+1)}} < +\infty.$

These results on zero sets also lead to complete parametric representations of these general area Nevanlinna type classes (see [48]).

Remark 2. Putting formally $\beta = 0$ or $\alpha = -1$ or $\beta_1 = -1$ in classes $(NA)_{p,\beta,\alpha}$, $N^{\infty,p}_{\alpha,\beta}$, $N_{\alpha,\beta_1,p}$, we arrive at other classes under discussion in this paper.

Similar results for spaces with more general weights studied in [50], [51].

In recent works of our colleagues (see [34], [39], [42]) the root sets of the Privaloff type spaces were provided (see also [29]):

$$\Pi_p(D) = \left\{ f \in H(D) : \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\ln^+ |f(re^{i\theta})|\right)^p \, d\theta < +\infty \right\}, \quad 0 < p \leq +\infty.$$

First we note that from Holder's inequality it is immediately follows the embedding $\Pi_p \subseteq N$ for $1 \leq p < +\infty$, and hence it is clear that for zero set of any function f from Π_p the classical Blaschke condition holds. Converse is also true: each sequence $\{z_k\}_1^\infty \subset D$ with Blaschke condition is zero set of certain function from Π_p $(1 \leq p < +\infty)$ class. As such function we can surely take a Blaschke product. But for $0 < p < 1$ the classical Blaschke condition is not necessary anymore.

In work [42] the following interesting result has found.

Theorem 9. *If $f \in \Pi_p$, $0 < p < 1$, $f(z_k) = 0$, $k = 1, 2, \ldots$, $f \neq 0$, then*

$$\sum_{k=1}^{+\infty} (1 - |z_k|)^{\frac{1}{p}+\varepsilon} < +\infty, \tag{8}$$

for any $\varepsilon > 0$.

Conversely, there exist a function $f \in \Pi_p$ and a sequence $\{z_k\}_1^\infty$, $z_k \in D$, $k = 1, 2, \ldots$, such that $f(z_k) = 0$, $k = 1, 2, \ldots$, $f \neq 0$, but

$$\sum_{k=1}^{+\infty} (1 - |z_k|)^{\frac{1}{p}} \ln^{\frac{1+\varepsilon}{p}} \left(\frac{1}{1 - |z_k|} \right) = +\infty$$

for any $\varepsilon > 0$.

Later in paper [34] it was shown that even stronger result is valid in this direction. It turns out that $(1 - |z_k|)^\varepsilon$ in assertion (8) can be changed to $\ln^{-\frac{1+\varepsilon}{p}}\left(\frac{1}{1-|z_k|}\right)$. This surely makes much narrower the obvious gap between two assertions, which can be seen in Theorem 9. Namely the following result is valid, as was stated in [33].

Theorem 10. *If $f \in \Pi_p$ $(0 < p < 1)$, $f(z_k) = 0$, $k = 1, 2, \ldots$, then*

$$\sum_{k=1}^{+\infty} (1 - |z_k|)^{\frac{1}{p}} \ln^{-\frac{1+\varepsilon}{p}} \left(\frac{1}{1 - |z_k|} \right) < +\infty, \tag{9}$$

for any $\varepsilon > 0$.

And moreover, if $f \in \Pi_p$ $(0 < p < 1)$, $f(z_k) = 0$, $k = 1, 2, \ldots$, $f(0) = 1$, then

$$\sum_{k=1}^{+\infty} \omega(1 - |z_k|)(1 - |z_k|)^2 < +\infty,$$

where $\omega \in C^{(1)}(0,1] \cap \Omega$ *satisfies the following conditions:*

$$\int_0^1 \frac{\omega^p(u)}{u^{2(1-p)}}\, \mathrm{d}u < +\infty,$$

$$\alpha_\omega = \lim_{t \to 0} \frac{\omega'(t) \cdot t}{\omega(t)} > -1,$$

where Ω *is a set of all measurable functions on* $(0,1]$ *with condition* (1).

F.A. Shamoyan later proved the following assertions on zeros of the Privaloff type spaces, noting that root sets are in the Stolz angle. We remind the reader that the angle with vertex $\mathrm{e}^{\mathrm{i}\theta}$, contained in D, having opening $\pi\delta$, $0 < \delta < 1$, and bisector $r\mathrm{e}^{\mathrm{i}\theta}$, $0 \le r < 1$, is said to be the Stolz angle $\Gamma_\delta(\theta)$.

Theorem 11. *(see [39]) If* $f \in \Pi_p$, $0 < p < 1$, $\{z_k\}_1^{+\infty} = Z_f$, *then*

$$n(r) \le \frac{c}{(1-r)^{1/p}}.$$

Conversely, if all points of a sequence $\{z_k\}_1^{+\infty}$ *are in finite numbers of Stolz angles, then from the following condition*

$$\int_0^1 n^p(r)\, \mathrm{d}r < +\infty,$$

it follows that it can be constructed function $f \in \Pi_p$, $0 < p < 1$, *whose zeros coincide with the points of given sequence* $\{z_k\}_1^{+\infty}$.

3 Zero sets of analytic area Nevanlinna type classes in the half plane and related domains

In this section among other things we present some recent interesting results on complete characterizations of zero sets of Nevanlinna type analytic spaces in upper half plane and in other general domains on the complex plane.

Classes of functions with bounded characteristic in the half plane were investigated in Kriloff's work (see [19]), some analogues of the Nevanlinna-Djrbashian classes in upper half plane were studied in interesting papers of A. Djrbashian and his coauthors later (see [13]).

In recent work of our colleague I.S. Kursina (Kipen) some analogues of S^p_ω classes in upper half plane also studied (see [20]).

Let C^+ be an upper half plane of complex plane $\mathbb{C}$. We consider the following classes of analytic functions:

$$S^p_\omega(C^+) := \left\{ f \in H(C^+) : \int_0^{+\infty} \omega(y) \left(\int_{-\infty}^{+\infty} \ln^+ |f(x+\mathrm{i}y)|\, \mathrm{d}x \right)^p \mathrm{d}y < +\infty \right\}.$$

In next theorem the full description of zero sets of $S^p_\omega(C^+)$ class is received. To formulate this result we consider the infinite product $\widetilde{B}_\alpha(z,z_k)$, introduced by A. Djrbashian and G. Mikaelyan (see [13]):

$$\widetilde{B}_\alpha(z,z_k) = \prod_{k=1}^{+\infty} \exp\left\{ - \int\limits_0^{2\Im z_k} \frac{t^\alpha \, \mathrm{d}t}{(t+\mathrm{i}(z_k - z))^{\alpha+1}} \right\}, \quad z \in C^+, \alpha > -1. \tag{10}$$

This product converges absolutely and uniformly in C^+ if

$$\sum_{k=1}^{+\infty} (\Im z_k)^{\alpha+1} < +\infty.$$

We also denote $\tilde{n}(y) = \mathrm{card}\{z_k : \Im z_k > y\}$, $\{z_k\}_1^\infty \in C^+$.

