

Geometry of Mus-Sasaki metric

Abderrahim Zagane, Mustapha Djaa

Abstract. In this paper, we introduce the Mus-Sasaki metric on the tangent bundle TM as a new natural metric non-rigid on TM . First we investigate the geometry of the Mus-Sasakian metrics and we characterize the sectional curvature and the scalar curvature.

1 Introduction

The geometry of the tangent bundle TM equipped with Sasaki metric has been studied by many authors Sasaki [16], K. Yano and S. Ishihara [18], P. Dombrowski [7], A. Salimov, A. Gezer and N. Cengiz (see [2], [8], [12], [13], [14]) etc. The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on TM . J. Cheeger and D. Gromoll has introduced the notion of Cheeger-Gromoll metric [3], this metric has been studied also by many authors (see [1], [4], [5], [6], [9], [10], [11], [15], [17]).

In this paper, we introduce the Mus-Sasaki metric on the tangent bundle TM as a new natural metric non-rigid on TM . First we investigate the geometry of the Mus-Sasakian metrics and we characterize the sectional curvature (Theorem 4 and Theorem 5) and the scalar curvature (Theorem 6 and Theorem 7).

1.1 Basic Notions and Definition on TM

Let (M, g) be an m -dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1, \dots, n}$ on M induces a local chart

$$(\pi^{-1}(U), x^i, y^j)_{i=1, \dots, n}$$

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on TM . Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g .

We have two complementary distributions on TM , the vertical distribution \mathcal{V} and the horizontal distribution \mathcal{H} , defined by:

$$\begin{aligned}\mathcal{V}_{(x,u)} &= \text{Ker}(d\pi_{(x,u)}) = \left\{ a^i \frac{\partial}{\partial y^i} \Big|_{(x,u)} : a^i \in \mathbb{R} \right\} \\ \mathcal{H}_{(x,u)} &= \left\{ a^i \frac{\partial}{\partial x^i} \Big|_{(x,u)} - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \Big|_{(x,u)} : a^i \in \mathbb{R} \right\},\end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M . The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \left\{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \right\} \tag{2}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1, \dots, n}$ is a local adapted frame on TTM .

Proposition 1. (see [18]) *Let (M, g) be a Riemannian manifold and R its tensor curvature, then for all vector fields $X, Y \in \Gamma(TM)$ and $p \in T^2M$ we have:*

$$\begin{aligned}[X^H, Y^H]_p &= [X, Y]_p^H - (R_x(X, Y)u)^V, \\ [X^H, Y^V]_p &= (\nabla_X Y)_p^V, \\ [X^V, Y^V]_p &= 0,\end{aligned}$$

where $p = (x, u)$.

Definition 1. Let (M, g) be a Riemannian manifold and $K: TM \rightarrow TM$ be a smooth bundle endomorphism of TM . Then the vertical and horizontal vector fields VK and HK are defined on TM by

$$\begin{aligned}VK: TM &\rightarrow TTM \\ (x, u) &\mapsto (K(u))^V, \\ HK: TM &\rightarrow TTM \\ (x, u) &\mapsto (K(u))^H,\end{aligned}$$

locally we have

$$VK = y^i K_i^j \frac{\partial}{\partial y^j} = y^i \left(K \left(\frac{\partial}{\partial x^i} \right) \right)^V \tag{3}$$

$$HK = y^i K_i^j \frac{\partial}{\partial x^j} - y^i y^k K_i^j \Gamma_{jk}^s \frac{\partial}{\partial y^s} = y^i \left(K \left(\frac{\partial}{\partial x^i} \right) \right)^H. \tag{4}$$

2 Mus-Sasaki metric.

Definition 2. Let (M, g) be a Riemannian manifold and $f: M \times \mathbb{R} \rightarrow (0, +\infty)$. On the tangent bundle TM , we define a Mus-Sasaki metric noted g_f^S by

$$\begin{aligned} g_f^S(X^H, Y^H)_{(x,u)} &= g_x(X, Y) \\ g_f^S(X^H, Y^V)_{(x,u)} &= 0 \\ g_f^S(X^V, Y^V)_{(x,u)} &= f(x, r)g_x(X, Y) \end{aligned}$$

where $X, Y \in \Gamma(TM)$, $(x, u) \in TM$ and $r = g(u, u)$. The function f is called twisting function.

Note that, if $f = 1$ then g_f^S is the Sasaki metric [18].

Lemma 1. Let (M, g) be a Riemannian manifold, then for all $x \in M$ and $u = u^i \frac{\partial}{\partial x^i} \in T_x M$, we have the following

$$\begin{aligned} X^H(g(u, u))_{(x,u)} &= 0 \\ X^H(g(Y, u))_{(x,u)} &= g(\nabla_X Y, u)_x \\ X^V(g(u, u))_{(x,u)} &= 2g(X, u)_x \\ X^V(g(Y, u))_{(x,u)} &= g(X, Y)_x \end{aligned}$$

Proof. Locally, Lemma 1 follows from formulas (1) and (2). □

From Lemma 2 we obtain

Lemma 2. Let (M, g) be a Riemannian manifold, $F: \mathbb{R}^2 \rightarrow [0, +\infty]$, $\alpha: M \rightarrow (0, +\infty)$ and $\beta: \mathbb{R} \rightarrow (0, +\infty)$ be smooth functions. If $f(x, r) = F(\alpha(x), \beta(r))$, then we have the following

$$\begin{aligned} X^V(f)_{(x,u)} &= 2\beta'(r)g_x(X, u) \frac{\partial F}{\partial t}(\alpha(x), \beta(r)) \\ X^H(f)_{(x,u)} &= g_x(\text{grad}_M \alpha, X) \frac{\partial F}{\partial s}(\alpha(x), \beta(r)) = X(\alpha) \frac{\partial F}{\partial s}(\alpha(x), \beta(r)) \end{aligned}$$

where $(x, u) \in TM$ and $r = g_x(u, u)$.

