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A Study on ϕ-recurrence τ-curvature tensor in (k,μ)-contact metric manifolds

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Abstract. In this paper we study ϕ-recurrence τ-curvature tensor in (k,μ)-contact metric manifolds.

1 Introduction

In [11], S. Tanno introduced the notion of k-nullity distribution of a contact metric manifold as a distribution such that the characteristic vector field ξ of the contact metric manifold belongs to the distribution. The contact metric manifold with ξ belonging to the k-nullity distribution is called $N(k)$-contact metric manifold and such a manifold is also studied by various authors. Generalizing this notion in 1995, D.E. Blair, T. Koufogiorgos and B.J. Papantoniou [2] introduced the notion of a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution, where k and μ are real constants. In particular, if $\mu = 0$ then the notion of (k,μ)-nullity distribution reduces to the notion of k-nullity distribution.

In [13], M.M. Tripathi and et al. introduced the τ-curvature tensor which consists of known curvatures like conformal, concircular, projective, M-projective, W_i-curvature tensor $(i = 0,\dots,9)$ and W_j^*-curvature tensor $(j = 0,1)$. Further, in [14] and [15] M.M. Tripathi and et al. studied τ-curvature tensor in K-contact, Sasakian and semi-Riemannian manifolds. Later in [6] the authors studied some properties of τ-curvature tensor and they obtained some interesting results.

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The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. In the context of contact geometry the notion of ϕ-symmetry is introduced and studied by E. Boeckx, P. Buecken and L. Vanhecke [3] with several examples. As a weaker version of local symmetry, T. Takahashi [12] introduced the notion of locally ϕ-symmetry on a Sasakian manifold. Generalizing the notion of ϕ-symmetry De et al. [5] introduced the notion of ϕ-recurrent Sasakian manifold. In [4], the authors studied ϕ-recurrent $N(k)$-contact metric manifolds. Motivated by all these work in this paper we study the ϕ-recurrent τ-curvature tensor in (k,μ)-contact metric manifold.

2 Preliminaries

A $(2n+1)$-dimensional differential manifold M is said to have an almost contact structure (ϕ,ξ,η) if it carries a tensor field ϕ of type $(1,1)$, a vector field ξ and 1-form η satisfying

$$\phi^2 = -I + \eta\otimes\xi, \qquad \eta(\xi)=1, \qquad \eta\circ\phi = 0, \qquad \phi\xi = 0. \tag{1}$$

Let g be a compatible Riemannian metric with almost contact structure (ϕ,ξ,η) such that,

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2}$$

$$g(\phi X, Y) = -g(X,\phi Y) \qquad\qquad g(X,\xi) = \eta(X), \tag{3}$$

where X,Y are vector fields defined on M. Then the structure (ϕ,ξ,η,g) on M is said to have an almost contact metric structure and the manifold M equipped with this structure is called an almost contact metric manifold.

An almost contact metric structure (ϕ,ξ,η,g) becomes a contact metric structure if

$$\mathrm{d}\eta(X,Y) = g(X,\phi Y),$$

for all vector fields X,Y on M.

Given a contact metric manifold $M(\phi,\xi,\eta,g)$, we define a $(1,1)$ tensor field h by $h=\frac{1}{2}\mathfrak{L}\phi$, where $\mathfrak{L}$ denotes the Lie differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. Also we have $\mathrm{tr}(h) = \mathrm{tr}(\phi h) = 0$ and $h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection on M, then the following relation holds:

$$\nabla_X\xi = -\phi X - \phi h X. \tag{4}$$

In contact metric manifold $M(\phi,\xi,\eta,g)$, the (k,μ)-nullity distribution is

$$p\to N_p(k,\mu) = \Big\{Z\in T_pM : R(X,Y)Z = k\big[g(Y,Z)X - g(X,Z)Y\big] + \mu\big[g(Y,Z)hX - g(X,Z)hY\big]\Big\}, \tag{5}$$

for all vector fields $X,Y\in T_pM$ and k,μ are real numbers and R is the curvature tensor. Hence, if the characteristic vector field ξ belongs to the (k,μ)-nullity distribution, then we have

$$R(X,Y)\xi = k\big[\eta(Y)X - \eta(X)Y\big] + \mu\big[\eta(Y)hX - \eta(X)hY\big]. \tag{6}$$