Theorem 12. *(see [20]) Let $\omega \in \Omega$, $\beta_\omega \in (0,1)$, $f \in S^p_\omega(C^+)$ $(0 < p < +\infty)$ and assume there exists a finite limit*

$$a_f = \lim_{t \to +\infty} t \ln |f(\mathrm{i}t)|, \tag{11}$$

and assume $0 \le a_f < +\infty$.

1. *If $\{z_k\} = Z_f$, then*
$$\sum_{k=1}^{+\infty} \frac{\tilde{n}_k^p \omega(\frac{1}{2^k})}{2^{k(p+1)}} < +\infty, \tag{12}$$
where $\tilde{n}_k = \tilde{n}\left(\frac{1}{2^k}\right)$.

2. *If $\{z_k\}$ is an arbitrary sequence in C^+, satisfying condition (12), $\alpha > \frac{\alpha_\omega + 1}{p}$, then the product $\widetilde{B}_\alpha(z,z_k)$ converges uniformly in C^+ and belongs to $S^p_\omega(C^+)$ class, $0 < p < +\infty$.*

We introduce also the following class of functions in upper half plane:

$$A_\alpha(C^+) := \left\{ f \in H(C^+) : \int\limits_{-\infty}^{+\infty} \ln^+ |f(x+\mathrm{i}y)| \, \mathrm{d}x \le \frac{c_f}{y^\alpha} \right\}, \alpha > 0.$$

The complete description of zeros of this class was obtained by I. Kursina in [20]. We formulate her result.

Theorem 13. *The following assertions are valid.*

1. *If $\{z_k\} = Z_f$, $f \in A_\alpha$, and if there exists a finite limit (11), and if $0 \le a_f < +\infty$, then*
$$\tilde{n}(y) \le \frac{\mathrm{const}}{y^{\alpha+1}}. \tag{13}$$

2. *If $\{z_k\}$ is an arbitrary sequence in C^+, satisfying condition* (13), *then the product $\widetilde{B}_\beta(z, z_k)$ converges uniformly in C^+ and uniformly on compact subsets of D and belongs to A_α class for all $\beta > \alpha$.*

In [49] for all values of parameters $0 < p < \infty$, $\alpha > -1$ we introduced the following classes of analytic functions:

$$(N^p_\alpha)_1(C^+) := \Big\{ f \in H(C^+) : \int_0^\infty \Big(\int_0^y \int_{-\infty}^\infty \ln^+ |f(x + i\tilde{y})| \, dx \, d\tilde{y} \Big)^p y^\alpha \, dy < \infty \Big\};$$

$$(N^p_{\alpha,\beta})_2(C^+) := \left\{ f \in H(C^+) : \Big(\sup_{0<y<\infty} \int_0^y \Big(\int_{-\infty}^\infty \ln^+ |f(x+i\tilde{y})| \, dx \Big)^p \tilde{y}^\alpha \, d\tilde{y} \Big) y^\beta < \infty \right\},$$

where $\beta > 0$. These are the Banach spaces for all $p \geq 1$, $\alpha > -1$ and complete metric spaces for all $0 < p < 1$.

The following statements are valid.

Theorem 14. *Let $0 < p < \infty$, $\beta > -1$. If $\{z_k\}_1^\infty = Z_f$, $f \in (N^p_\beta)_1$ and*

$$\limsup_{t\to\infty} \ln |f(it)| < \infty \,,$$

then

$$\sum_{k=1}^\infty \frac{\tilde{n}_k^p}{2^{k(\beta+2p+1)}} < \infty. \tag{14}$$

Conversely, if (14) *is valid, then $\widetilde{B}_\gamma(z, z_k) \in (N^p_\beta)_1$ for all $\gamma > \frac{\beta+1}{p}$.*

Theorem 15. *Let $0 < p < \infty$, $\alpha > -1$, $\beta > 0$. The following assertions are equivalent:*

1. *$\exists f \in (N^p_{\alpha,\beta})_2 : \{z_k\} = Z_f$,*

2. *$\tilde{n}(y) \leq \frac{C_3}{y^{\alpha+\beta+p+1}}$.*

Natural analogues of S^p_ω meromorphic classes in the finite plane were introduced and studied in a work of E.N. Shubabko (she found factorization and described zero sets). Also in that work of E.N. Shubabko (see [46]) some new classes of meromorphic function in the finite complex plane $\mathbb{C}$ were studied. These are classes for which

$$\int_1^{+\infty} T(r, f) \exp(-\sigma r^\alpha) r^\beta \, dr < +\infty, \qquad (\alpha > 0, \sigma > 0, \beta \in \mathbb{R}).$$

We denote them by $S(\alpha, \sigma, \beta)$. Let $S(\alpha, \sigma) = \bigcup_{\beta\in\mathbb{R}} S(\alpha, \sigma, \beta)$.

The necessary condition on zeros of function of these classes are

$$\int_1^{+\infty} n(r,f)\exp(-\sigma r^{\alpha})r^{\beta-\alpha}\,\mathrm{d}r < +\infty. \tag{15}$$

She also obtained the following result: if (15) converges, then we can construct a function g, such that

$$\int_1^{+\infty} T(r,g)\exp(-\sigma_0 r^{\alpha})r^{\beta_0}\,\mathrm{d}r < +\infty.$$

where $\alpha > 0, \sigma_0 > \frac{\sigma}{\alpha}\exp(\alpha - 1), \beta_0 < \beta - 2\alpha.$

In recent work of E.N. Shubabko this result was specified in the following direction: Let $\alpha \geq 1$. For each sequence $\{z_k\}_{k=1}^{\infty}$ ($|z_k| > 0$, $|z_k| \uparrow +\infty$), satisfying the condition (15), there exists a function $f \in H(\mathbb{C})$, such that $Z_f = \{z_k\}$ and

$$\int_1^{+\infty} T(r,g)\exp(-\sigma r^{\alpha})r^{\beta_0}\,\mathrm{d}r < +\infty,$$

$\beta_0 < \beta - 3\alpha.$

Thus, the following statement is valid.

Theorem 16. *(see [46]) Let $\alpha \geq 1$, $\sigma > 0$. The following assertions are equivalent:*

1. $\exists f \in S(\alpha,\sigma) : \{z_k\} = Z_f$,
2. $\int_1^{+\infty} T(r,g)\exp(-\sigma r^{\alpha})r^{\beta_0}\,\mathrm{d}r < +\infty$ *for certain* $\beta \in \mathbb{R}$.

In I. Kursina's thesis some new classes of analytic function in a circular ring were introduced and complete description of zero sets of those classes were given. Based on that she obtained complete characterization of even more general classes in general domains. Then also complete factorization theorems were also provided by her (see [20]). To present her results we need some definitions.

For $0 < R_1 < R_2 < +\infty$ we denote by

$$K = K(R_1,R_2) = \{z \in \mathbb{C} :\ R_1 < |z| < R_2\}$$

the circular ring in the complex plane. For any $f \in H(K)$ we define the Nevanlinna characteristic as follows

$$T(r,f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln^+|f(re^{\mathrm{i}\varphi})|\,\mathrm{d}\varphi,\ R_1 < r < R_2.$$

For all $0 < p < +\infty$ we consider the class

$$S^p_{\omega_1,\omega_2}(K) := \left\{ f \in H(K) : \int_{R_1}^{R_2} \omega_1(r-R_1)\omega_2(R_2-r)T^p(r,f)\,\mathrm{d}r < +\infty \right\},$$

where $\omega_j \in \Omega$, $\Omega = \Omega(K)$, $\omega_j \in L^1(0, C)$, $j = 1, 2$, $C \geq R_2 - R_1$.

