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Approach of q-Derivative Operators to Terminating q-Series Formulae

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Abstract. The q-derivative operator approach is illustrated by reviewing several typical summation formulae of terminating basic hypergeometric series.

1 Introduction and Motivation

The q-derivative operator is a useful tool for proving q-series identities (cf. Carlitz [10], Chu [16], [18] and Liu [28]). It is defined by

$$\mathcal{D}_x f(x) := \frac{f(x) - f(qx)}{x} \quad \text{and} \quad \mathcal{D}^n f = \mathcal{D}(\mathcal{D}^{n-1}) f \quad \text{for} \quad n = 2, 3, \dots$$

with the convention that $\mathcal{D}_x^0 f(x) = f(x)$ for the identity operator. One can show, by means of the induction principle, the following explicit formula

$$\mathcal{D}_x^n f(x) = x^{-n} \sum_{k=0}^{n} q^k \frac{(q^{-n}; q)_k}{(q; q)_k} f(q^k x), \tag{1}$$

where the q-shifted factorial of x is given by $(x; q)_0 \equiv 1$ and

$$(x; q)_n = \prod_{k=0}^{n-1} (1 - xq^k) \quad \text{for} \quad n = 1, 2, \dots,$$

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with product and quotient forms being abbreviated, respectively, to

$$[a,b,\ldots,c;q]_n = (a;q)_n\,(b;q)_n\cdots(c;q)_n\,,$$

$$\left[\begin{matrix} a,\ b,\ \ldots,\ c \\ A,\, B,\ldots,\, C \end{matrix}\middle| q\right]_n = \frac{(a;q)_n\,(b;q)_n\cdots(c;q)_n}{(A;q)_n\,(B;q)_n\cdots(C;q)_n}.$$

According to (1), we can verify, by means of the q-binomial theorem and the q-Chu--Vandermonde formula, that for a monic polynomial $P_m(x)$ of degree $m \le n$, the following evaluation formulae for higher q-derivatives hold:

$$\mathcal{D}_x^n P_m(x) = \chi(m=n)(q;q)_n, \tag{2}$$

$$\mathcal{D}^n \frac{P_m(x)}{1-\lambda x} = \frac{\lambda^n (q;q)_n}{(\lambda;q)_{n+1}} P_m(1/\lambda), \tag{3}$$

where χ denotes the logical function with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$ otherwise.

The objective of this paper is to review several typical summation formulae of terminating basic hypergeometric series by means of the q-derivative operator. The approach will consist of the following three steps:

- First, for a given a q-series identity, identifying a parameter x as a variable and expressing the q-sum in terms of the higher q-derivatives displayed in (1).
- Then, evaluating the q-sum for particular values of x with the help of q-derivative properties (2) and/or (3).
- Finally, confirming the q-series identity via the fundamental theorem of algebra, i.e., "two polynomials of degrees $\le n$ are identical if they agree at $n+1$ distinct points".

Throughout the paper, the basic hypergeometric series (cf. Bailey [5] and Gasper-Rahman [23]) is defined by

$${}_{1+\ell}\phi_\ell\left[\begin{matrix} a_0,\, a_1,\ldots,\, a_\ell \\ b_1,\ldots,\, b_\ell \end{matrix}\middle| q;z\right] = \sum_{n=0}^{\infty} z^n \left[\begin{matrix} a_0,\, a_1,\ldots,\, a_\ell \\ q,\ b_1,\ldots,\, b_\ell \end{matrix}\middle| q\right]_n,$$

which becomes terminating if one of the numerator parameters $\{a_i\}_{0\le i\le \ell}$ results in q^{-m} with m being a nonnegative integer.

2 The q-Pfaff-Saalschütz theorem

As a warm-up, we start with the following fundamental formula; see [5, Chapter 8] and [31, §3.3].

Theorem 1 (The q-Pfaff-Saalschütz theorem).

$${}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\quad b \\ c,\ q^{1-n}ab/c \end{matrix}\middle| q;q\right] = \left[\begin{matrix} c/a,\ c/b \\ c,\ \ c/ab \end{matrix}\middle| q\right]_n = a^n \left[\begin{matrix} c/a,\ q^{1-n}b/c \\ c,\ \ q^{1-n}ab/c \end{matrix}\middle| q\right]_n. \tag{4}$$

Proof. Multiplying across this equation by $(q^{1-n}ab/c;q)_n$, we see that both sides are polynomials of degree n in b. To prove the identity, it suffices to show that the equality holds for $n+1$ distinct values of b. First it is trivial to see that both sides are equal to 1 for $b=1$. Then for $b=q^{m-1}c$ with $1 \le m \le n$, the right member equals zero. The corresponding left member can be written as the q-binomial sum

$$ {}_3\phi_2\left[\begin{matrix} q^{-n},\, a,\, q^{m-1}c \\ c,\, q^{m-n}a \end{matrix}\middle| q;q\right] = \sum_{k=0}^{n} q^k \frac{(q^{-n};q)_k}{(q;q)_k} \frac{(q^{m-n+k}a;q)_{n-m}(q^k c;q)_{m-1}}{(q^{m-n}a;q)_{n-m}(c;q)_{m-1}}. $$