In the following, we consider $f(x, r) = F(\alpha(x), \beta(r))$, where $F: \mathbb{R}^2 \rightarrow [0, +\infty]$, $\alpha: M \rightarrow (0, +\infty)$ and $\beta: \mathbb{R} \rightarrow (0, +\infty)$ are smooth functions.

Lemma 3. Let (M, g) be a Riemannian manifold. If $f(x, r) = F(\alpha(x), \beta(r))$ and ∇ (resp. $\tilde{\nabla}$) denote the Levi-Civita connection of (M, g) (resp. (TM, g_f^S)), then we have:

$$\begin{aligned} g_f^S(\tilde{\nabla}_{X^H} Y^H, Z^H) &= g_f^S((\nabla_X Y)^H, Z^H) \\ g_f^S(\tilde{\nabla}_{X^H} Y^H, Z^V) &= -\frac{1}{2}g_f^S((R(X, Y)u)^V, Z^V) \end{aligned}$$

$$\begin{aligned}
g_f^S(\tilde{\nabla}_{X^H}Y^V, Z^H) &= g_f^S\left(\frac{f}{2}(R(u, Y)X)^H, Z^H\right) \\
g_f^S(\tilde{\nabla}_{X^H}Y^V, Z^V) &= g_f^S\left(\frac{1}{2}X(\alpha)\frac{\partial \ln F}{\partial s}Y^V + (\nabla_X Y)^V, Z^V\right) \\
g_f^S(\tilde{\nabla}_{X^V}Y^H, Z^H) &= g_f^S\left(\left(\frac{f}{2}R(u, X)Y\right)^H, Z^H\right) \\
g_f^S(\tilde{\nabla}_{X^V}Y^H, Z^V) &= g_f^S\left(\frac{1}{2}Y(\alpha)\frac{\partial \ln F}{\partial s}X^V, Z^V\right) \\
g_f^S(\tilde{\nabla}_{X^V}Y^V, Z^H) &= g_f^S\left(-\frac{1}{2}g(X, Y)\frac{\partial F}{\partial s}(\text{grad}(\alpha))^H, Z^H\right) \\
g_f^S(\tilde{\nabla}_{X^V}Y^V, Z^V) &= \beta'\frac{\partial \ln F}{\partial t}g_f^S(g(X, u)Y^V + g(Y, u)X^V - g(X, Y)U^V, Z^V)
\end{aligned}$$

The proof of Lemma 3 follows directly from Kozul formula, Definition 2, Lemma 1 and Lemma 2.

As a direct consequence of Lemma 3, we get the following theorem.

Theorem 1. *Let (M, g) be a Riemannian manifold. If $f(x, r) = F(\alpha(x), \beta(r))$ and ∇ (resp. $\tilde{\nabla}$) denote the Levi-Civita connection of (M, g) (resp. (TM, g_f^S)), then we have:*

$$\begin{aligned}
(\tilde{\nabla}_{X^H}Y^H)_p &= (\nabla_X Y)_p^H - \frac{1}{2}(R_x(X, Y)u)^V \\
(\tilde{\nabla}_{X^H}Y^V)_p &= (\nabla_X Y)_p^V + \frac{f(x, r)}{2}(R_x(u, Y)X)^H + \frac{1}{2}X(\alpha)\frac{\partial \ln F}{\partial s}(\alpha(x), \beta(r))Y_p^V \\
(\tilde{\nabla}_{X^V}Y^H)_p &= \frac{f(x, r)}{2}(R_x(u, X)Y)^H + \frac{1}{2}Y(\alpha)\frac{\partial \ln F}{\partial s}(\alpha(x), \beta(r))X_p^V \\
(\tilde{\nabla}_{X^V}Y^V)_p &= \beta'(r)\frac{\partial \ln F}{\partial t}(\alpha(x), \beta(r))[g_x(Y, U)X_p^V + g_x(X, U)Y_p^V - g_x(X, Y)U_p^V] \\
&\quad - \frac{1}{2}g_x(X, Y)\frac{\partial F}{\partial s}(\alpha(x), \beta(r))(\text{grad}_M \alpha)_p^H
\end{aligned}$$

for all vector fields $X, Y \in \Gamma(TM)$ and $p = (x, u) \in TM$, where R denotes the curvature tensor of (M, g) .

From Definition 1 and Theorem 1 we have:

Proposition 2. *Let (M, g) be a Riemannian manifold and $\tilde{\nabla}$ be the Levi-Civita connection of the tangent bundle (TM, g_f^S) equipped with the Mus-Sasaki metric. If K is a tensor field of type $(1, 1)$ on M , then:*

$$\begin{aligned}
(\tilde{\nabla}_{X^H}HK)_{(x, u)} &= H(\nabla_X K)(x, u) - \frac{1}{2}\left(R_x(X_x, K_x(u))u\right)^V \\
(\tilde{\nabla}_{X^H}VK)_{(x, u)} &= V(\nabla_X K)(x, u) + \frac{f(x, r)}{2}\left(R_x(u, K_x(u))X_x\right)^H \\
&\quad + \frac{1}{2}X(\alpha)\frac{\partial \ln F}{\partial s}(\alpha(x), \beta(r))(VK)_p
\end{aligned}$$

$$\begin{aligned}
 (\tilde{\nabla}_{X^V} HK)_{(x,u)} &= (K(X))_{(x,u)}^H + \frac{f(x,r)}{2} \left(R_x(u, X_x) K_x(u) \right)^H \\
 &\quad + \frac{1}{2} g_x(\text{grad}_M \alpha, K_x(u)) \frac{\partial \ln F}{\partial s}(\alpha(x), \beta(r)) X_p^V \\
 (\tilde{\nabla}_{X^V} VK)_{(x,u)} &= (K(X))_{(x,u)}^V - \frac{1}{2} g_x(X, K_x(u)) \frac{\partial F}{\partial s}(\alpha(x), \beta(r)) (\text{grad}_M \alpha)_p^H \\
 &\quad + \beta'(r) \frac{\partial \ln F}{\partial t}(\alpha(x), \beta(r)) \left[g_x(K_x(u), u) X_p^V \right. \\
 &\quad \left. + g_x(X, U)(VK)_p - g_x(X, K_x(u)) U_p^V \right]
 \end{aligned}$$

where $(x, u) \in TM$ and $X \in \Gamma(TM)$.