We have the following theorem:

Theorem 17. *(see [20]) Let $0 < p < +\infty$, $\omega_j \in \Omega$, $\beta_{\omega_j} \in (0,1)$, $\beta_{\omega_j} > \frac{\alpha_{\omega_j}+1}{p}$, $j = 1, 2$. The following assertions are equivalent:*

1. *$\{z_k\} = Z_f \cap K(R_1, r_0)$, $\{w_k\} = Z_f \cap K(r_0, R_2)$ for certain $f \in S^p_{\omega_1,\omega_2}$;*
2. *the following conditions are valid*

$$\sum_{k=1}^{+\infty} \frac{\tilde{\nu}_k^p \omega_1(\frac{1}{2^k})}{2^{k(p+1)}} < +\infty, \qquad \sum_{k=1}^{+\infty} \frac{\nu_k^p \omega_2(\frac{1}{2^k})}{2^{k(p+1)}} < +\infty, \tag{16}$$

where

$$\tilde{\nu}_k = \operatorname{card}\Big\{z_k : |z_k| > R_1\Big(1 - \frac{1}{2^k}\Big)\Big\},$$

and

$$\nu_k = \operatorname{card}\Big\{w_k : |w_k| < R_2\Big(1 - \frac{1}{2^k}\Big)\Big\},$$

$k = 1, 2, \ldots$.

In paper [47] the second author extends the results of work [48] to the case of circular ring. We must to say that the theory of analytic function spaces in circular rings was developed in [58]. Note that similar problems were considered in [18], [22] for other analytic area Nevanlinna type spaces in circular rings. However, the classes, introduced by I.S Kursina and by second author in circular rings are larger and we can consider their theorems as direct extension of previously known results.

Consider the following class of analytic function in the simple connected domain G:

$$S_\omega(G) := \left\{ f \in H(G) \iint_G \omega(\operatorname{dist}(z, \partial G)) \ln^+ |f(z)| \, \mathrm{d}m_2(z) < +\infty \right\},$$

where $\operatorname{dist}(z, \partial G)$ is a distance from the point z to the boundary of G, $\mathrm{d}m_2$ is the planar Lebesgue measure in G.

In the same work of I.S. Kursina a complete description of zero sets of these classes of analytic function were provided. The following result is valid.

Theorem 18. *Let G be a simply connected domain, ψ be the function, that provide the conformal mapping from G onto D, such that*

$$|\psi'(z)| \leq C_4, \quad \omega \in \Omega, \quad \beta_\omega \in (0,1)\,.$$

The following assertions are equivalent:

1. $\exists f \in S_\omega(G) : \{z_k\} = Z_f$,

2.
$$\sum_{k=1}^{+\infty} \omega(\operatorname{dist}(z_k, \partial G))(\operatorname{dist}(z_k, \partial G))^2 < +\infty. \tag{17}$$

It is worth to mention that an effective approach to the construction of the factorization of functions of Nevanlinna type in multiply connected domains was proposed in 1974 by Matevosyan in [21].

O. Prichodko in [28] studied root sets of some new Nevanlinna type classes in simply connected domain G with nonsmooth boundary whose boundary consists of several smooth arcs forming in the junction points the non-zero angles of $\frac{\pi}{\beta_j}$, $j = 1, 2, \ldots, n$. Regions, having such break at a boundary point, will be called angular domains.

We remind the reader that S_α class is called the plane Nevanlinna class in D. It is naturally arises the question, how the geometric properties of the boundary of G influences on the properties of the root sets. It turns out that the effect is essential.

Consider the class

$$S_\alpha(G_{\vec{\beta}}) := \left\{ f \in H(G_{\vec{\beta}}) : \int_{G_{\vec{\beta}}} (\operatorname{dist}(z, \partial G))^\alpha \ln^+ |f(z)| \, \mathrm{d}m_2(z) < +\infty, \right\}, \quad \alpha > -1,$$

where $\operatorname{dist}(z, \partial G_{\vec{\beta}})$ is a distance from the point z to the boundary of $G_{\vec{\beta}}$, $\mathrm{d}m_2(z)$ is the planar Lebesgue measure in $G = G_{\vec{\beta}}$.

The following statement is valid.

Theorem 19. *If $f \in S_\alpha(G_{\vec{\beta}})$, $\frac{1}{2} < \beta \leq 1$ and $\{z_k\}_{k=1}^{+\infty} = Z_f$, then the following condition is valid*
$$\sum_{k=1}^{+\infty} (1 - |\psi(z_k)|)^{\frac{\alpha+2}{\beta}} < +\infty, \tag{18}$$
where ψ is a function, that provide the conformal mapping from $G_{\vec{\beta}}$ onto D. Conversely, if $\{z_k\}_{k=1}^{+\infty}$ is an arbitrary points from $G_{\vec{\beta}}$, for which (18) *is valid, then it can be constructed a function $f \in S_\alpha(G_{\vec{\beta}})$, such that $\{z_k\}_{k=1}^{+\infty} = Z_f$.*

4 On parametric representations of area Nevanlinna type classes in the unit disk and other domains

The results on zero sets of classes of analytic functions in the unit disk and various simply connected domains of the complex plane allow to construct factorization and parametric representation of the corresponding classes of functions. This section provides some vital direct applications of the main results of previous sections.

Some important theorems on factorization presented in the previous sections (see Theorem 1 and 3).

Now we present a factorization of $S^\infty_{\alpha,a} = S^\infty_\alpha \cap H(D)$ class:

Theorem 20. *(see [45]). The $S^{\infty}_{\alpha,a}$ class coincides with the class of analytic functions in D, allowing the following representation:*

$$f(z) = c_\lambda z^\lambda \pi_\beta(z, a_k) \exp\left\{ \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{\psi(\mathrm{e}^{\mathrm{i}\theta})}{(1 - z\mathrm{e}^{-\mathrm{i}\theta})^{\beta+2}} \,\mathrm{d}\theta \right\}, \quad z \in D, \alpha > 0, \qquad (19)$$

for all $\beta > \alpha - 1$, where $\pi_\beta(z, a_k)$ is the Djrbashian infinite product with zeros in the points of the sequence $\{a_k\}_{k=1}^{+\infty}$, $\psi(\mathrm{e}^{\mathrm{i}\theta})$ is a certain function from O. Besov's class $B^{\beta-\alpha+1}_{1,\infty}$, $\lambda \in \mathbb{Z}_+$, $c_\lambda \in \mathbb{C}$, the sequence $\{a_k\}_{k=1}^{+\infty}$ satisfies the condition (6).