In view of (1) and (2), the last sum vanishes for $1 \le m \le n$ because it results in a multiple of the nth q-derivative of the polynomial $(cx;q)_{m-1}(q^{m-n}ax;q)_{n-m}$ of degree $n-1$ in x. Therefore we have validated the equality for $n+1$ distinct values of b and completed proof of (4). $\square$

3 The q-Watson Formula

In this section we give a proof of an identity that was first found by Andrews [2, Theorem 1] in 1976 as a terminating q-series analogue of the Watson formula; see also Gasper-Rahman [23, II-17].

Theorem 2. *Let $b = q^{-\delta-2n}$ with $\delta = 0$ or 1 and $n \in \mathbb{N}_0$. Then the following terminating series identity holds:*

$$ {}_4\phi_3\left[\begin{matrix} a,\, b,\, \sqrt{c},\, -\sqrt{c} \\ c,\, \sqrt{qab},\, -\sqrt{qab} \end{matrix}\middle| q;q\right] = (1-\delta)\left[\begin{matrix} q,\ qc/a \\ q/a,\ qc \end{matrix}\middle| q^2\right]_n. \tag{5} $$

For different proofs, see Chu [19, Corollary 7] and Verma-Jain [32, Eq. 1.1].

Proof. Multiplying across the equality (5) by $(qc;q^2)_n$, we can rewrite the resulting equation equivalently as

$$ \sum_{k=0}^{\delta+2n} \frac{(q^{-\delta-2n};q)_k}{(q;q)_k} \frac{(a;q)_k(c;q^2)_k(qc;q^2)_n}{(c;q)_k(q^{1-\delta-2n}a;q^2)_k} q^k = (1-\delta)\frac{(q;q^2)_n(qc/a;q^2)_n}{(q/a;q^2)_n}. \tag{6} $$

According to the relation

$$ \frac{(c;q^2)_k(qc;q^2)_n}{(c;q)_k} = \begin{cases} \dfrac{(qc;q^2)_n(c;q)_{2k}}{(qc;q^2)_k(c;q)_k}, & k \le n; \\[2ex] \dfrac{(c;q^2)_k(c;q)_{2n}}{(c;q^2)_n(c;q)_k}, & k > n; \end{cases} $$

both sides of (6) are polynomials of degree $\le n$ in c. In order to prove (5), it suffices to show that the equality holds for $n+1$ distinct values of c.

Let $\mathcal{S}(c)$ be the ${}_4\phi_3$-series displayed in (5). Then for $c = q^{1-\delta-2m}a$ with $1 \le m \le n$, the right member equals zero. The corresponding left member can be restated as the following sum:

$$ \begin{aligned} \mathcal{S}(q^{1-\delta-2m}a) &= \sum_{k=0}^{\delta+2n} \frac{(q^{-\delta-2n};q)_k}{(q;q)_k} \frac{(a;q)_k(q^{1-\delta-2m}a;q^2)_k}{(q^{1-\delta-2m}a;q)_k(q^{1-\delta-2n}a;q^2)_k} q^k \\ &= \sum_{k=0}^{\delta+2n} \frac{(q^{-\delta-2n};q)_k}{(q;q)_k} \frac{(q^{1-\delta-2m+k}a;q)_{\delta-1+2m}(q^{1-\delta-2n+2k}a;q^2)_{n-m}}{(q^{1-\delta-2m}a;q)_{\delta-1+2m}(q^{1-\delta-2n}a;q^2)_{n-m}} q^k. \end{aligned} $$

In view of (1) and (2), the last sum vanishes for $1-\delta \le m \le n$ because it results in a multiple of the $(\delta+2n)$th q-derivatives of the polynomial

$$(q^{1-\delta-2m}ax;q)_{\delta-1+2m}(q^{1-\delta-2n}ax^2;q^2)_{n-m}$$

of degree $\delta-1+2n$ in x. Therefore, (5) is confirmed when $\delta=1$ because it holds for $n+1$ distinct values of $c \in \{q^{-2m}a\}_{0\le m\le n}$.