3 Curvatures of Mus-Sasaki metric.

Theorem 2. Let (M, g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If R (resp. \tilde{R}) denote the Riemann curvature tensor of M (resp. TM), then we have the following formulae.

$$\begin{aligned}
 \tilde{R}_{(x,u)}(X^H, Y^H)Z^H &= \left\{ \frac{1}{2}(\nabla_Z R)(X, Y)u + \frac{1}{2}Z(\alpha) \frac{\partial F}{\partial s}(\alpha(x), \beta(r))R(X, Y) \right. \\
 &\quad \left. - \frac{1}{4}X(\alpha) \frac{\partial \ln F}{\partial s}(R(Y, Z)u + \frac{1}{4}Y(\alpha) \frac{\partial \ln F}{\partial s}R(X, Z)u) \right\}_x^V \\
 &\quad + \left\{ R(X, Y)Z + \frac{f(x,r)}{4}R(u, R(Z, Y)u)X \right. \\
 &\quad \left. + \frac{f(x,r)}{4}R(u, R(X, Z)u)Y + \frac{f(x,r)}{2}R(u, R(X, Y)u)Z \right\}_x^H \quad (5)
 \end{aligned}$$

$$\begin{aligned}
 \tilde{R}(X^H, Y^H)Z^V &= \left\{ R(X, Y)Z - \frac{f(x,r)}{4}R(X, R(u, Z)Y)u \right. \\
 &\quad \left. + \frac{f(x,r)}{4}R(Y, R(u, Z)X)u \right. \\
 &\quad \left. + \beta'(r) \frac{\partial \ln F}{\partial t} [g(Z, u)R(X, Y)u - g(Z, R(X, Y)u)u] \right\}^V \quad (6) \\
 &\quad + \left\{ \frac{f(x,r)}{2} [(\nabla_X R)(u, Z)Y - (\nabla_Y R)(u, Z)X \right. \\
 &\quad \left. + \frac{1}{2}X(\alpha) \frac{\partial \ln F}{\partial s}R(u, Z)Y - \frac{1}{2}Y(\alpha) \frac{\partial \ln F}{\partial s}R(u, Z)X] \right. \\
 &\quad \left. - \frac{1}{2}g(R(X, Y)u, Z) \frac{\partial \ln F}{\partial s} \text{grad}_M(\alpha) \right\}^H
 \end{aligned}$$

$$\begin{aligned}
\tilde{R}(X^H, Y^V)Z^V &= \left\{ \beta' X(\alpha) \frac{\partial^2 \ln F}{\partial s \partial t} \left[g(Z, U)Y + g(Y, U)Z - g(Y, Z)U \right] \right. \\
&\quad - \beta' \frac{\partial \ln F}{\partial t} g_x(Y, Z)(\nabla_X U) - \frac{f}{4} g(\text{grad}_M \alpha, R(u, Z)X) \frac{\partial \ln F}{\partial s} Y \\
&\quad - \left. \beta' X(\alpha) g(Y, u) \frac{\partial^2 \ln F}{\partial t \partial s} Z + \frac{1}{4} g(Y, Z) \frac{\partial F}{\partial s} (R(X, \text{grad}_M \alpha)u) \right\}^V \\
&\quad + \left\{ \frac{f}{2} \beta' \frac{\partial \ln F}{\partial t} \left[g(Z, u)(R(u, Y)X) + g(Y, u)(R(u, Z)X) \right] g \right. \\
&\quad - \frac{1}{2} X(\alpha) g_x(Y, Z) \frac{\partial^2 F}{\partial s^2} (\text{grad}_M \alpha) \\
&\quad - \frac{1}{2} g(Y, Z) \frac{\partial F}{\partial s} (\nabla_X \text{grad}_M \alpha) - \beta' g(Y, u) \frac{\partial F}{\partial t} (R(u, Z)X) \\
&\quad - \frac{f}{2} (R(Y, Z)X)^H - \frac{f^2}{4} (R(u, Y)R(u, Z)X) \\
&\quad \left. + \frac{f}{4} X(\alpha) g(Y, Z) \left(\frac{\partial \ln F}{\partial s} \right)^2 (\text{grad}_M \alpha) \right\}^H
\end{aligned} \tag{7}$$

$$\begin{aligned}
\tilde{R}(X^H, Y^V)Z^H &= \left\{ \frac{1}{2} R(X, Z)Y - \frac{f(x, r)}{4} R(X, R(u, Y)Z)u + \frac{1}{2} X(Z(\alpha)) \frac{\partial \ln F}{\partial s} Y \right. \\
&\quad + \frac{1}{2} X(\alpha) Z(\alpha) \left(\frac{\partial^2 \ln F}{\partial s^2} + \frac{1}{2} \left(\frac{\partial \ln F}{\partial s} \right)^2 \right) Y - \frac{1}{2} (\nabla_X Z)(\alpha) \frac{\partial \ln F}{\partial s} Y \\
&\quad + \left. \frac{\beta'(r)}{2} \frac{\partial \ln F}{\partial t} \left[g(Y, u)R(X, Z)u - g(Y, R(X, Z)u)U \right] \right\}^V \\
&\quad + \left\{ \frac{1}{2} X(\alpha) \frac{\partial F}{\partial s} R(u, Y)Z + \frac{f(x, r)}{4} Z(\alpha) \frac{\partial \ln F}{\partial s} R(u, Y)X \right. \\
&\quad \left. + \frac{f(x, r)}{2} (\nabla_X R)(u, Y)Z - \frac{1}{4} g(Y, R(X, Z)u) \frac{\partial F}{\partial s} \text{grad}_M(\alpha) \right\}^H
\end{aligned} \tag{8}$$