Remark 3. Note that factorization of the classes S^p_ω and S^∞_ω can also be constructed via B_β infinite products (see [45]), introduced by M. Djrbashian in his fundamental work [11] (see Chapter 9):

$$B_\beta\left(z, z_k\right) = \prod_{k=1}^{+\infty} \left(1 - \frac{z}{z_k}\right) \exp\left\{-w_\beta\left(z, z_k\right)\right\},$$

where

$$w_\beta\left(z, \zeta\right) = \sum_{k=1}^{+\infty} \frac{\Gamma\left(\beta + k + 2\right)}{\Gamma\left(\beta + 2\right) \Gamma\left(k + 1\right)} \left\{ \left(\bar{\zeta} z\right)^k \int\limits_{|\zeta|}^{1} \frac{\left(1 - x\right)^\beta}{x^{k+1}} \,\mathrm{d}x - \left(\frac{z}{\zeta}\right)^k \int\limits_{0}^{|\zeta|} \left(1 - x\right)^\beta x^{k-1} \,\mathrm{d}x \right\},$$

for all z, $\zeta \in D$.

The B_β products are absolutely and uniformly convergent and uniformly on compact subsets of D if and only if the series $\sum\limits_{k=1}^{+\infty} (1 - |z_k|)^{\beta+2}$ is convergent. Moreover, B_β and π_β products have significant differences. It was first mentioned by M. Djrbashian and then by F.A. Shamoyan (see e.g. [36]). We introduce now the result of E. Shubabko and F.A. Shamoyan (see [45], also [46]):

Theorem 21. *Let $\alpha > 0$, $\alpha < \beta < \alpha + 5$. The S^∞_α class coincides with the class of analytic functions in D, allowing the following representation:*

$$f(z) = \mathrm{e}^{\mathrm{i}\gamma + m k_\beta} z^m B_\beta(z, z_k) \exp\left(\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \left(\frac{2}{(1 - z\mathrm{e}^{-\mathrm{i}\varphi})^{\beta+1} - 1} \right) \psi(\mathrm{e}^{\mathrm{i}\varphi}) \,\mathrm{d}\varphi \right),$$

where the sequence $\{z_k\}_{k=1}^{+\infty} \subset D$ satisfies the condition (6), *$\psi(\mathrm{e}^{\mathrm{i}\theta})$ is a certain function from O. Besov's class $B^{\beta-\alpha}_{1,\infty}$, and*

$$\psi(\mathrm{e}^{\mathrm{i}\varphi}) = \lim_{r \to 1-0} \frac{1}{\Gamma(\beta)} \int\limits_{0}^{r} (r - t)^{\beta-1} \ln |f(t\mathrm{e}^{\mathrm{i}\varphi})| \,\mathrm{d}t,$$

$m \in \mathbb{Z}$, $\gamma \in \mathbb{R}$, $k_\beta = \beta \sum\limits_{k=1}^{+\infty} \frac{1}{k(k+\beta)}$.

In the paper [20] it was constructed a factorization of $S^p_\omega(C^+)$ and $A_\alpha(C^+)$ classes. We present these results:

Theorem 22. *Let $\omega \in \Omega$, $\beta_\omega \in (0,1)$, $f \in S^p_\omega(C^+)(0 < p < +\infty)$ and there exists a finite limit (11), and $0 \le a_f < +\infty$.*

The class of functions $S^p_\omega(C^+)$ coincides with the class of functions $f \in H(C^+)$, allowing the following representation

$$f(z) = \widetilde{B}_\alpha(z, z_k) \exp(h(z)), \quad z \in C^+,$$

where $\{z_k\}$ is an arbitrary sequence from C^+, satisfying the condition (12), $\widetilde{B}_\alpha(z, z_k)$ is infinite product of the form (10) with zeros in the points of the sequence $\{z_k\}$, h is analytic function in C^+, such as

$$\int_0^{+\infty} \omega(y) \left(\int_{-\infty}^{+\infty} |h(x + \mathrm{i}y)| \,\mathrm{d}x \right)^p \mathrm{d}y < +\infty.$$

Theorem 23. *(see [20]) Let $\alpha > 0$, $\beta > \alpha$, $f \in A_\alpha$ and there exists a finite limit (11), and $0 \le a_f < +\infty$.*

The class of functions A_α coincides with the class of functions $f \in H(C^+)$, allowing the following representation

$$f(z) = \widetilde{B}_\beta(z, z_k) \exp(h(z)), \quad z \in C^+,$$

where $\{z_k\}$ is an arbitrary sequence from C^+, satisfying the condition (13), h is analytic function in C^+, such as

$$\int_{-\infty}^{+\infty} |h(x + \mathrm{i}y)| \,\mathrm{d}x \le \frac{c_5}{y^\alpha}.$$

Also I. Kursina obtained a factorization of $S_{\omega_1,\omega_2}(K)$ and $S_\omega(G)$ classes.

Theorem 24. *Let $0 < p < +\infty$, $\omega_j \in \Omega$, $\beta_{\omega_j} \in (0,1)$, $\beta_{\omega_j} > \frac{\alpha_{\omega_j}+1}{p}$, $j = 1,2$. $S_{\omega_1,\omega_2}(K)$ class coincides with the class of functions $f \in H(K)$, allowing the following representation:*

$$f(z) = c_m z^m \pi_{\beta_1}\left(\frac{R_1}{z_k}, \frac{R_1}{z}\right) \pi_{\beta_2}\left(\frac{w_k}{R_2}, \frac{z}{R_2}\right) \exp\left\{ h_1\left(\frac{R_1}{z}\right) + h_2\left(\frac{z}{R_2}\right) \right\},$$

where $z \in K(R_1, R_2)$, $c_m \in \mathbb{C}$, $m \in \mathbb{Z}_+$, functions $h_j \in H(D)$, $j = 1,2$, satisfies the conditions

$$\int_{R_1}^{R_2} \omega_1(r - R_1) \left(\int_{-\pi}^{\pi} \left| h_1\left(\frac{R_1}{r\mathrm{e}^{\mathrm{i}\theta}}\right) \right| \mathrm{d}\theta \right)^p \mathrm{d}r < +\infty,$$

$$\int\limits_{R_1}^{R_2} \omega_2(R_2-r)\left(\int\limits_{-\pi}^{\pi}\left|h_2\left(\frac{re^{i\theta}}{R_2}\right)\right|\,d\theta\right)^p dr < +\infty,$$

the sequences $\{z_k\}$, $\{w_k\}$ *satisfy the conditions* (16).

Theorem 25. *(see [20]) Let* $\omega \in \Omega$, $\beta_\omega \in (0,1)$, $m \geq \alpha_\omega$. *Class* $S_\omega(G)$ *coincides with the class of analytic functions allowing the following representation:*

$$f(z) = c_\lambda z^\lambda \pi_m(z_k, z)\exp(h(z)),\ z \in G,\ f \in H(G),$$

where $\{z_k\}$ *is an arbitrary sequence from* D, *satisfying the condition* (17),

$$\pi_m(z_k,z) = \prod_{k=1}^{+\infty}\left(1-\frac{1-|\psi(z_k)|^2}{1-\overline{\psi(z_k)}\psi(z)}\right)\exp\left\{\sum_{j=1}^{m+1}\frac{1}{j}\left(\frac{1-|\psi(z_k)|^2}{1-\overline{\psi(z_k)}\psi(z)}\right)^j\right\},$$

where $m \in \mathbb{Z}_+$, $m \geq 0$ *and function* $h \in H(G)$ *satisfies the condition*

$$\iint\limits_G \omega(\operatorname{dist}(z,\partial G))|h(z)|\,dm_2(z) < +\infty.$$

In the recent work of E.N. Shubabko [46] the factorization for weighted class of entire functions $S(\alpha,\sigma)$ was introduced.