When $\delta=0$ and $m=0$, reformulating the last sum and then evaluating it by (3),

$$\begin{aligned}\sum_{k=0}^{2n} q^k \frac{(q^{-2n};q)_k}{(q;q)_k}\frac{(q^{1-2n+2k}a;q^2)_n}{(q^{1-2n}a;q^2)_n}\frac{1-a}{1-q^k a} &= a^{2n}\frac{(q^{1-2n}/a;q^2)_n}{(q^{1-2n}a;q^2)_n}\frac{(q;q)_{2n}}{(qa;q)_{2n}}\\ &= \left[\begin{matrix} q, & q^2 \\ q/a, & q^2a\end{matrix}\middle| q^2\right]_n\end{aligned}$$

which coincides with the right member of (5) under the conditions $\delta=0$ and $c=qa$. Hence, we have validated (5) also when $\delta=0$ for $n+1$ distinct values of $c\in\{q^{1-2m}a\}_{0\le m\le n}$. This completes the proof of the theorem. $\square$

4 Two Balanced ${}_4\phi_3$-Series

Unlike the q-Pfaff-Saalschütz theorem, there exist two summation formulae of balanced ${}_4\phi_3$-series with one less free parameter. They are, in fact, the q-analogues of the particular case $b=2$ of the following well-known Hagen-Rothe convolution identity (cf. Chu [20] and Gould [25]):

$$\sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\binom{c-bk}{n-k}=\binom{a+c}{n}.$$

The first one is due to Al-Salam and Verma [1]; see also Andrews [4, Eq. 7.6], Chu [11, Eq. 5.3b] and [30, Eq. 17.7.12].

Theorem 3.

$${}_4\phi_3\left[\begin{matrix} q^{-2n}, a, qa, q^{2n}b^2 \\ b, qb, q^2a^2\end{matrix}\middle| q^2;q^2\right] = a^n\left[\begin{matrix} -q, & b/a \\ -qa, & b\end{matrix}\middle| q\right]_n. \tag{7}$$

Proof. Multiplying both sides of (7) by $(-qa;q)_n$ and observing that

$$\frac{(a;q)_{2k}(-qa;q)_n}{(q^2a^2;q^2)_k}=\frac{(a;q)_{2k}(-q^{k+1}a;q)_{n-k}}{(qa;q)_k},$$

we infer that the resulting equation is a polynomial identity of degree n in a. In order to prove (7), we need only to validate it for $n+1$ distinct values of a. First of all, (7) is obviously valid for $a=1$. Then denote by $\mathcal{S}(a)$ the ${}_4\phi_3$-series in (7). For $a=q^{m-1}b$ with $1\le m\le n$, the right member of (7) is equal to zero. The corresponding left member can be reformulated as

$$\begin{aligned}\mathcal{S}(q^{m-1}b) &= \sum_{k=0}^{n} \frac{(q^{-2n};q^2)_k}{(q^2;q^2)_k} \frac{(q^{m-1}b;q)_{2k}}{(b;q)_{2k}} \frac{(q^{2n}b^2;q^2)_k}{(q^{2m}b^2;q^2)_k} q^{2k} \\ &= \sum_{k=0}^{n} \frac{(q^{-2n};q^2)_k}{(q^2;q^2)_k} \frac{(q^{2k}b;q)_{m-1}}{(b;q)_{m-1}} \frac{(q^{2m+2k}b^2;q^2)_{n-m}}{(q^{2m}b^2;q^2)_{n-m}} q^{2k},\end{aligned}$$

which vanishes because in base q^2, it results in the nth q-derivative of the polynomial $(bx;q)_{m-1}(q^{2m}b^2x;q^2)_{n-m}$ of degree $n-1$. This confirms (7). □

The second balanced ${}_4\phi_3$-series identity is due to Andrews [3, Eq. 4.3] & [4, Eq. 7.7]; see also Gessel-Stanton [24, Eq. 4.22], Chu [12, Eq. 4.3d] and [30, Eq. 17.7.13].

Theorem 4 (Variant of Theorem 3).

$${}_4\phi_3\left[\begin{matrix}q^{-2n},\, a,\, qa,\, q^{2n-2}b^2\\ b,\, qb,\, \quad a^2\end{matrix}\,\middle|\, q^2;q^2\right] = a^n \left[\begin{matrix}-q,\, b/a\\ -a,\, \;b\end{matrix}\,\middle|\, q\right]_n \frac{1-q^{n-1}b}{1-q^{2n-1}b}.$$

In exactly the same manner, this identity can be proved after the equation having been multiplied by $(-a;q)_n$ across.

5 Four Terminating Well-Poised Series

This section will be devoted to four terminating well-poised series identities of Dixon's type. The first one is essentially due to Jackson [26, Eq. 2]. See Bailey [6, Eq. 2], Bressoud [8, Eq. 2]), Carlitz [9, Eq. 1.2], Chu [21], Verma-Joshi [33, Eq. 3.10] for different proofs, and Bailey [7, Eqs. 2.2 and 1.2], Chu [17], Verma-Jain [32, Eq. 5.5] for the nonterminating form.