Thus a contact metric manifold satisfying (6) is called a (k,μ)-contact metric manifold. In particular, if $\mu = 0$, then the notion of (k,μ)-nullity distribution reduces to the notion of k-nullity distribution introduced by S. Tanno [11]. In a (k,μ)-contact metric manifold the following relations hold [2], [9]:

$$
\begin{aligned}
h^2 &= (k-1)\phi^2, \qquad k \leq 1, && (7)\\
(\nabla_X \phi)Y &= g(X+hX,Y)\xi - \eta(Y)[X+hX], \\
(\nabla_X h)Y &= \big[(1-k)g(X,\phi Y) + g(X,h\phi Y)\big]\xi \\
&\quad + \eta(Y)\big[h(\phi X + \phi h X)\big] - \mu\eta(X)\phi h Y, \\
R(\xi,X)Y &= k\big[g(X,Y)\xi - \eta(Y)X\big] + \mu\big[g(hX,Y)\xi - \eta(Y)hX\big], \\
\eta(R(X,Y)Z) &= k\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big] \\
&\quad + \mu\big[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)\big], \\
S(X,\xi) &= 2nk\eta(X), && (8)\\
S(X,Y) &= \big[2(n-1) - n\mu\big]g(X,Y) + \big[2(n-1)+\mu\big]g(hX,Y) \\
&\quad + \big[2(1-n) + n(2k+\mu)\big]\eta(X)\eta(Y), \qquad n \geq 1, && (9)\\
QX &= \big[2(n-1) - n\mu\big]X + \big[2(n-1)+\mu\big]hX \\
&\quad + \big[2(1-n) + n(2k+\mu)\big]\eta(X)\xi, \qquad n \geq 1, && (10)\\
r &= 2n\big[2n-2+k-n\mu\big], \\
S(\phi X,\phi Y) &= S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n-2+\mu)g(hX,Y), && (11)
\end{aligned}
$$

where S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator, that is, $S(X,Y) = g(QX,Y)$ and r is the scalar curvature of the manifold. From (3), it follows that

$$(\nabla_X \eta)Y = g(X+hX,\phi Y).$$

Definition 1. A (k,μ)-contact metric manifold M is said to be locally ϕ-symmetric if

$$\phi^2\big((\nabla_W R)(X,Y)Z\big) = 0,$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced by T. Takahashi [12] for Sasakian manifolds.

A field that is at every point and for every direction proportional to its covariant differential is called recurrent. Based on this concept we define the following definition:

Definition 2. A (k,μ)-contact metric manifold M is said to be ϕ-recurrent if and only if there exists a non zero 1-form A such that

$$\phi^2\big((\nabla_W R)(X,Y)Z\big) = A(W)R(X,Y)Z,$$

for all arbitrary vector fields X, Y, Z, W which are not necessarily orthogonal to ξ.

If the 1-form A vanishes identically, then the manifold is said to be a locally ϕ-symmetric manifold.

Definition 3. A (k,μ)-contact metric manifold M is said to be ϕ-τ-recurrent if and only if there exists a non zero 1-form A such that

$$\phi^2\big((\nabla_W\tau)(X,Y)Z\big) = A(W)\tau(X,Y)Z,$$

for all arbitrary vector fields X, Y, Z, W which are not necessarily orthogonal to ξ.