Theorem 26. *Let* $\alpha \geq 1$, $\sigma > 0$, $S(\alpha,\sigma)$ *class coincides with the class of entire functions allowing the following representation:*

$$f(z) = z^n W(z_k, z)\exp(h(z)),\ z \in \mathbb{C},$$

where $\{z_k\}$ *satisfies the condition* (15) *for a certain* $\beta \in \mathbb{R}$, $W(z,z_k)$ *is the infinite product of the Weierstrass type*

$$W(z_k,z) = \prod_{k=1}^{+\infty}\left(1-\frac{z}{z_k}\right)\exp\left(\sum_{j=1}^{p_k}\frac{1}{j}\left(\frac{z}{z_k}\right)^j\right), \qquad z \in \mathbb{C},$$

constructed on zeros of function f, $p_k = [\max\{\gamma|z_k|^\alpha + \delta, 1\}]$, $\gamma = \alpha\sigma$, $\delta \in \mathbb{R}$ *(for* $\alpha = 1$, $\delta \geq 1-\beta$*),* $h(z)$ *is entire function, satisfying the condition*

$$M_h(r) = \max_{|z|\leq r}|h(z)| \leq c_h r^\mu \exp\sigma r^\alpha$$

for certain $\mu \in \mathbb{R}$, n *is multiplicity of zero of function* f *in origin.*

5 On classes with the restrictions on the alpha-characteristic function of M. Djrbashian

In 1964 M. Djrbashian posed the problem to generalize the theory of R. Nevanlinna. In his important work [10] he introduced a new characteristic function $T_\alpha(r,f)$. The following M. Djrbashian, we call it α-characteristic. We introduce that definition: let $f \in M(D)$, $\alpha > -1$

$$T_\alpha(r,f) = m_\alpha(r,f) + N_\alpha(r,f), \tag{20}$$

where

$$m_\alpha(r,f) = \frac{r^{-(\alpha+1)}}{2\pi \cdot \Gamma(\alpha+1)} \int_{-\pi}^{\pi} \left(\int_0^r (r-t)^\alpha \ln|f(te^{\mathrm{i}\varphi})| \, \mathrm{d}t \right)^+ \mathrm{d}\varphi,$$

$$N_\alpha(r,f) = \frac{r^{-(\alpha+1)}}{\Gamma(\alpha+2)} \int_0^r (r-t)^{\alpha+1} \cdot \frac{n(t,\infty) - n(0,\infty)}{t} \, \mathrm{d}t + \frac{n(0,\infty)}{\Gamma(\alpha+2)} (\ln r - k_\alpha), \tag{21}$$

where Γ is the Euler function, $k_\alpha = \alpha \sum\limits_{k=1}^{+\infty} \frac{1}{k(k+\alpha)}$.

We denote the analogue of counting function of zeros by

$$n_\alpha(r,f) = \frac{r^{-(\alpha+1)}}{\Gamma(\alpha+2)} \int_0^r (r-t)^{\alpha+1} \cdot \frac{n(t) - n(0)}{t} \, \mathrm{d}t.$$

In the same vital paper M. Djrbashian introduced a class of meromorphic functions in D with bounded α-characteristic, described zero sets and obtained a parametric representation of the specified class of functions. We denote that class by N_α, i.e.

$$N_\alpha := \left\{ f \in M(D) : \sup_{0<r<1} T_\alpha(r,f) < +\infty \right\}.$$

These results were fundamental for the construction of a new theory of classes of meromorphic functions (see [12]). Note that $S_\alpha \subset N_\alpha$, and this inclusion is strict (see [10], [33], [38]).

From above it is natural to define the following class of meromorphic functions:

$$N^p_{\alpha,\gamma}(D) := \left\{ f \in M(D) : \int_0^1 (1-r)^\gamma T^p_\alpha(r,f) \, \mathrm{d}r < +\infty \right\}$$

for all $< p < +\infty$, $\alpha > -1$, $\gamma > -1$.

It is necessary to get the characterization of the root sets and to construct the factorization of the class $N^p_{\alpha,\gamma}$. Further we will assume that $N^p_{\alpha,\gamma}$ class consists only of analytic functions in D.

In the work [32] this problem was finally solved. The following results are valid:

Theorem 27. *The sequence $\{z_k\}_{k=1}^{+\infty}$ from D be zero set of certain function $f \in N^p_{\alpha,\gamma}$ for all $0 < p < +\infty$, if and only if the following series is convergent*

$$\sum_{k=1}^{+\infty} \frac{n^p_{\alpha,k}}{2^{k(\gamma+1)}} < +\infty, \tag{22}$$

where $n_{\alpha,k} = n_\alpha\left(1 - \frac{1}{2^k}, f\right)$.

Theorem 28. *Let $0 < p < +\infty$, $\alpha > -1$, $\gamma > -1$, $\beta > \alpha + 1 + \frac{\gamma+1}{p}$. The following assertions are equivalent*

1. $f \in N^p_{\alpha,\gamma}$;
2. $f(z)$ *can be represented as:*

$$f(z) = c_\lambda z^\lambda \pi_\beta(z, z_k) \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\psi(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta}{(1 - \mathrm{e}^{-\mathrm{i}\theta} z)^{\beta+1}}\right), \quad z \in D, \tag{23}$$

where $\{z_k\}_{k=1}^{+\infty}$ is an arbitrary sequence from D, satisfying the condition (22), *$\pi_\beta(z, z_k)$ is the Djrbashian infinite product with zeros $\{z_k\}_{k=1}^{+\infty}$, $\psi \in B^s_{1,p}$, $s = \beta - (\alpha+1) - \frac{\gamma+1}{p}$, $\lambda \in \mathbb{Z}$, $c_\lambda \in \mathbb{C}$.*

Recent results of F.A. Shamoyan and his pupils continues investigations in this direction. They are devoted to comparison of classes with restrictions on the Nevanlinna characteristic and classes with restrictions on the α-characteristic of M.M. Djrbashian (see [2], [41]).

We denote the following function

$$\int_{-\pi}^{\pi} \left(\int_0^1 \omega_1(1-x) \ln|f(rx\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}x \right)^+ \mathrm{d}\theta$$

by $T_{\omega_1}(r, f)$. For $\omega_1(t) = t^\alpha$, $\alpha > -1$, the T_{ω_1} characteristic coincides with α-characteristic.