Theorem 5. *For $\delta = 0$ or 1 and $n \in \mathbb{N}_0$, there holds the terminating series identity:*

$$\begin{aligned}&{}_3\phi_2\left[\begin{matrix}q^{-2n}, & b, & d\\ & q^{1-2n}/b, & q^{1-2n}/d\end{matrix}\,\middle|\, q;q^{1+\delta-n}/bd\right]\\ &\qquad = q^{n(\delta-1)} \left[\begin{matrix}b,\, d\\ q,\, bd\end{matrix}\,\middle|\, q\right]_n \left[\begin{matrix}q,\, bd\\ b,\, d\end{matrix}\,\middle|\, q\right]_{2n}. \qquad (8)\end{aligned}$$

Proof. Multiplying across (8) by $(q^nb;q)_n$, we may rewrite the resulting equation equivalently as

$$\sum_{k=0}^{2n} \frac{(q^{-2n};q)_k}{(q;q)_k} \frac{(b;q)_k(q^nb;q)_n(d;q)_k}{(q^{1-2n}/b;q)_k(q^{1-2n}/d;q)_k}\Big(\frac{q^{1+\delta-n}}{bd}\Big)^k = \frac{(q^{n+1};q)_n(q^nbd;q)_n}{q^{n(1-\delta)}(q^nd;q)_n}. \qquad (9)$$

Observing the relation

$$\begin{aligned}\frac{(b;q)_k(q^nb;q)_n}{b^k(q^{1-2n}/b;q)_k} &= (-1)^k q^{2nk-\binom{k+1}{2}} \frac{(b;q)_k(q^nb;q)_n}{(q^{2n-k}b;q)_k}\\ &= (-1)^k q^{2nk-\binom{k+1}{2}} \frac{(b;q)_k(b;q)_{2n-k}}{(b;q)_n},\end{aligned}$$

we assert that (9) is a polynomial identity of degree n in b. In order to prove (8), it suffices to show that the equality holds for $n+1$ distinct values of b.

Let $\mathcal{S}(b)$ be the ${}_3\phi_2$-series displayed in (8). Then for $b = q^{m-2n}/d$ with $1 \leq m \leq n$, the right member equals zero. The corresponding left member can be written as the expression

$$\begin{aligned}\mathcal{S}(q^{m-2n}/d) &= \sum_{k=0}^{2n} \frac{(q^{-2n};q)_k}{(q;q)_k} \left[\begin{matrix} d, & q^{m-2n}/d \\ q^{1-m}d, & q^{1-2n}/d \end{matrix}\middle| q\right]_k q^{k(1+\delta-m+n)} \\ &= \sum_{k=0}^{2n} \frac{(q^{-2n};q)_k}{(q;q)_k} \left[\begin{matrix} q^{1+k-m}d, & q^{1+k-2n}/d \\ q^{1-m}d, & q^{1-2n}/d \end{matrix}\middle| q\right]_{m-1} q^{k(1+\delta-m+n)}.\end{aligned}$$

In view of (1) and (2), the last sum vanishes for $1 \leq m \leq n$ because it results in a multiple of the $(2n)$th q-derivative of the polynomial

$$(q^{1-m}xd;q)_{m-1}(q^{1-2n}x/d;q)_{m-1}x^{\delta-m+n}$$

of degree $\delta - 2 + m + n < 2n$ in x.

When $m = 0$, the last sum can be reformulated by partial fractions and then evaluated by (3) as

$$\begin{aligned}&\sum_{k=0}^{2n} q^k \frac{(q^{-2n};q)_k}{(q;q)_k} \frac{(1-d)(1-q^{-2n}/d)}{(1-q^k d)(1-q^{k-2n}/d)} q^{k(\delta+n)} \\ &\quad = \frac{(1-d)(1-q^{-2n}/d)}{1-q^{2n}d^2} \sum_{k=0}^{2n} q^k \frac{(q^{-2n};q)_k}{(q;q)_k} \left\{ \frac{q^{k(\delta+n)}}{1-q^{k-2n}/d} - \frac{q^{k(\delta+n)+2n}d^2}{1-q^k d} \right\} \\ &\quad = \frac{(1-d)(1-q^{-2n}/d)}{1-q^{2n}d^2} \left\{ \frac{(q;q)_{2n}(q^{2n}d)^{\delta-n}}{(q^{-2n}/d;q)_{2n+1}} - \frac{(q;q)_{2n}q^{2n}d^{n-\delta+2}}{(d;q)_{2n+1}} \right\}.\end{aligned}$$

The above expression can be easily simplified to

$$(q^{\delta-1}d)^n \frac{(q;q)_{2n}}{(d;q)_{2n}} \frac{1-d}{1-q^n d},$$

which agrees with the right member of (8) specified by $b = q^{-2n}/d$. Therefore for $n+1$ distinct values of $b \in \{q^{m-2n}/d\}_{0 \leq m \leq n}$, we have validated (8), which completes the proof. □

The next formula serves as a counterpart of (8) whose nonterminating version was found by Bailey [7, Eq. 2.3]. For different proofs, the reader may refer to Carlitz [9, Eq. 2.12], Chu [17], Chu-Wang [21] and Verma-Joshi [33, Eq. 3.13].