The τ-curvature tensor [13] is given by

$$\begin{aligned}\tau(X,Y)Z = {} & a_0R(X,Y)Z + a_1S(Y,Z)X + a_2S(X,Z)Y + a_3S(X,Y)Z \\ & + a_4g(Y,Z)QX + a_5g(X,Z)QY + a_6g(X,Y)QZ \\ & + a_7r[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{12}$$

where $a_0,\ldots,a_7$ are all constants on M. For different values of $a_0,\ldots,a_7$ the τ-curvature tensor reduces to the curvature tensor R, quasi-conformal curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor, pseudo-projective curvature tensor, projective curvature tensor, M-projective curvature tensor, W_i-curvature tensors $(i = 0,\ldots,9)$, W_j^*-curvature tensors $(j = 0,1)$.

3 ϕ-τ-recurrent (k,μ)-contact metric manifold

In this section, we define ϕ-τ-recurrent (k,μ)-contact metric manifold

$$\phi^2\big((\nabla_W\tau)(X,Y)Z\big) = A(W)\tau(X,Y)Z, \tag{13}$$

for all vector fields X,Y,Z,W. By using (9) and (10) in (12), we get

$$\begin{aligned}\tau(X,Y)Z = {} & a_0R(X,Y)Z + a_1[\alpha g(Y,Z)X + \beta g(hY,Z)X + \gamma\eta(Y)\eta(Z)X] \\ & + a_2[\alpha g(X,Z)Y + \beta g(hX,Z)Y + \gamma\eta(X)\eta(Z)Y] \\ & + a_3[\alpha g(X,Y)Z + \beta g(hX,Y)Z + \gamma\eta(X)\eta(Y)Z] \\ & + a_4g(Y,Z)\big[\alpha X + \beta hX + \gamma\eta(X)\xi\big] \\ & + a_5g(X,Z)\big[\alpha Y + \beta hY + \gamma\eta(Y)\xi\big] \\ & + a_6g(X,Y)\big[\alpha Z + \beta hZ + \gamma\eta(Z)\xi\big] \\ & + a_7r\big[g(Y,Z)X - g(X,Z)Y\big],\end{aligned} \tag{14}$$

where $\alpha = [2(n-1) - n\mu]$, $\beta = [2(n-1) + \mu]$ and $\gamma = [2(1-n) + n(2k+\mu)]$.

Differentiating (14) with respect to W, we obtain

$$\begin{aligned}(\nabla_W\tau)(X,Y)Z &= a_0(\nabla_W R)(X,Y)Z \\ &+ a_1\Big[\beta g((\nabla_W h)Y,Z)X + \gamma\big\{(\nabla_W\eta)Y\eta(Z)X + (\nabla_W\eta)Z\eta(Y)X\big\}\Big] \\ &+ a_2\Big[\beta g((\nabla_W h)X,Z)Y + \gamma\big\{(\nabla_W\eta)X\eta(Z)Y + (\nabla_W\eta)Z\eta(X)Y\big\}\Big] \\ &+ a_3\Big[\beta g((\nabla_W h)X,Y)Z + \gamma\big\{(\nabla_W\eta)X\eta(Y)Z + (\nabla_W\eta)Y\eta(X)Z\big\}\Big] \\ &+ a_4 g(Y,Z)\Big[\beta(\nabla_W h)X + \gamma\big\{(\nabla_W\eta)(X)\xi + \eta(X)\nabla_W\xi\big\}\Big] \\ &+ a_5 g(X,Z)\Big[\beta(\nabla_W h)Y + \gamma\big\{(\nabla_W\eta)(Y)\xi + \eta(Y)\nabla_W\xi\big\}\Big] \\ &+ a_6 g(X,Y)\Big[\beta(\nabla_W h)Z + \gamma\big\{(\nabla_W\eta)(Z)\xi + \eta(Z)\nabla_W\xi\big\}\Big] \\ &+ a_7(\nabla_W r)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{15}$$

By virtue of (1), (13), we have

$$-(\nabla_W\tau)(X,Y)Z + \eta\big((\nabla_W\tau)(X,Y)Z\big)\xi = A(W)\tau(X,Y)Z. \tag{16}$$