$$\begin{aligned}
\tilde{R}_{(x, u)}(X^V, Y^V)Z^H &= \left\{ \beta' \frac{\partial F}{\partial t} \left[g(X, U)R(u, Y)Z - g(Y, U)R(u, X)Z \right] + fR(X, Y)Z \right. \\
&\quad + \left. \frac{f^2}{4} \left[R(u, X)R(u, Y)Z - R(u, Y)R(u, X)Z \right] \right\}_x^H \\
&\quad + \left\{ \beta' Z(\alpha) \frac{\partial^2 \ln F}{\partial s \partial t} \left[g(X, u)Y - g(Y, u)X \right] \right. \\
&\quad + \frac{f}{4} \frac{\partial \ln F}{\partial s} \left[g(\text{grad}_M \alpha, R(u, Y)Z)X \right. \\
&\quad \left. - g(\text{grad}_M \alpha, R(u, X)Z)Y \right] \right\}_x^V
\end{aligned} \tag{9}$$

$$\begin{aligned}
 \tilde{R}_{(x,u)}(X^V, Y^V)Z^V &= \left\{ 2 \left[\beta' \frac{\partial \ln F}{\partial t} + (\beta')^2 \frac{\partial^2 \ln F}{\partial t^2} \right] \left[g(X, u)g(Z, U)Y \right. \right. \\
 &\quad - g(X, u)g(Y, Z)U - g(Y, u)g(Z, U)X + g(Y, u)g(X, Z)U \left. \right] \\
 &\quad + \beta' \frac{\partial \ln F}{\partial t} \left[g(X, Z)Y - g(Y, Z)X \right] \\
 &\quad + (\beta')^2 \left(\frac{\partial \ln F}{\partial t} \right)^2 \left[g(Y, u)g(Z, U)X - g(Y, U)g(X, Z)U \right. \\
 &\quad - g(X, u)g(Z, U)Y + g(X, u)g(Y, Z)U \\
 &\quad \left. - g(Y, Z)\|u\|^2 X + g(X, Z)\|u\|^2 Y \right] \\
 &\quad + \frac{f}{4} \left(\frac{\partial \ln F}{\partial s} \right)^2 \|\text{grad}_M \alpha\|^2 \left[g(X, Z)Y - g(Y, Z)X \right] \Big\}_x^V \\
 &\quad + \left\{ \beta' \frac{\partial \ln F}{\partial t} \frac{\partial F}{\partial s} \left[g(Y, Z)g(X, u) - g(X, Z)g(Y, u) \right] \text{grad}_M \alpha \right. \\
 &\quad + \frac{f}{4} \frac{\partial F}{\partial s} \left[g(X, Z)R(u, Y) \text{grad}_M \alpha - g(Y, Z)R(u, X) \text{grad}_M \alpha \right] \\
 &\quad \left. + \beta' \frac{\partial^2 \ln F}{\partial s \partial t} \left[g(Y, u)g(X, Z) - g(X, u)g(Y, Z) \right] \text{grad}_M \alpha \right\}_x^H
 \end{aligned} \tag{10}$$

for all $(x, u) \in TM$ and $X, Y, Z \in \Gamma(TM)$.

Proof. Let

$$\begin{aligned}
 K_1: TM &\rightarrow TM \\
 u &\mapsto R(Y, Z)u.
 \end{aligned}$$

Then we have:

$$\begin{aligned}
 \tilde{\nabla}_{X^H} \tilde{\nabla}_{Y^H} Z^H &= \tilde{\nabla}_{X^H} (\nabla_Y Z)^H - \frac{1}{2} \tilde{\nabla}_{X^H} V K_1 \\
 &= (\nabla_X \nabla_Y Z)^H - \frac{1}{2} (R(X, \nabla_Y Z)u)^V - \frac{1}{2} (\nabla_X R(Y, Z)u)^V \\
 &\quad - \frac{f}{4} (R_x(u, R_x(Y, Z)u)X)^H - \frac{1}{4} X(\alpha) \frac{\partial \ln F}{\partial s} V K_1
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\nabla}_{X^H} \tilde{\nabla}_{Y^H} Z^H &= (\nabla_X \nabla_Y Z)^H - \frac{1}{2} (R(X, \nabla_Y Z)u)^V - \frac{1}{2} [(\nabla_X R)(Y, Z)u]^V \\
 &\quad - \frac{1}{2} [R(\nabla_X Y, Z)u + R(Y, \nabla_X Z)u]^V \\
 &\quad - \frac{f}{4} (R_x(u, R_x(Y, Z)u)X)^H - \frac{1}{4} X(\alpha) \frac{\partial \ln F}{\partial s} (R(Y, Z)u)^V
 \end{aligned} \tag{11}$$

$$\begin{aligned}
\tilde{\nabla}_{Y^H} \tilde{\nabla}_{X^H} Z^H &= (\nabla_Y \nabla_X Z)^H - \frac{1}{2} (R(Y, \nabla_X Z)u)^V - \frac{1}{2} [(\nabla_Y R)(X, Z)u]^V \\
&\quad - \frac{1}{2} [R(\nabla_Y X, Z)u + R(X, \nabla_Y Z)u]^V \\
&\quad - \frac{f}{4} (R_x(u, R_x(X, Z)u)Y)^H - \frac{1}{4} Y(\alpha) \frac{\partial \ln F}{\partial s} (R(X, Z)u)^V
\end{aligned} \tag{12}$$

$$\begin{aligned}
\tilde{\nabla}_{[X, Y]^H} Z^H &= (\nabla_{[X, Y]} Z)^H - \frac{1}{2} [R([X, Y], Z)u]^V - \frac{f(x, r)}{2} (R(u, R(X, Y)u)Z)^H \\
&\quad - \frac{1}{2} Z(\alpha) \frac{\partial \ln F}{\partial s} (R(X, Y)u)^V.
\end{aligned} \tag{13}$$

From (11), (12), (13) and the second Bianchi identity

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0$$

we obtain the formula (5).