Theorem 6. *For $\delta = 0$ or 1 and $n \in \mathbb{N}_0$, there holds the terminating series identity:*

$$\begin{aligned}&{}_3\phi_2 \left[\begin{matrix} q^{-1-2n}, & b, & d \\ & q^{-2n}/b, & q^{-2n}/d \end{matrix}\middle| q; q^{2\delta-n}/bd\right] \\ &\qquad = \left\{1 - q^{(2n+1)(2\delta-1)}\right\} \left[\begin{matrix} qb, & qd \\ q, & qbd \end{matrix}\middle| q\right]_n \left[\begin{matrix} q, & qbd \\ qb, & qd \end{matrix}\middle| q\right]_{2n}. \qquad (10)\end{aligned}$$

Proof. Multiplying across (10) by $(q^{n+1}b;q)_n$, we may rewrite the resulting equation equivalently as

$$\sum_{k=0}^{2n+1}\frac{(q^{-1-2n};q)_k}{(q;q)_k}\frac{(b;q)_k(q^{n+1}b;q)_n(d;q)_k}{(q^{-2n}/b;q)_k(q^{-2n}/d;q)_k}\Big(\frac{q^{2\delta-n}}{bd}\Big)^k$$
$$=\Big(1-q^{(2n+1)(2\delta-1)}\Big)\frac{(q^{n+1};q)_n(q^{n+1}bd;q)_n}{(q^{n+1}d;q)_n}. \quad (11)$$

Observing that

$$\frac{(b;q)_k(q^{n+1}b;q)_n}{b^k(q^{-2n}/b;q)_k}=(-1)^kq^{2nk-\binom{k}{2}}\frac{(b;q)_k(q^{n+1}b;q)_n}{(q^{1+2n-k}b;q)_k}$$
$$=(-1)^kq^{2nk-\binom{k}{2}}\frac{(b;q)_k(b;q)_{1+2n-k}}{(b;q)_{n+1}},$$

we assert that both sides of (11) are polynomials of degree n in b. In order to prove (10), it suffices to show that the equality holds for $n+1$ distinct values of b.

Let $\mathcal{S}(b)$ be the ${}_3\phi_2$-series displayed in (10). Then for $b=q^{-m-n}/d$ with $1\le m\le n$, the right member equals zero. The corresponding left member can be written as the expression

$$\mathcal{S}(q^{-m-n}/d)=\sum_{k=0}^{2n+1}\frac{(q^{-1-2n};q)_k}{(q;q)_k}\left[\begin{matrix}d, & q^{-m-n}/d\\ q^{m-n}d, & q^{-2n}/d\end{matrix}\middle|\,q\right]_k q^{k(2\delta+m)}$$
$$=\sum_{k=0}^{2n+1}\frac{(q^{-1-2n};q)_k}{(q;q)_k}\left[\begin{matrix}q^{k+m-n}d, & q^{k-2n}/d\\ q^{m-n}d, & q^{-2n}/d\end{matrix}\middle|\,q\right]_{n-m} q^{k(2\delta+m)}.$$

In view of (1) and (2), the last sum vanishes for $1\le m\le n$ because it is a multiple of the $(2n+1)$th q-derivative of the polynomial

$$(q^{m-n}xd;q)_{n-m}(q^{-2n}x/d;q)_{n-m}x^{2\delta-1+m}$$

of degree $2n+2\delta-m-1<2n+1$ in x.

When $m=n+1$, we can rewrite the last sum by partial fractions and then evaluate it by (3) as

$$\sum_{k=0}^{2n+1}q^k\frac{(q^{-1-2n};q)_k}{(q;q)_k}\frac{(1-d)(1-q^{-1-2n}/d)}{(1-q^kd)(1-q^{k-1-2n}/d)}q^{k(2\delta+n)}$$
$$=\frac{(1-d)(1-q^{-2n-1}/d)}{1-q^{1+2n}d^2}$$
$$\times\sum_{k=0}^{2n+1}q^k\frac{(q^{-1-2n};q)_k}{(q;q)_k}\Big\{\frac{q^{k(2\delta+n)}}{1-q^{k-1-2n}/d}-\frac{q^{k(2\delta+n)+1+2n}d^2}{1-q^kd}\Big\}$$
$$=\frac{(1-d)(1-q^{-1-2n}/d)}{1-q^{1+2n}d^2}$$
$$\times\Big\{\frac{(q;q)_{2n+1}(q^{1+2n}d)^{2\delta-1-n}}{(q^{-1-2n}/d;q)_{2n+2}}-\frac{(q;q)_{2n+1}q^{1+2n}d^{3+n-2\delta}}{(d;q)_{2n+2}}\Big\}.$$