By taking an inner product with U, we obtain

$$-g((\nabla_W\tau)(X,Y)Z,U) + \eta\big((\nabla_W\tau)(X,Y)Z\big)g(\xi,U) = A(W)g(\tau(X,Y)Z,U). \tag{17}$$

Let $\{e_i : i = 1,2,\ldots,2n+1,\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = e_i$ in (17) and taking summation over i, we get

$$-g((\nabla_W\tau)(e_i,Y)Z,e_i) + \eta\big((\nabla_W\tau)(e_i,Y)Z\big)g(\xi,e_i) = A(W)g(\tau(e_i,Y)Z,e_i). \tag{18}$$

By using (15) in (18), we obtain

$$\begin{aligned}&-a_0(\nabla_W S)(Y,Z) - [2na_1 + a_2 + a_5]\Big[\beta\Big\{[(1-k)g(W,\phi Y) + g(W,h\phi Y)]\eta(Z) \\ &+ \eta(Y)g(h(\phi W + \phi hW),Z) - \mu\eta(W)g(\phi hY,Z)\Big\} + \gamma\Big\{g(W+hW,\phi Y)\eta(Z) \\ &+ \eta(Y)g(W+hW,\phi Z)\Big\}\Big] - (a_3+a_6)\Big[\beta\Big\{[(1-k)g(W,\phi Z) + g(W,h\phi Z)]\eta(Y) \\ &+ \eta(Z)g(h(\phi W + \phi hW),Y) - \mu\eta(W)g(\phi hZ,Y)\Big\} + \gamma\Big\{g(W+hW,\phi Z)\eta(Y) \\ &+ \eta(Z)g(W+hW,\phi Y)\Big\}\Big] - a_7(\nabla_W r)[2ng(Y,Z)] + a_0\eta\big((\nabla_W R)(\xi,Y)Z\big) \\ &+ a_2\big[\beta g(h(\phi W + \phi hW),Z)\eta(Y) + \gamma g(W+hW,\phi Z)\eta(Y)\big] \\ &+ a_3\big[\beta g(h(\phi W + \phi hW),Y)\eta(Z) + \gamma g(W+hW,\phi Y)\eta(Z)\big] \\ &+ a_5\big[\beta\{(1-k)g(W,\phi Y) + g(W,h\phi Y)\}\eta(Z) + \gamma\eta(Z)g(W+hW,\phi Y)\big] \\ &+ a_6\big[\beta\{(1-k)g(W,\phi Z) + g(W,h\phi Z)\}\eta(Y) + \gamma\eta(Y)g(W+hW,\phi Z)\big] \\ &+ a_7(\nabla_W r)[g(Y,Z) - \eta(Y)\eta(Z)] \\ = {}& A(W)[a_4 + 2na_7]rg(Y,Z) + A(W)\big[a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6\big]S(Y,Z).\end{aligned} \tag{19}$$

Putting $Z = \xi$ in (19) and on simplification, we get

$$\begin{aligned} & - a_0(\nabla_W S)(Y,\xi) - (2na_1 + a_2 + a_6)\big[\beta\big\{(1-k)g(W,\phi Y) + g(W,h\phi Y)\big\} \\ & + \gamma g(W + hW, \phi Y)\big] - 2na_7(\nabla_W r)\eta(Y) \\ = {} & A(W)\eta(Y)\big[[a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]2nk + [a_4 + 2na_7]r\big]. \end{aligned} \tag{20}$$

We know that

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi). \tag{21}$$

By using (4), (8) in (21), we obtain

$$(\nabla_W S)(Y,\xi) = S(Y,\phi W) + S(Y,\phi hW) - 2nkg(Y,\phi W) - 2nkg(Y,\phi hW). \tag{22}$$