Consider the following classes of functions:

$$N^p_{\omega_1,\,\omega_2}(D) := \left\{ f \in H(D) : \int_0^1 \omega_2(1-r) T^p_{\omega_1}(r, f)\,\mathrm{d}r < +\infty \right\},$$

and

$$S^p_{\omega_1,\omega_2}(D) := \left\{ f \in H(D) : \int_0^1 \omega_2(1-r)\omega_1(1-r)^p (1-r)^p T^p(r, f)\,\mathrm{d}r < +\infty \right\},$$

where ω_1, $\omega_2 \in \Omega$, $0 < p < +\infty$.

In [41] the full description of zero sets and factorization of the above classes $N^p_{\omega_1,\,\omega_2}$ found for all ω_1, $\omega_2 \in \Omega$, $0 < p < +\infty$.

The following result is valid.

Theorem 29. *Let $\omega_1, \omega_2 \in \Omega$. The sequence $\{z_k\}_1^\infty$ from D be zero set of certain function $f \in N^p_{\omega_1,\,\omega_2}$ for all $0 < p < +\infty$, if and only if the following series is convergent*

$$\sum_{k=1}^{+\infty} \frac{n_k^p\, \omega_2(\frac{1}{2^k}) \omega_1^p(\frac{1}{2^k})}{2^{k(2p+1)}} < +\infty.$$

The next result was also established.

Theorem 30. *For all* $\omega_1,\, \omega_2 \in \Omega$, $0 < p < +\infty$, *the following inclusion is valid*

$$S^p_{\omega_1,\,\omega_2} \subset N^p_{\omega_1,\,\omega_2}.$$

We denote by $N^p_{\alpha,\,\beta}$ and $S^p_{\alpha,\,\beta}$ the classes $N^p_{\omega_1,\,\omega_2}$ and $S^p_{\omega_1,\,\omega_2}$ respectively for $\omega_1(t) = t^\alpha$, $\omega_2(t) = t^\beta$, $\alpha,\, \beta > -1$.

From above statements the next statement follows:

Theorem 31. *Let* $0 < p < +\infty$, $\alpha,\, \beta > -1$. *Then the classes* $N^p_{\alpha,\,\beta}$ *and* $S^p_{\alpha,\,\beta}$ *are equivalent.*

6 Some results on classes of subharmonic functions

In this section we consider general weighted spaces of subharmonic functions of Bergman type and solve some similar type problems for such type spaces. In particular the problem of parametric representation will be completely solved. for such type classes.

In the recent works of O. Ohlupina new classes of subharmonic functions in the disk and halfplane of type S^p_α, S^∞_α were investigated (see [24], [26]), i.e. a complete descriptions of zero sets and parametric representations of those classes of functions were found.

To formulate further results we need some more notations. Let $u \in SH(D)$, $u^+ = \max(u, 0)$. The Nevanlinna characteristic of subharmonic function u we call the function

$$T(r, u) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} u^+ \left(re^{\mathrm{i}\varphi}\right) \mathrm{d}\varphi.$$

The following I.I. Privaloff, we denote by A the class of all functions u, subharmonic in D, such that

$$\sup_{0<r<1} T(r, u) < +\infty. \tag{24}$$

The following statement was established in the work [31]:
A class coincides with the class of subharmonic functions in D, *allowing the following representation:*

$$u(z) = \frac{1}{2\pi} \int\limits_D \ln \left| \frac{\zeta - z}{1 - \bar{\zeta} z} \right| \mathrm{d}\mu(\zeta) + \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} \,\mathrm{d}\psi(\theta), \tag{25}$$

where $\mu(\zeta)$ *is an arbitrary nonnegative Borel measure in the unit disk, such that*

$$\int\limits_D (1 - |\zeta|) \,\mathrm{d}\mu(\zeta) < +\infty,$$

ψ *is an arbitrary function with bounded variation on* $[-\pi; \pi]$.

If $u(z) = \ln|f(z)|$, $z \in D$, where $f \in H(D)$, then the representation (25) coincides with the Poison-Jensen formula for the function of bounded type (see [16]). The problem of representation of subharmonic functions, for which the condition (24) is not valid, occurs naturally. The expansion of the F.A. Shamoyan's results to the class of subharmonic function is given by K. Avetisyan in [1] in the case $p = 1$. However, his methods did not allow him to obtain similar results in the classes of subharmonic functions in the unit disk whose characteristic belongs to L^p-weight classes or has exponential growth approaching the unit circle.

Let $\alpha > 0$. Consider $SH_\alpha(D)$ class of subharmonic functions u in D, such that

$$T(r,u) \leq \frac{C_u}{(1-r)^\alpha}, \quad 0 \leq r < 1,$$

where C_u is a certain positive constant, depending on u.

It is clear that $SH_0(D) = A$. For $\alpha > 0$ the I.I. Privaloff's method for $SH_0(D)$, is failed, since it is possible that functions from $SH_\alpha(D)$ class have no the boundary values on the unit circle. It is needed to build a new method for further study. The approach used in [37] helps O.V. Ohlupina to prove the analogue of above representation for $SH_0(D)$ class for all scale of classes $SH_\alpha(D)$ $(\alpha > 0)$.

For fixed $z, \zeta \in D$, $\zeta \neq 0$, $\beta > -1$ we denote the factor in the M.M. Djrbashian product by

$$A_\beta(z,\zeta) = \left(1 - \frac{z}{\zeta}\right) \exp\left\{ -\frac{2(\beta+1)}{\pi} \int\limits_D \frac{\left(1-|t|^2\right)^\beta \ln\left|1-\frac{t}{\zeta}\right|}{\left(1-\bar{t}z\right)^{\beta+2}} \, \mathrm{d}m_2(t) \right\},$$

where $\mathrm{d}m_2(t)$ denotes the normalized Lebesgue measure in the unit disk on the complex plane.

The following statement is valid.

Theorem 32. *(see [26]) $SH_\alpha(D)$ class coincides with the class of u functions, subharmonic in D, allowing the following representation:*

$$u(z) = \int\limits_D \ln|A_\beta(z,\zeta)| \, \mathrm{d}\mu(\zeta) + \mathrm{Re}\left\{ \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{\psi(\mathrm{e}^{\mathrm{i}\theta}) \, \mathrm{d}\theta}{(1-\mathrm{e}^{-\mathrm{i}\theta}z)^{\beta+1}} \right\}$$

where $z \in D$, $\psi(\mathrm{e}^{i\theta})$ is an arbitrary real value function from the class $B^{1,\infty}_{\beta-\alpha+1}$, $\beta > \alpha$, $\alpha > -1$, $\mu(\zeta)$ is the nonnegative Borel measure in D, satisfying the condition: $n(r) \leq \frac{C_1}{(1-r)^{\alpha+1}}$, where $n(r) = \mu(D_r)$.

In [24] a full description of the class of subharmonic functions in the unit disk with the Nevanlinna characteristic from L^p-spaces $(0 < p < +\infty)$ was given.