The above expression can be further simplified to

$$d^n \frac{(q;q)_{2n}}{(d;q)_{2n}} \frac{1-d}{1-q^{1+2n}d} \Big\{1 - q^{(2n+1)(2\delta-1)}\Big\},$$

which agrees with the right member of (10) specified by $b = q^{-1-2n}/d$. Therefore for $n+1$ distinct values of $b \in \{q^{m-2n}/d\}_{1 \le m \le n+1}$, we have validated (10), which completes the proof. □

There is also the following more general well-poised ${}_5\phi_4$-series identity discovered by Jackson [26, Eq. 1], that different proofs can be found in Bailey [7, Eq. 3.1], Bressoud [8, Eq. 1], Chu [14, §2] and Verma-Joshi [33, Eq. 3.8].

Theorem 7. *For $\delta = 0$ or 1 and $n \in \mathbb{N}_0$, there holds the terminating series identity:*

$$\begin{aligned}{}_5\phi_4 &\left[\begin{matrix} q^{-2n}, & b, & c, & d, & q^{1-3n}/bcd \\ & q^{1-2n}/b, & q^{1-2n}/c, & q^{1-2n}/d, & q^n bcd \end{matrix} \middle| q; q^{1+\delta}\right] \\ &= q^{n(\delta-1)} \left[\begin{matrix} b, \ c, \ d, \ bcd \\ q, bc, bd, \ cd \end{matrix} \middle| q\right]_n \left[\begin{matrix} q, bc, bd, \ cd \\ b, \ c, \ d, \ bcd \end{matrix} \middle| q\right]_{2n}. \end{aligned} \tag{12}$$

Proof. Multiplying across (12) by $(q^n b;q)_n (q^n bcd;q)_n$, we may rewrite the resulting equation equivalently as

$$\begin{aligned}\sum_{k=0}^{2n} \frac{(q^{-2n};q)_k}{(q;q)_k} &\frac{[b,c,d,q^{1-3n}/bcd;q]_k (q^n b;q)_n (q^n bcd;q)_n}{[q^{1-2n}/b, q^{1-2n}/c, q^{1-2n}/d, q^n bcd;q]_k} q^{k(1+\delta)} \\ &= q^{n(\delta-1)} \frac{(q^{n+1};q)_n (q^n bc;q)_n (q^n bd;q)_n (q^n cd;q)_n}{(q^n c;q)_n (q^n d;q)_n}. \end{aligned} \tag{13}$$

According to the relation

$$\begin{aligned}&\frac{(b;q)_k (q^n b;q)_n}{(q^{1-2n}/b;q)_k} \frac{(q^{1-3n}/bcd;q)_k (q^n bcd;q)_n}{(q^n bcd;q)_k} \\ &= \Big(\frac{q^{-n}}{cd}\Big)^k \frac{(b;q)_k (q^n b;q)_n}{(q^{2n-k}b;q)_k} \frac{(q^{3n-k}bcd;q)_k (q^n bcd;q)_n}{(q^n bcd;q)_k} \\ &= \Big(\frac{q^{-n}}{cd}\Big)^k \frac{(b;q)_k (b;q)_{2n-k}}{(b;q)_n} \frac{(bcd;q)_{2n} (bcd;q)_{3n}}{(bcd;q)_{n+k} (bcd;q)_{3n-k}},\end{aligned}$$

both sides of (13) become polynomials of degree $2n$ in b. In order to prove (12), it suffices to show that the equality holds for $2n+1$ distinct values of b.

First, for $b = 1$ in (12), the ${}_5\phi_4$-series becomes $1 + q^{n(2\delta-1)}$ because only the two extreme terms survive. This coincides with the corresponding right member.

Let $\mathcal{S}(b)$ be the ${}_5\phi_4$-series displayed in (12). Then for $b = q^{m-2n}/c$ with $1 \le m \le n$, the right member equals zero. The corresponding left member can be written as the expression

$$\begin{aligned}\mathcal{S}(q^{m-2n}/c) &= \sum_{k=0}^{2n}\frac{(q^{-2n};q)_k}{(q;q)_k}\left[\begin{matrix} c, & d, & q^{m-2n}/c, & q^{1-m-n}/d \\ q^{1-m}c, & q^{m-n}d, & q^{1-2n}/c, & q^{1-2n}/d\end{matrix}\middle| q\right]_k q^{k(1+\delta)} \\ &= \sum_{k=0}^{2n}\frac{(q^{-2n};q)_k}{(q;q)_k}\left[\begin{matrix} q^{1+k-m}c, & q^{1+k-2n}/c \\ q^{1-m}c, & q^{1-2n}/c\end{matrix}\middle| q\right]_{m-1} \\ &\qquad\qquad\times\left[\begin{matrix} q^{k+m-n}d, & q^{1+k-2n}/d \\ q^{m-n}d, & q^{1-2n}/d\end{matrix}\middle| q\right]_{n-m} q^{k+k\delta}.\end{aligned}$$