Substituting (22) in (20), we get

$$\begin{aligned} & - a_0\Big\{S(Y,\phi W) + S(Y,\phi hW) - 2nkg(Y,\phi W) - 2nkg(Y,\phi hW)\Big\} \\ & - (2na_1 + a_2 + a_6)\big[\beta\{(1-k)g(W,\phi Y) + g(W,h\phi Y)\} + \gamma g(W + hW,\phi Y)\big] \\ & - 2na_7(\nabla_W r)\eta(Y) \\ = {} & A(W)\eta(Y)\big[[a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]2nk + [a_4 + 2na_7]r\big]. \end{aligned} \tag{23}$$

Replacing Y by ϕY in (23) and simplifying, we have

$$\begin{aligned} & - a_0S(Y,W) - a_0S(Y,hW) + \big[2a_0\beta + 2nka_0 + (2na_1 + a_2 + a_6)(\beta+\gamma)\big]g(hW,Y) \\ & + \big[2nka_0 + (2na_1 + a_2)\{\beta(1-k) + \gamma\} + a_6\gamma - (2a_0 + a_6)\beta(k-1)\big]g(W,Y) \\ & + \big[(2a_0 + a_6)\beta(k-1) - (2na_1 + a_2)\{\beta(1-k) + \gamma\} - a_6\gamma\big]\eta(W)\eta(Y) = 0. \end{aligned} \tag{24}$$

Replacing W by hW in (24) and by virtue of (7), (9) and on simplification, we get

$$\begin{aligned} & - a_0S(Y,hW) + a_0(k-1)S(Y,W) \\ & + (k-1)\big[2a_0\beta + (2na_1 + a_2 + a_6)(\beta+\gamma)\big]\eta(W)\eta(Y) \\ & + \big[2nka_0 + (2na_1 + a_2)\{\beta(1-k) + \gamma\} + a_6\gamma - (2a_0 + a_6)\beta(k-1)\big]g(hW,Y) \\ & - (k-1)\big[2a_0\beta + 2nka_0 + (2na_1 + a_2 + a_6)(\beta+\gamma)\big]g(W,Y) = 0. \end{aligned} \tag{25}$$

Subtracting (24) and (25) and by virtue of (9), we obtain

$$g(Y,hW) = \frac{E}{F}g(Y,W) + \frac{G}{F}\eta(Y)\eta(W), \tag{26}$$

where

$$\begin{aligned} E &= [a_0(\beta - 2nk) + (2na_1 + a_2 + a_6)\gamma] \\ F &= [a_0(2\beta - \alpha) + \beta(2na_1 + a_2)] \end{aligned}$$

and

$$G = \gamma(-a_0 + 2na_1 + a_2 + a_6).$$

By substituting (26) in (24), we get

$$S(Y,W) = \left[\frac{NE}{a_0F} + \frac{P}{a_0}\right] g(Y,W) + \left[\frac{GN}{Fa_0} + \frac{Q}{a_0}\right] \eta(Y)\eta(W),$$

where

$$N = \big[[2nk - \alpha + 2\beta]a_0 + (2na_1 + a_2 + a_6)(\beta + \gamma)\big],$$
$$P = \big[2nka_0 - \beta(k-1)[a_0 + 2na_1 + a_2 + a_6] + \gamma[2na_1 + a_2 + a_6]\big]$$

and

$$Q = \big[\beta(k-1)[a_0 + 2na_1 + a_2 + a_6] - \gamma[2na_1 + a_2 + a_6]\big].$$

Hence, we state the following:

Theorem 1. *A ϕ-τ-recurrent (k,μ)-contact metric manifold is an η-Einstein manifold with $a_0 \neq 0$.*

4 η-τ-Ricci-recurrent (k,μ)-contact metric manifold

Definition 4. A (k,μ)-contact metric manifold M is said to be η-τ-Ricci-recurrent if it satisfies the condition

$$(\nabla_X S_\tau)(\phi Y, \phi Z) = A(X) S_\tau(\phi Y, \phi Z), \tag{27}$$

for all vector fields X, Y, Z on M

From (12), we have

$$S_\tau(Y,Z) = [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]S(Y,Z) + r[a_4 + 2na_7]g(Y,Z). \tag{28}$$