For all $0 < p < +\infty$, $\omega \in \Omega$ we consider the following classes:

$$SH^p_\omega(D) = \left\{ u \in SH(D) : \left(\int\limits_0^1 \omega(1-r) \left(\int\limits_{-\pi}^{\pi} u^+\left(r\mathrm{e}^{\mathrm{i}\varphi}\right) \, \mathrm{d}\varphi \right)^p \mathrm{d}r \right)^{\frac{1}{p}} < +\infty \right\}.$$

Denote

$$L_{p,\omega}(D) = \left\{ \psi \in L'(D) : \left(\int_0^1 \omega(1-r) \left(\int_{-\pi}^{\pi} \left|\psi(re^{i\varphi})\right| d\varphi \right)^p dr \right)^{\frac{1}{p}} < +\infty \right\}.$$

In the following theorem parametric representation of new classes of subharmonic functions, having, generally speaking, unlimited Nevanlinna characteristic $T(r,u)$, but belonging to the space $L_{p,\omega}(D)$, namely

$$\int_0^1 T^p(r,u)\omega(1-r)\, dr < +\infty,$$

will be given.

Theorem 33. *A subharmonic function belongs to the class $SH^p_\omega(D)$, $0 < p < +\infty$, $\omega \in \Omega$, if it allows the following representation in D:*

$$u(z) = \int_D \ln|A_\beta(z,\zeta)|\, d\mu(\zeta) + h(z), \tag{26}$$

where β is sufficiently large number, depending on ω: $\beta > 1 + \frac{\alpha_\omega}{p}$, $0 \le \alpha_\omega < +\infty$, $\mu(\zeta)$ is the nonnegative Borel measure in D, such that:

$$\int_0^1 \omega(1-r)(1-r)^p n^p(r)\, dr < +\infty\,,$$

$n(r) = \mu(D_r)$, $h(z)$ is harmonic function in D, satisfying the condition

$$\int_0^1 \omega(1-r)\left(\int_{-\pi}^{\pi} \left|h(re^{i\varphi})\right| d\varphi \right)^p dr < +\infty.$$

And the reverse is also true: if $f \in SH^p_\omega(D)$, then (26) holds.

Consider the class $SH^p_\alpha(C^+)$ of subharmonic functions u in C^+, such that:

1. $\displaystyle\int_0^{+\infty} y^{\alpha-1}\left(\int_{-\infty}^{+\infty} u^+(x+iy)\, dx \right)^p dy < +\infty;$

2. $\displaystyle\sup_{y>y_0} \int_{-\infty}^{+\infty} |u\,(x+iy)|\, dx \le C_{y_0} < +\infty\,, \quad \forall y_0 > 0;$

3. $\displaystyle\limsup_{y\to+\infty} yu(iy) \ge 0.$

We introduce the factor of A.M. Djrbashian and G. Mikaelyan (see [13]):

$$a_\beta(z,\zeta) = \exp\left\{ -\int_0^{2\mathrm{Im}\zeta} \frac{r^\beta \, \mathrm{d}r}{(r + \mathrm{i}\zeta - \mathrm{i}z)^{\beta+1}} \right\}, \quad z \in \mathbb{C},$$

where a principal branch of power function is taken, $\zeta \in C^+$, $-1 < \beta < +\infty$. For $\beta = 0$: $a_0(z,\zeta) = \frac{\zeta - z}{\bar{\zeta} - z}$.

Parametric representation of $SH^p_\alpha(C^+)$ class of function will be given in next theorem.

Theorem 34. *(see [25]) A subharmonic function belongs to the class $SH^p_\alpha(C^+)$, $0 < p < +\infty$, $0 < \alpha < +\infty$, if it allows the following representation in C^+:*

$$u(z) = \int_{C^+} \ln |a_\beta(z,\zeta)| \, \mathrm{d}\mu(\zeta) + h(z), \tag{27}$$

where $h(z)$ is harmonic function in C^+, satisfying the condition:

$$\int_0^{+\infty} y^{\alpha-1} \left(\int_{-\infty}^{+\infty} |h(x+\mathrm{i}y)| \, \mathrm{d}x \right)^p \mathrm{d}y < +\infty,$$

$\mu(\zeta)$ is the nonnegative Borel measure in C^+, such that

$$\int_0^{+\infty} y^{\alpha+p-1} n^p(y) \, \mathrm{d}y < +\infty,$$

where $n(y) = \mu(C^+_y)$, $C^+_y = \{z : \mathrm{Im}\, z > y\}$,$\beta > \frac{\alpha-1}{p} + 1$. And the reverse is also true: if $f \in SH^p_\alpha(C^+)$, then (27) holds.

7 Some final remarks for the problem of the description of zero sets and construction of factorization representation of Nevanlinna type classes

In this section among other things we consider some other nice but less effective infinite products similar to Djrbashian type products and also known general scheme for finding simple conditions on zeros in simplest Nevanlinna type spaces in the unit disk with the help of Djrbashian product and Jensen formula will be provided. This scheme based on an estimate of Djrbashian product provided for the first time by F. Shamoyan back in 1978 served later as a base for many other proofs of similar type theorems in this direction, it is very flexible and allows various extentions and further generalizations for other analytic function spaces of Nevanlinna type. This type of work was presented by various authors later as it was partially indicated in previous sections of this survey. Other interesting new approaches to this problem of zero sets for this type of function spaces will be also discussed in this section.

E. Beller and S. Shvedenko constructed and used products of similar Djrbashian type products for factorization of analytic area Nevanlinna classes. They are simpler than Djrbashian products and they have the following form

$$P_\beta(z, z_k) = \prod_{k=1}^{\infty} \left(1 - \left(\frac{1 - |z_k|^2}{1 - \bar{z}_k z} \right)^{2+\beta} \right), \qquad \beta > \alpha.$$

These products for all $\beta > \alpha$ have also the property

$$\|f\|_{N_\alpha} = \int_D \ln^+ |P_\beta(z, z_k)| (1 - |z|)^\alpha \, \mathrm{d}m_2(z) < \infty.$$

Moreover, $f(z) = z^m \cdot P_\beta(z, z_k) \cdot f_0(z)$, $\beta > 0$ for any $f \in N$, and $P_\beta(z, z_k)$ is a product, based on zeros of f, f_0 is free of zeros in D (see [54]).

This type of factorization is valid for all S_α also, for $\beta > \alpha$, $\alpha > -1$ (see ibid.).

These products, as later was found, are defective in the sense that for values $\alpha > 4$ they have more zeros, than the given sequence, and hence cannot be used in many problems where π_α products can be used.

Note, $P_\beta \in S_\alpha$ if $\beta > \alpha$; $\pi_m \in S_\alpha$ if $m > \alpha$. P_β type products were used by E. Beller also for factorization of some similar to S_α classes of analytic functions.

Note, in proving sharp theorems on zero of analytic Nevanlinna type spaces in the unit disk π_α products were constantly used in the following simple manner. Shortly speaking, the problem is to find condition (K) on a sequence $\{z_k\}$ from unit disk, so that $\|\pi_\alpha\|_X \le c \, \|(z_k)\|_K$, where X is the certain fixed Nevanlinna type space, for all $\alpha > \alpha_0$ in a certain fixed analytic area Nevanlinna type space in the unit disk.