The other formulae are obtained by a similar calculation. \square

From Definition 2 and Theorem 2 we get

Theorem 3. *Let (M, g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If R (resp. \tilde{R}) denote the Riemann curvature tensor of M (resp. TM), then we have*

$$\begin{aligned}
g_f^S(\tilde{R}(X^H, Y^H)Y^H, X^H) &= g(R(X, Y)Y, X) - \frac{3f}{4} \|R(Y, X)u\|^2 \\
g_f^S(\tilde{R}(X^H, Y^V)Y^V, X^H) &= \frac{f}{4} |X(\alpha)|^2 \|Y\|^2 \left(\frac{\partial \ln F}{\partial s} \right)^2 - \frac{1}{2} |X(\alpha)|^2 \|Y\|^2 \frac{\partial^2 F}{\partial s^2} \\
&\quad - \frac{1}{2} \|Y\|^2 \frac{\partial F}{\partial s} g(\nabla_X \text{grad}_M \alpha, X) + \frac{f^2}{4} \|R(u, Y)X\|^2 \\
g_f^S(\tilde{R}(X^V, Y^V)Y^V, X^V) &= 2f \left[\beta'' \frac{\partial \ln F}{\partial t} + (\beta')^2 \frac{\partial^2 \ln F}{\partial t^2} \right] \left[2g(X, u)g(Y, u)g(X, Y) \right. \\
&\quad \left. - \|Y\|^2 |g(X, u)|^2 - \|X\|^2 |g(Y, u)|^2 \right] \\
&\quad + f\beta' \frac{\partial \ln F}{\partial t} \left[|g(X, Y)|^2 - \|X\|^2 \|Y\|^2 \right] \\
&\quad + f(\beta')^2 \left(\frac{\partial \ln F}{\partial t} \right)^2 \left[\|X\|^2 |g(Y, u)|^2 - 2g(X, u)g(Y, u)g(X, Y) \right. \\
&\quad \left. + \|Y\|^2 |g(X, u)|^2 - \|X\|^2 \|Y\|^2 \|u\|^2 + |g(X, Y)|^2 \|u\|^2 \right] \\
&\quad + \frac{f^2}{4} \left(\frac{\partial \ln F}{\partial s} \right)^2 \|\text{grad}_M \alpha\|^2 \left[|g(X, Y)|^2 - \|X\|^2 \|Y\|^2 \right].
\end{aligned}$$

for any $X, Y \in \Gamma(TM)$.

By Theorem 3 and definition of sectional curvature tensor we get

Theorem 4. *Let (M, g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If K (resp. \tilde{K}) denote the sectional curvature tensor of M (resp. TM), then for any orthonormal vector fields $X, Y \in \Gamma(TM)$, we have*

$$\begin{aligned}\tilde{K}(X^H, Y^H) &= K(X, Y) - \frac{3f}{4} \|R(X, Y)u\|^2. \\ \tilde{K}(X^H, Y^V) &= \frac{f}{4} \|R(u, Y)X\|^2 + \frac{1}{4} |X(\alpha)|^2 \left(\frac{\partial \ln F}{\partial s} \right)^2 - \frac{1}{2f} |X(\alpha)|^2 \frac{\partial^2 F}{\partial s^2} \\ &\quad - \frac{1}{2f} \frac{\partial F}{\partial s} g(\nabla_X \text{grad}_M \alpha, X) \\ \tilde{K}(X^V, Y^V) &= -\frac{2}{f} \beta'' \frac{\partial \ln F}{\partial t} \left[|g(X, u)|^2 + |g(Y, u)|^2 \right] - \frac{1}{4f^2} \left(\frac{\partial F}{\partial s} \right)^2 \|\text{grad}_M \alpha\|^2 \\ &\quad - \frac{(\beta')^2}{f} \left(\frac{\partial \ln F}{\partial t} \right)^2 \left[\|u\|^2 - |g(X, u)|^2 - |g(Y, u)|^2 \right] - \frac{\beta'}{f} \frac{\partial \ln F}{\partial t}\end{aligned}$$

Theorem 5. *Let (M, g) be a Riemannian manifold of constant sectional curvature λ and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If \tilde{K} denote the sectional curvature tensor of TM , then for any orthonormal vector fields $X, Y \in \Gamma(TM)$, we have*

$$\begin{aligned}\tilde{K}(X^H, Y^H) &= \lambda - \frac{3f\lambda^2}{4} \left[g(X, u)^2 + g(Y, u)^2 \right] \\ \tilde{K}(X^H, Y^V) &= \frac{f\lambda^2}{4} g(u, X)^2 + \frac{1}{4} |X(\alpha)|^2 \left(\frac{\partial \ln F}{\partial s} \right)^2 - \frac{1}{2f} |X(\alpha)|^2 \frac{\partial^2 F}{\partial s^2} \\ &\quad - \frac{1}{2f} \frac{\partial F}{\partial s} g(\nabla_X \text{grad}_M \alpha, X) \\ \tilde{K}(X^V, Y^V) &= -\frac{2}{f} \left[\beta'' \frac{\partial \ln F}{\partial t} + (\beta')^2 \frac{\partial^2 \ln F}{\partial t^2} \right] \left[|g(X, u)|^2 + |g(Y, u)|^2 \right] - \frac{\beta'}{f} \frac{\partial \ln F}{\partial t} \\ &\quad - \frac{(\beta')^2}{f} \left(\frac{\partial \ln F}{\partial t} \right)^2 \left[\|u\|^2 - |g(X, u)|^2 - |g(Y, u)|^2 \right] \\ &\quad - \frac{1}{4} \left(\frac{\partial \ln F}{\partial s} \right)^2 \|\text{grad}_M \alpha\|^2\end{aligned}$$