In view of (1) and (2), the last sum vanishes for $1 \le m \le n$ because it results in a multiple of the $(2n)$th q-derivative of the following polynomial

$$x^{\delta}[q^{1-m}xc, q^{1-2n}x/c; q]_{m-1}[q^{m-n}xd, q^{1-2n}x/d; q]_{n-m}$$

of degree $\delta - 2 + 2n < 2n$ in x.

Because (12) is symmetric with respect to c and d, it holds also for $b = q^{m-2n}/d$ with $1 \le m \le n$. Therefore for $2n+1$ distinct values of

$$b \in \{1\} \cup \{q^{m-2n}/c\}_{1\le m\le n} \cup \{q^{m-2n}/d\}_{1\le m\le n},$$

we have validated (12), which completes the proof. □

Finally, we record the following counterpart of (12) due to Bailey [7, Eq. 3.2] for which the reader can find different proofs in Carlitz [9, Eq. 3.4], Chu [14, §2] and Verma-Joshi [33, Eq. 3.12].

Theorem 8. *For $\delta = 0$ or 1 and $n \in \mathbb{N}_0$, there holds the terminating series identity:*

$$\begin{aligned}&{}_5\phi_4\left[\begin{matrix} q^{-1-2n}, & b, & c, & d, & q^{-1-3n}/bcd \\ & q^{-2n}/b, & q^{-2n}/c, & q^{-2n}/d, & q^{1+n}bcd\end{matrix}\middle| q; q^{1+2\delta}\right] \\ &= (-q^{1+2n})^{\delta-1}(q;q)_{2n+1}\left[\begin{matrix} qb, & qc, & qd, & qbcd \\ q, & qbc, & qbd, & qcd\end{matrix}\middle| q\right]_n \left[\begin{matrix} & qbc, & qbd, & qcd \\ qb, & qc, & qd, & qbcd\end{matrix}\middle| q\right]_{2n}. \qquad (14)\end{aligned}$$

Proof. Multiplying across (14) by $(q^{n+1}b;q)_n(q^{n+1}bcd;q)_n$, we may rewrite the resulting equation equivalently as

$$\begin{aligned}&\sum_{k=0}^{2n}\frac{(q^{-1-2n};q)_k}{(q;q)_k}\frac{[b,c,d,q^{-1-3n}/bcd;q]_k(q^{n+1}b;q)_n(q^{n+1}bcd;q)_n}{[q^{-2n}/b,q^{-2n}/c,q^{-2n}/d,q^{1+n}bcd;q]_k}q^{k(1+2\delta)} \\ &\qquad= (-q^{1+2n})^{\delta-1}\frac{(q^{n+1};q)_{n+1}(q^{n+1}bc;q)_n(q^{n+1}bd;q)_n(q^{n+1}cd;q)_n}{(q^{n+1}c;q)_n(q^{n+1}d;q)_n}. \qquad (15)\end{aligned}$$

Observing that

$$\begin{aligned}&\frac{(b;q)_k(q^{n+1}b;q)_n}{(q^{-2n}/b;q)_k}\frac{(q^{-1-3n}/bcd;q)_k(q^{n+1}bcd;q)_n}{(q^{n+1}bcd;q)_k} \\ &\qquad= \Big(\frac{q^{-n-1}}{cd}\Big)^k\frac{(b;q)_k(q^{n+1}b;q)_n}{(q^{2n+1-k}b;q)_k}\frac{(q^{2+3n-k}bcd;q)_k(q^{n+1}bcd;q)_n}{(q^{n+1}bcd;q)_k} \\ &\qquad= \Big(\frac{q^{-n-1}}{cd}\Big)^k\frac{(b;q)_k(b;q)_{1+2n-k}}{(b;q)_{n+1}}\frac{(bcd;q)_{2n+1}(bcd;q)_{3n+2}}{(bcd;q)_{n+k+1}(bcd;q)_{3n+2-k}}\end{aligned}$$

we assert that both sides of (15) are polynomials of degree $2n$ in b. In order to prove (14), it suffices to show that the equality holds for $2n+1$ distinct values of b.

First, for $b=1$ in (14), the corresponding ${}_5\phi_4$-series reduces to $1-q^{(2n+1)(2\delta-1)}$ because only the two extreme terms survive. It is trivial to check that the right member has the same value in this case.