The most important property here is that π_α has zeros precisely in the points of the sequence $\{z_k\}$ and that if

$$\sum_{k=1}^{\infty} (1 - |z_k|)^{2+\alpha} < \infty, \qquad \alpha > -1, \tag{28}$$

then

$$\ln |\pi_\alpha(z, z_k)| \le c_6 \sum_{k \ge 1} \frac{(1 - |z_k|)^{2+\alpha}}{|1 - \bar{z}_k z|^{2+\alpha}}, \qquad z \in D, \tag{29}$$

as established in [37] (see also [40], [46]).

Note, from here we have for $\beta > \alpha$

$$\begin{aligned}
\|\pi_\beta\|_{S_\alpha} &= \int_D (1 - |z|)^\alpha \ln^+ |\pi_\beta(z, z_k)| \, \mathrm{d}m_2(z) \\
&\le c_7 \sum_{k \ge 1} (1 - |z_k|)^{\alpha+2} \int_D \frac{(1 - |z|)^\alpha (1 - |z_k|)^{\beta-\alpha} \, \mathrm{d}m_2(z)}{|1 - \bar{z}_k z|^{2+\beta}} \\
&\le c_8 \sum_{k \ge 1} (1 - |z_k|)^{\alpha+2} < \infty,
\end{aligned}$$

(see [38], also [40], [46]).

If we replace the norm of S_α by other norm with more complicated form (see for definitions of spaces above), then using a chain of classical estimates and inequalities of function theory, we arrive at desired final result. This scheme was used in [48], [46], [49], [51]. Condition (28) however should be changed to the other one.

At the final step we must show that the infinite product $\pi_\beta(z, z_k)$ under this new condition on $\{z_k\}_{k=1}^{\infty}$ converges uniformly in D. The scheme of prove of the reverse assertion is based fully on classical Jensen formula in the unit disk, properties of counting $n(r)$ function and in many cases can be obtained immediately after applications of several inequalities from classical function theory. We gave an typical example here for analytic area Nevanlinna spaces with mixed norm, which illustrates the idea.

Let $f \in S^p_{\omega,a}$, $f(0) = 1$, $f(z_k) = 0$, $k = 1, 2, \dots$. Then from Jensen's inequality we have

$$\int_0^1 \omega(1-r)\Big(\int_0^r \Big(\frac{n(u)}{u}\Big)\,\mathrm{d}u\Big)^p \mathrm{d}r \le \frac{1}{(2\pi)^p}\int_0^1 \omega(1-r)\,(T^p(r))\,\mathrm{d}r = M_\omega(f) < \infty.$$

Hence

$$\begin{aligned} M_\omega(f) &\ge \frac{1}{(2\pi)^p}\int_{\frac12}^1 \omega(1-r)\Bigg(\int_{r-\left(\frac{1-r}{2}\right)}^{r} n(t)\,\mathrm{d}t\Bigg)^p \mathrm{d}r \\ &\ge c(\pi)\int_{\frac12}^1 \omega(1-r)\Big[n\Big(\frac{3r-1}{2}\Big)(1-r)\Big]^p \mathrm{d}r \\ &= c(p)\int_{\frac14}^1 \omega\Big(\frac{2}{3}(1-\rho)\Big)\,(n^p(\rho))\,(1-\rho)^p\,\mathrm{d}\rho\,. \end{aligned}$$

$$M_\omega(f) \ge c_1(p)\int_0^1 (\omega(1-\rho))(n(\rho)^p)(1-\rho)^p\,\mathrm{d}\rho \ge c_2(p)\sum_{k=1}^{+\infty} n_k^p\Bigg(\frac{\omega\left(\frac{1}{2^k}\right)}{2^{k(p+1)}}\Bigg).$$

The last estimate follows from properties of ω and $n(r)$ directly (see [39], [40], [46]).

Hence if

$$\frac{\sum_{k=1}^{+\infty} n_k^p\,\omega\left(\frac{1}{2^k}\right)}{2^{k(p+1)}} < \infty,$$

then $M_\omega(f) < \infty$.

Similar arguments lead to similar assertions.

Approximately at the same time after the appearance of [37], G. Heipler and E. Beller provided closely related results in this direction of research (see [3], [4],

[17]). We mention also separately [27], where new promising approaches were provided to this classical problem, related with the description of zero sets of analytic Nevanlinna classes in the unit disk. Unfortunately, these approaches were out of consideration of experts in this area since then. We add more comments. In [27] authors give complete characterization of positive Borel measures of smooth Ω domains in $\mathbb{R}^n$, so that $\Delta u = \mu$ for $u \in L^p(\Omega)$, $1 \le p < \infty$.

In $n = 2$ case this problem is closely related with the problem of the zeros distribution of holomorphic functions, since for $\mu = (2\pi)\left(\sum\limits_n \delta_{a_n}\right)$ all solutions of the equation $\Delta u = \mu$ take form $u = \ln|f|$ with $f \in H$ $(|z| < 1)$, vanishing exactly on the points a_{n_0}. This gives complete characterizations of zeros of area Nevanlinna classes for all $p \ge 1$, $\ln|f| \in L^p(\Omega)$ (see ibid.).

We refer the reader to [27] for details concerning this new interesting approach related with the Laplace equation.

It is known, that $\left(\ln^+|f|\right)^p$ is subharmonic function, for $f \in H(D)$, and $p \ge 1$. This fact in combination with some embedding theorems for analytic Lizorkin--Triebel spaces with mixed norm spaces leads to some new embedding for general classes of functions of the following type

$$LF^{p,q}_{\alpha}(D) = \left\{ f \in H(D) : \int\limits_{-\pi}^{\pi} \left(\int\limits_0^1 \left(\ln^+\left|f(re^{i\varphi})\right|\right)^q (1-r)^{\alpha}\, dr \right)^{\frac{p}{q}} d\varphi < \infty \right\},$$

$$LA^{p,q}_{\alpha}(D) = \left\{ f \in H(D) : \int\limits_0^1 \left(\int\limits_{-\pi}^{\pi} \left(\ln^+\left|f(re^{i\varphi})\right|\right)^q d\varphi \right)^{\frac{p}{q}} (1-r)^{\alpha}\, dr < \infty \right\},$$

where $1 \le p, q < \infty$, $\alpha > -1$. These are Banach spaces of functions. The embedding theorems were obtained for analytic classes $LF^{p,q}_{\alpha}$ and $LF^{p_1,q_1}_{\alpha_1}$ (and $LA^{p,q}_{\alpha}$ and $LA^{p_1,q_1}_{\alpha_1}$) for various indices $p, q, p_1, q_1, \alpha, \alpha_1$ recently in [51].

Note also, that using these new embedding and known results for zero sets we get directly various assertion for $LF^{p,q}_{\alpha}$ and $LA^{p,q}_{\alpha}$ spaces for $p \ne q$. This type of approach was developed in paper [51].

We refer the reader to [51] for introduction and new results on zeros of area Nevanlinna type spaces of analytic functions with mixed norm in the unit disk.

Weights, considered in this paper, are of special type. Studying of such type problems for some other weighted analytic area Nevanlinna type spaces is a new and interesting direction of investigations.

Finnaly we mention some recent interesting results of I. Chyzhikov on this topic (see [5], [8], [6], [7] and various references there).

We refer the reader for the second part of this review concerning new interesting results in analytic area Nevanlinna type spaces of several variables and related problems to [52].

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