Replacing $Y = \phi Y$ and $Z = \phi Z$ in (28), we obtain

$$\begin{aligned} S_\tau(\phi Y, \phi Z) = {} & [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]S(\phi Y, \phi Z) \\ & + [a_4 + 2na_7] r g(\phi Y, \phi Z). \end{aligned} \tag{29}$$

Differentiating (29) with respect to X, we get

$$\begin{aligned} (\nabla_X S_\tau)(\phi Y, \phi Z) = {} & [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6](\nabla_X S)(\phi Y, \phi Z) \\ & + [a_4 + 2na_7](\nabla_X r) g(\phi Y, \phi Z). \end{aligned} \tag{30}$$

By using (30) and (29) in (27), we have

$$L(\nabla_X S)(\phi Y, \phi Z) + M(\nabla_X r) g(\phi Y, \phi Z) = A(X)\{LS(\phi Y, \phi Z) + Mrg(\phi Y, \phi Z)\}, \tag{31}$$

where $L = [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]$ and $M = [a_4 + 2na_7]$.

Now, differentiating (11), we have

$$\begin{aligned} (\nabla_X S)(\phi Y, \phi Z) = {} & (\nabla_X S)(Y,Z) - 2nk\big[-\eta(Z)g(Y,\phi X) - \eta(Z)g(Y,\phi hX) \\ & - \eta(Y)g(Z,\phi X) - \eta(Y)g(Z,\phi hX)\big] \\ & - 2[2n - 2 + \mu]\big[(1-k)g(X,\phi Y)\eta(Z) \\ & + g(X, h\phi Y)\eta(Z) + \eta(Y)g(h(\phi X + \phi hX), Z) \\ & - \mu\eta(X)g(\phi hY, Z)\big] \\ & + \eta(Y)S(X + hX, \phi Z) + \eta(Z)S(\phi Y, X + hX). \end{aligned} \tag{32}$$

Substituting (32) in (31) and on simplification, we obtain

$$\begin{aligned}(\nabla_X S)(Y,Z) = & -2nk\big[\eta(Z)g(Y,\phi X) + \eta(Z)g(Y,\phi hX) + \eta(Y)g(Z,\phi X) \\ & + \eta(Y)g(Z,\phi hX)\big] + 2[2n-2+\mu]\big[(1-k)g(X,\phi Y)\eta(Z) \\ & + g(X,h\phi Y)\eta(Z) + \eta(Y)g(h(\phi X+\phi hX),Z) - \mu\eta(X)g(\phi hY,Z)\big] \\ & - \eta(Y)S(X+hX,\phi Z) - \eta(Z)S(\phi Y, X+hX) \\ & - \frac{M}{L}(\nabla_X r)g(\phi Y,\phi Z) + A(X)\{S(Y,Z) - 2nk\eta(Y)\eta(Z) \\ & - 2[2n-2+\mu]g(hY,Z) + \frac{Mr}{L}g(\phi Y,\phi Z)\}. \end{aligned} \tag{33}$$

Let $\{e_i : i = 1, 2, \ldots, 2n+1\}$ be an orthonormal frame field at any point of the manifold. Then contracting Y and Z in (33), we have

$$\mathrm{d}r(X) = A(X)\left[r - \frac{2nkL}{(L+2nM)}\right]. \tag{34}$$

Again, contracting over X and Z in (33), we get

$$\begin{aligned}\frac{1}{2}\mathrm{d}r(Y) = & \left[-2nk\,\mathrm{tr}(\phi) + 2(2n-2+\mu)\,\mathrm{tr}(h\phi h) - \mathrm{tr}(Q\phi) - \mathrm{tr}(Qh\phi) + \frac{(\xi r)M}{L}\right]\eta(Y) \\ & - \left[2nk + \frac{rM}{L}\right]A(\xi)\eta(Y) - \frac{M}{L}dr(Y) + S(Y,\rho) - 2(2n-2+\mu)A(hY) \\ & + \frac{rM}{L}A(Y). \end{aligned} \tag{35}$$