Let $\mathcal{S}(b)$ be the ${}_5\phi_4$-series displayed in (14). Then for $b=q^{-m-n}/c$ with $1\le m\le n$, the right member equals zero. The corresponding left member can be written as the expression

$$\begin{aligned}\mathcal{S}(q^{-m-n}/c) &= \sum_{k=0}^{2n+1}\frac{(q^{-1-2n};q)_k}{(q;q)_k}\left[\begin{matrix}c, & d, & q^{-m-n}/c, & q^{m-2n-1}/d\\ q^{m-n}c, & q^{1-m}d, & q^{-2n}/c, & q^{-2n}/d\end{matrix}\middle| q\right]_k q^{k+2k\delta}\\ &= \sum_{k=0}^{2n+1}\frac{(q^{-1-2n};q)_k}{(q;q)_k}\left[\begin{matrix}q^{k+m-n}c, & q^{k-2n}/c\\ q^{m-n}c, & q^{-2n}/c\end{matrix}\middle| q\right]_{n-m}\\ &\qquad\qquad\qquad\times\left[\begin{matrix}q^{1+k-m}d, & q^{k-2n}/d\\ q^{1-m}d, & q^{-2n}/d\end{matrix}\middle| q\right]_{m-1} q^{k+2k\delta}.\end{aligned}$$

In view of (1) and (2), the last sum vanishes for $1\le m\le n$ because it results in a multiple of the $(2n+1)$th q-derivative of the polynomial

$$x^{2\delta}[q^{m-n}xc, q^{-2n}x/c;q]_{n-m}[q^{1-m}xd, q^{-2n}x/d;q]_{m-1}$$

of degree $2\delta-2+2n<2n+1$ in x.

Analogously, (14) is valid also for $b=q^{-m-n}/d$ with $1\le m\le n$ for its symmetry with respect to c and d. In conclusion, we have shown (14) for $2n+1$ distinct values of $b\in\{1\}\cup\{q^{-m-n}/c\}_{1\le m\le n}\cup\{q^{-m-n}/d\}_{1\le m\le n}$, which completes the proof. □

6 Gasper's q-Karlsson-Minton Formula

Finally, we examine the q-analogue of Gasper [22] (see also Chu [13]) for a classical hypergeometric sum due to Minton [29] and subsequently extended by Karlsson [27].

Theorem 9. *For nonnegative integers m_i and n with $n\ge\sum_{i=1}^{\ell}m_i$, we have*

$${}_{\ell+2}\phi_{\ell+1}\left[\begin{matrix}q^{-n}, & \lambda, & \{q^{m_i}a_i\}_{i=1}^{\ell}\\ & q\lambda, & \{a_i\}_{i=1}^{\ell}\end{matrix}\middle| q;q\right]=\lambda^n\frac{(q;q)_n}{(q\lambda;q)_n}\prod_{i=1}^{\ell}\frac{(a_i/\lambda;q)_{m_i}}{(a_i;q)_{m_i}}. \tag{16}$$

Its nonterminating form and extensions can be found in Gasper [22] and Chu [13], [15]. However, we believe that the proof given here is the simplest.

Proof. According to the relation

$$\frac{(q^{m_i}a_i;q)_k}{(a_i;q)_k}=\frac{(q^k a_i;q)_{m_i}}{(a_i;q)_{m_i}}$$

we may express (16) equivalently as the equality

$$\sum_{k=0}^{n} q^k \frac{(q^{-n};q)_k}{(q;q)_k} \frac{\prod_{i=1}^{\ell}(q^k a_i;q)_{m_i}}{1-q^k\lambda} = \lambda^n \frac{(q;q)_n}{(\lambda;q)_{n+1}} \prod_{i=1}^{\ell}(a_i/\lambda;q)_{m_i}. \tag{17}$$

Writing the last sum in terms of q-derivatives (1) and then evaluating it, by (3), as

$$\mathcal{D}^n \frac{\prod_{i=1}^{\ell}(a_i x;q)_{m_i}}{1-\lambda x}\Big|_{x=1} = \lambda^n \frac{(q;q)_n}{(\lambda;q)_{n+1}} \prod_{i=1}^{\ell}(a_i/\lambda;q)_{m_i},$$

we confirm (17) and so Gasper's summation formula (16). □

7 Concluding comments

It should be pointed out that the approach presented in this paper works only for some q-series identities. For example, we have failed to verify the following q-Whipple formula due to Andrews [2, Theorem 2] (cf. Chu [19, Corollary 10] and Verma-Jain [32, Eq. 1.2]):

$${}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}, \sqrt{c}, -\sqrt{c} \\ -q, \quad e, \quad qc/e \end{matrix}\middle| q;q\right] = q^{\binom{n+1}{2}} \frac{(q^{-n}e;q^2)_n (q^{1-n}c/e;q^2)_n}{(e;q)_n (qc/e;q)_n},$$

even though this will evidently become a polynomial identity of degree n in c if multiplying it by the factorial $(qc/e;q)_n$.

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