The proof of Theorem 5 is deduced from the Theorem 4 and the following equations

$$\begin{aligned}R(X, Y)Z &= \lambda(g(X, Z)Y - g(Y, Z)X) \\ \|R(X, Y)u\|^2 &= \lambda^2 [g(X, u)^2 + g(Y, u)^2] \\ \|R(u, Y)X\|^2 &= \lambda^2 g(u, X)^2\end{aligned}$$

for all $(x, u) \in TM$ and $X, Y, Z \in \Gamma(TM)$

Theorem 6. Let (M, g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If σ (resp. $\tilde{\sigma}$) denote the scalar curvature of M (resp. TM), then for any local orthonormal frame (E_1, \dots, E_m) on M , we have

$$\begin{aligned} \tilde{\sigma} = & \sigma + \frac{2-3f}{4} \sum_{i,j=1}^m \|R(E_i, E_j)u\|^2 - \frac{m}{f^2} \frac{\partial F}{\partial s} \text{trace}[g(\nabla_{\text{grad}_M} \alpha, *)] \\ & + \frac{m}{f^2} \|\text{grad}(\alpha)\|^2 \left[\frac{1}{2f} \left(\frac{\partial F}{\partial s} \right)^2 - \frac{\partial^2 F}{\partial s^2} - \frac{(m-1)}{4f^2} \left(\frac{\partial F}{\partial s} \right)^2 \right] \\ & - \frac{4(m-1)}{f^3} \left[\beta'' \frac{\partial \ln F}{\partial t} + (\beta')^2 \frac{\partial^2 \ln F}{\partial t^2} + \frac{(m-2)}{4} (\beta')^2 \left(\frac{\partial \ln F}{\partial t} \right)^2 \right] \|u\|^2 \\ & - \frac{m(m-1)}{f^3} \beta' \frac{\partial \ln F}{\partial t} \end{aligned}$$

Proof. Let (E_1, \dots, E_m) be local orthonormal frame on M , then

$$\left(E_1^H, \dots, E_m^H, \frac{1}{\sqrt{f}} E_1^V, \dots, \frac{1}{\sqrt{f}} E_m^V \right)$$

are a local orthonormal frame on TM . Locally we obtain

$$\begin{aligned} \sum_{i,j=1}^m \|R(u, E_i)E_j\|^2 &= \sum_{i,j=1}^m g(R(u, E_i)E_j, R(u, E_i)E_j) \\ &= \sum_{i,j,k,l,s=1}^m u_k u_l g(R(E_k, E_i)E_j, E_s) g(R(E_l, E_i)E_j, E_s) \\ &= \sum_{i,j,k,l,s=1}^m u_k u_l g(R(E_j, E_s)E_k, E_i) g(R(E_j, E_s)E_l, E_i) \\ &= \sum_{i,j,s=1}^m g(R(E_j, E_s)u, g(R(E_j, E_s)u, E_i)E_i) \\ &= \sum_{i,j=1}^m \|R(E_i, E_j)u\|^2 \end{aligned} \tag{14}$$

From Theorem 4 and definition of scalar curvature, we have

$$\tilde{\sigma} = \sum_{i,j=1}^m \tilde{K}(E_i^H, E_j^H) + \frac{2}{f} \sum_{i,j=1}^m \tilde{K}(E_i^H, E_j^V) + \frac{1}{f^2} \sum_{i \neq j=1}^m \tilde{K}(E_i^V, E_j^V)$$

$$\begin{aligned}
 \tilde{\sigma} = \sigma & - \sum_{i,j=1}^m \frac{3f}{4} \|R(E_i, E_j)u\|^2 + \frac{2}{f} \sum_{i,j=1}^m \left[\frac{f}{4} \|R(u, E_j)E_i\|^2 + \frac{1}{4} |E_i(\alpha)|^2 \left(\frac{\partial \ln F}{\partial s} \right)^2 \right. \\
 & - \frac{1}{2f} |E_i(\alpha)|^2 \frac{\partial^2 F}{\partial s^2} - \frac{1}{2f} \frac{\partial F}{\partial s} g(\nabla_{E_i} \text{grad}_M \alpha, E_i) \Big] \\
 & - \frac{1}{f^2} \sum_{i \neq j=1}^m \left\{ \frac{2}{f} \left[\beta'' \frac{\partial \ln F}{\partial t} + (\beta')^2 \frac{\partial^2 \ln F}{\partial t^2} \right] \left[|g(E_i, u)|^2 + |g(E_j, u)|^2 \right] \right. \\
 & + \frac{\beta'}{f} \frac{\partial \ln F}{\partial t} + \frac{(\beta')^2}{f} \left(\frac{\partial \ln F}{\partial t} \right)^2 \left[\|u\|^2 - |g(E_i, u)|^2 - |g(E_j, u)|^2 \right] \\
 & \left. + \frac{1}{4} \left(\frac{\partial \ln F}{\partial s} \right)^2 \|\text{grad}_M \alpha\|^2 \right\} \tag{15}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\sigma} = \sigma & - \sum_{i,j=1}^m \frac{3f}{4} \|R(E_i, E_j)u\|^2 + \frac{1}{2} \sum_{i,j=1}^m \|R(u, E_j)E_i\|^2 + \frac{m}{2f} \|\text{grad}(\alpha)\|^2 \left(\frac{\partial \ln F}{\partial s} \right)^2 \\
 & - \frac{m}{f^2} \|\text{grad}(\alpha)\|^2 \frac{\partial^2 F}{\partial s^2} - \frac{m}{f^2} \frac{\partial F}{\partial s} \text{trace}[g(\nabla_{\text{grad}_M} \alpha, *)] \\
 & - \frac{1}{f^2} \left\{ \frac{4(m-1)}{f} \left[\beta'' \frac{\partial \ln F}{\partial t} + (\beta')^2 \frac{\partial^2 \ln F}{\partial t^2} \right] \|u\|^2 + m(m-1) \frac{\beta'}{f} \frac{\partial \ln F}{\partial t} \right. \\
 & \left. + \frac{(\beta')^2}{f} \left(\frac{\partial \ln F}{\partial t} \right)^2 (m-1)(m-2) \|u\|^2 + \frac{m(m-1)}{4} \left(\frac{\partial \ln F}{\partial s} \right)^2 \|\text{grad}_M \alpha\|^2 \right\}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\sigma} = \sigma & - \sum_{i,j=1}^m \frac{3f}{4} \|R(E_i, E_j)u\|^2 + \frac{1}{2} \sum_{i,j=1}^m \|R(u, E_j)E_i\|^2 + \frac{m}{2f} \|\text{grad}(\alpha)\|^2 \left(\frac{\partial \ln F}{\partial s} \right)^2 \\
 & - \frac{m}{f^2} \|\text{grad}(\alpha)\|^2 \frac{\partial^2 F}{\partial s^2} - \frac{m}{f^2} \frac{\partial F}{\partial s} \text{trace}[g(\nabla_{\text{grad}_M} \alpha, *)] \\
 & - \frac{4(m-1)}{f^3} \left[\beta'' \frac{\partial \ln F}{\partial t} + (\beta')^2 \frac{\partial^2 \ln F}{\partial t^2} \right] \|u\|^2 - \frac{m(m-1)}{f^3} \beta' \frac{\partial \ln F}{\partial t} \\
 & - \frac{(m-1)(m-2)}{f^3} (\beta')^2 \left(\frac{\partial \ln F}{\partial t} \right)^2 \|u\|^2 - \frac{m(m-1)}{4f^2} \left(\frac{\partial \ln F}{\partial s} \right)^2 \|\text{grad}_M \alpha\|^2
 \end{aligned}$$