By using (34) in (35) and on simplification, we get

$$\begin{aligned}A(Y)\left[\frac{r}{2} - \frac{nk(L+2M)}{(L+2nM)}\right] = & \Big[-2nk\,\mathrm{tr}(\phi) + 2(2n-2+\mu)\,\mathrm{tr}(h\phi h) \\ & - \mathrm{tr}(Q\phi) - \mathrm{tr}(Qh\phi) - \frac{2nk(L+M+2nM)}{(L+2nM)}A(\xi)\Big]\eta(Y) \\ & + S(Y,\rho) - 2(2n-2+\mu)A(hY). \end{aligned} \tag{36}$$

Replacing $Y = hY$ in (36) and by virtue of (9), we obtain

$$A(hY) = \frac{2(L+2nM)}{[(r-2\alpha)(L+2nM) - 2nk(L+2M)]}[\beta(k-1)\{A(Y) - A(\xi)\eta(Y)\}], \tag{37}$$

where $\alpha = [2(n-1) - n\mu]$ and $\beta = [2(n-1) + \mu]$.

Substituting (37) in (36), we have

$$A(Y)\left[\frac{r}{2}-\frac{nk(L+2M)}{(L+2nM)}+\frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM)-2nk(L+2M)]}\right]$$
$$=[-2nk\,\mathrm{tr}(\phi)+2\beta\,\mathrm{tr}(h\phi h)-\mathrm{tr}(Q\phi)-\mathrm{tr}(Qh\phi)]\eta(Y)$$
$$+\left[\frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM)-2nk(L+2M)]}-\frac{2nk(L+M+2nM)}{(L+2nM)}\right]A(\xi)\eta(Y)$$
$$+S(Y,\rho). \quad (38)$$

Putting $Y=\xi$ in (38), we get

$$A(\xi)\left[\frac{r}{2}-\frac{nkL}{(L+2nM)}\right]=[-2nk\,\mathrm{tr}(\phi)+2\beta\,\mathrm{tr}(h\phi h)-\mathrm{tr}(Q\phi)-\mathrm{tr}(Qh\phi)]. \quad (39)$$

From (39) and (38), we have

$$\begin{aligned}S(Y,\rho)=&\left[\frac{r}{2}-\frac{nk(L+2M)}{(L+2nM)}+\frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM)-2nk(L+2M)]}\right]g(Y,\rho)\\&+\left[-\frac{r}{2}+\frac{nkL}{(L+2nM)}-\frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM)-2nk(L+2M)]}\right.\\&\left.+\frac{2nk(L+M+2nM)}{(L+2nM)}\right]\eta(Y)\eta(\rho).\end{aligned} \quad (40)$$

From (40), we have

$$\begin{aligned}QY=&\left[\frac{r}{2}-\frac{nk(L+2M)}{(L+2nM)}+\frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM)-2nk(L+2M)]}\right]Y\\&+\left[-\frac{r}{2}+\frac{nkL}{(L+2nM)}-\frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM)-2nk(L+2M)]}\right.\\&\left.+\frac{2nk(L+M+2nM)}{(L+2nM)}\right]\eta(Y)\xi.\end{aligned} \quad (41)$$

Hence, we state the following:

Theorem 2. *If the Ricci tensor of a (k,μ)-contact metric manifold is η-τ-Ricci--recurrent then its Ricci tensor along the associated vector field of the 1-form is given by* (40) *and the eigen value of the Ricci tensor with respect to the characteristic vector ξ is given by (41).*

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