Using formula (14) we deduce

$$\begin{aligned}
 \tilde{\sigma} = \sigma & + \frac{2-3f}{4} \sum_{i,j=1}^m \|R(E_i, E_j)u\|^2 + \frac{m}{2f} \|\text{grad}(\alpha)\|^2 \left(\frac{\partial \ln F}{\partial s} \right)^2 \\
 & - \frac{m}{f^2} \|\text{grad}(\alpha)\|^2 \frac{\partial^2 F}{\partial s^2} - \frac{m}{f^2} \frac{\partial F}{\partial s} \text{trace}[g(\nabla_{\text{grad}_M} \alpha, *)] \\
 & - \frac{4(m-1)}{f^3} \left[\beta'' \frac{\partial \ln F}{\partial t} + (\beta')^2 \frac{\partial^2 \ln F}{\partial t^2} \right] \|u\|^2 - \frac{m(m-1)}{f^3} \beta' \frac{\partial \ln F}{\partial t} \\
 & - \frac{(m-1)(m-2)}{f^3} (\beta')^2 \left(\frac{\partial \ln F}{\partial t} \right)^2 \|u\|^2 - \frac{m(m-1)}{4f^2} \left(\frac{\partial \ln F}{\partial s} \right)^2 \|\text{grad}_M \alpha\|^2
 \end{aligned}$$

$$\begin{aligned}
\tilde{\sigma} &= \sigma + \frac{2-3f}{4} \sum_{i,j=1}^m \|R(E_i, E_j)u\|^2 - \frac{m}{f^2} \frac{\partial F}{\partial s} \text{trace}[g(\nabla_{\text{grad}_M} \alpha, *)] \\
&\quad + \frac{m}{f^2} \|\text{grad}(\alpha)\|^2 \left[\frac{1}{2f} \left(\frac{\partial F}{\partial s} \right)^2 - \frac{\partial^2 F}{\partial s^2} - \frac{(m-1)}{4f^2} \left(\frac{\partial F}{\partial s} \right)^2 \right] \\
&\quad - \frac{4(m-1)}{f^3} \left[\beta'' \frac{\partial \ln F}{\partial t} + (\beta')^2 \frac{\partial^2 \ln F}{\partial t^2} + \frac{(m-2)}{4} (\beta')^2 \left(\frac{\partial \ln F}{\partial t} \right)^2 \right] \|u\|^2 \\
&\quad - \frac{m(m-1)}{f^3} \beta' \frac{\partial \ln F}{\partial t} \quad \square
\end{aligned}$$

Corollary 1. *Let (M, g) be a Riemannian manifold of constant sectional curvature λ and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If $\tilde{\sigma}$ denote the scalar curvature of TM , then for any local orthonormal frame (E_1, \dots, E_m) on M , we have*

$$\begin{aligned}
\tilde{\sigma} &= (m-1)\lambda \left[m + \frac{(2-3f)}{2} \lambda \|u\|^2 \right] - \frac{m}{f^2} \frac{\partial F}{\partial s} \text{trace}[g(\nabla_{\text{grad}_M} \alpha, *)] \\
&\quad + \frac{m}{f^2} \|\text{grad}(\alpha)\|^2 \left[\frac{1}{2f} \left(\frac{\partial F}{\partial s} \right)^2 - \frac{\partial^2 F}{\partial s^2} - \frac{(m-1)}{4f^2} \left(\frac{\partial F}{\partial s} \right)^2 \right] \\
&\quad - \frac{4(m-1)}{f^3} \left[\beta'' \frac{\partial \ln F}{\partial t} + (\beta')^2 \frac{\partial^2 \ln F}{\partial t^2} + \frac{(m-2)}{4} (\beta')^2 \left(\frac{\partial \ln F}{\partial t} \right)^2 \right] \|u\|^2 \\
&\quad - \frac{m(m-1)}{f^3} \beta' \frac{\partial \ln F}{\partial t} \quad (16)
\end{aligned}$$

Proof. Taking account that $\sigma = m(m-1)\lambda$ and for any vector fields $X, Y, Z \in TM$

$$R(X, Y)Z = \lambda(g(Y, Z)Y - g(X, Z)Y)$$

then we obtain

$$\begin{aligned}
\sum_{i,j=1}^m \|R(E_i, E_j)u\|^2 &= \lambda^2 \sum_{\substack{i \neq j \\ i,j=1}}^m \|g(E_j, u)E_i - g(E_i, u)E_j\|^2 \\
&= \lambda^2 \sum_{\substack{i \neq j \\ i,j=1}}^m (\|g(E_j, u)\|^2 + \|g(E_i, u)\|^2) \\
&= 2(m-1)\lambda^2 \|u\|^2 \quad (17)
\end{aligned}$$

From Theorem 6 and formula (17) we deduce

$$\begin{aligned}
\tilde{\sigma} &= (m-1)\lambda \left[m + \frac{(2-3f)}{2} \lambda \|u\|^2 \right] - \frac{m}{f^2} \frac{\partial F}{\partial s} \text{trace}[g(\nabla_{\text{grad}_M} \alpha, *)] \\
&\quad + \frac{m}{f^2} \|\text{grad}(\alpha)\|^2 \left[\frac{1}{2f} \left(\frac{\partial F}{\partial s} \right)^2 - \frac{\partial^2 F}{\partial s^2} - \frac{(m-1)}{4f^2} \left(\frac{\partial F}{\partial s} \right)^2 \right] \\
&\quad - \frac{4(m-1)}{f^3} \left[\beta'' \frac{\partial \ln F}{\partial t} + (\beta')^2 \frac{\partial^2 \ln F}{\partial t^2} + \frac{(m-2)}{4} (\beta')^2 \left(\frac{\partial \ln F}{\partial t} \right)^2 \right] \|u\|^2 \\
&\quad - \frac{m(m-1)}{f^3} \beta' \frac{\partial \ln F}{\partial t} \quad (18)
\end{aligned}$$

□

Theorem 7. Let (M, g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If $f = \text{const}$ then (TM, g_f^S) has a constant scalar curvature if and only if $f = \frac{3}{2}$ and (M, g) has a constant scalar curvature, or (M, g) is flat.

Proof. From Theorem 6 we have

$$\tilde{\sigma} = \sigma + \frac{(2-3f)}{4} \sum_{i,j=1}^m \|R(E_i, E_j)u\|^2$$

then $\tilde{\sigma}$ is constant if and only if $\sigma = \text{const}$ and $f = \frac{3}{2}$ or $R = 0$. □

Corollary 2. Let (M, g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If $f = \frac{3}{2}$ then $\tilde{\sigma} = \sigma$.

References

- [1] M.T.K. Abbassi, M. Sarih: On Natural Metrics on Tangent Bundles of Riemannian Manifolds. *Archivum Mathematicum* 41 (2005) 71–92.
- [2] N. Cengiz, A.A. Salimov: Diagonal lift in the tensor bundle and its applications. *Appl. Math. Comput.* 142 (2–3) (2003) 309–319.
- [3] J. Cheeger, D. Gromoll: On the structure of complete manifolds of nonnegative curvature. *Ann. of Math.* 96 (2) (1972) 413–443.
- [4] M. Djaa, N.E.H. Djaa, R. Nasri: Natural Metrics on T^2M and Harmonicity. *International Electronic Journal of Geometry* 6 (1) (2013) 100–111.
- [5] M. Djaa, J. Gancarzewicz: The geometry of tangent bundles of order r. *Boletín Academia, Gaceta de Ciencias* 4 (1985) 147–165.
- [6] N.E.H. Djaa, S. Ouakkas, M. Djaa: Harmonic sections on the tangent bundle of order two. *Annales Mathematicae et Informaticae* 38 (2011) 15–25.
- [7] P. Dombrowski: On the Geometry of the Tangent Bundle. *J. Reine Angew. Math.* 210 (1962) 73–88.
- [8] A. Gezer: On the Tangent Bundle With Deformed Sasaki Metric. *International Electronic Journal of Geometry* 6 (2) (2013) 19–31.
- [9] S. Gudmundsson, E. Kappos: On the Geometry of the Tangent Bundle with the Cheeger-Gromoll Metric. *Tokyo J. Math.* 25 (1) (2002) 75–83.
- [10] O. Kowalski and M. Sekizawa: On Riemannian Geometry Of Tangent Sphere Bundles With Arbitrary Constant Radius. *Archivum Mathematicum* 44 (2008) 391–401.
- [11] E. Musso, F. Tricerri: Riemannian Metrics on Tangent Bundles. *Ann. Mat. Pura Appl.* 150 (4) (1988) 1–19.
- [12] A.A. Salimov, F. Agca: Some Properties of Sasakian Metrics in Cotangent Bundles. *Mediterranean Journal of Mathematics* 8 (2) (2011) 243–255.
- [13] A.A. Salimov, A. Gezer: On the geometry of the $(1, 1)$ -tensor bundle with Sasaki type metric. *Chinese Annals of Mathematics* 32 (3) (2011) 369–386.

- [14] A.A. Salimov, A. Gezer, K. Akbulut: Geodesics of Sasakian metrics on tensor bundles. *Mediterr. J. Math* 6 (2) (2009) 135–147.
- [15] A.A. Salimov, S. Kazimova: Geodesics of the Cheeger-Gromoll Metric. *Turk. J. Math.* 33 (2009) 99–105.
- [16] S. Sasaki: On the differential geometry of tangent bundles of Riemannian manifolds II. *Tohoku Math. J.* 14 (1962) 146–155.
- [17] M. Sekizawa: Curvatures of Tangent Bundles with Cheeger-Gromoll Metric. *Tokyo J. Math.* 14 (2) (1991) 407–417.
- [18] K. Yano, S. Ishihara: *Tangent and Cotangent Bundles*. Marcel Dekker. INC., New York (1973).

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