Communications in Mathematics 26 (2018) 137–145

DOI: 10.2478/cm-2018-0010

A new class of almost complex structures on tangent bundle of a Riemannian manifold

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Abstract. In this paper, the standard almost complex structure on the tangent bunle of a Riemannian manifold will be generalized. We will generalize the standard one to the new ones such that the induced $(0,2)$-tensor on the tangent bundle using these structures and Liouville 1-form will be a Riemannian metric. Moreover, under the integrability condition, the curvature operator of the base manifold will be classified.

1 Introduction

Let (M,g) be a Riemannian manifold and ∇ represents the Levi-Civita connection of g and suppose $\pi\colon TM \to M$ is the tangent bundle of M. Furthermore, we denote by X^h and X^v the horizontal and vertical lifts of any vector field X on M, respectively. There are many papers ([1], [3], [5], [10], [8], [9]) which are on differential geometric structures on tangent and cotangent bundles like the Riemannian metrics, harmonic sections, almost complex structures, connections and so on.

Almost complex structures are some important structures in differential geometry. These structures obtained many applications in physics, signal processing and information geometry. Kählerian manifolds as a special class of complex manifolds plays an important role in signal processing. Choi and Mullhaupt [4] proved a correspondence between the information geometry of a signal filter and a Kähler manifold; the information geometry of a minimum-phase linear system with a finite complex spectrum norm is a Kähler manifold. In [11], the authors investigated

2010 MSC: 32Q60, 58A30
Key words: Almost complex structure, curvature operator, integrability, tangent bundle
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the necessary conditions for a divergence function on a manifold M such that the manifold $M \times M$ admits a Kählerian structure. Lisi [6] investigated the applications of pseudo-holomorphic curves to problems in Hamiltonian dynamics using the structures of symplectic manifolds.

The classical almost complex structure $J_{1,0} \colon TTM \to TTM$ is defined by

$$J_{1,0}(X^h) = X^v, \quad J_{1,0}(X^v) = -X^h.$$

for vector field X on M. In [2], Aguilar generalized this structure to a class of almost complex structures and called them isotropic almost complex structures $J_{\delta,\sigma}$ with definition

$$J_{\delta,\sigma}(X^h) = \alpha X^v + \sigma X^h, \quad J_{\delta,\sigma}(X^v) = -\sigma X^v - \delta X^h,$$

for functions $\alpha, \delta, \sigma \colon TM \to \mathbb{R}$ which satisfy $\alpha\delta - \sigma^2 = 1$. He showed that there exists an integrable isotropic almost complex structure on an open subset $\mathcal{A} \subset TM$ if and only if the sectional curvature of $(\pi(\mathcal{A}), g)$ is constant.

To make a metrical discussion, let Θ be the Liouville 1-form on TM and $J_{\delta,\sigma}$ be an almost isotropic structure. If α is a positive valued function on TM then the $(0,2)$-tensor

$$g_{\delta,\sigma}(A, B) = \mathrm{d}\Theta(J_{\delta,\sigma} A, B), \qquad A, B \in TTM$$

will be a Riemannian metric on TM. Now, let J be an almost complex structure on the tangent bundle of a Riemannian manifold. The $(0,2)$-tensor

$$G(A, B) = \mathrm{d}\Theta(JA, B), \qquad A, B \in TTM,$$

is not a symmetric tensor, in general. So, we would like to generalize $J_{1,0}$ to a class of structures $\bar{J}$ with definition

$$\bar{J}(X^h) = \alpha X^v + J_1(X^h), \quad \bar{J}(X^v) = J_2(X^v) - \delta X^h, \tag{1}$$

such that the tensor $G(A, B) = \mathrm{d}\Theta(\bar{J}A, B)$ is a Riemannian metric where

$$\alpha, \delta \colon TM \to \mathbb{R}^+$$

are smooth mappings, $J_1 \colon \mathcal{H}TM \to \mathcal{H}TM$ and $J_2 \colon \mathcal{V}TM \to \mathcal{V}TM$ are linear bundle maps where $\mathcal{H}TM$ is the horizontal sub-bundle and $\mathcal{V}TM$ is the vertical sub-bundle of TTM, respectively.

As a relevance between the generalized structure $\bar{J}$ and isotropic almost complex structure $J_{\delta,\sigma}$ it can be said that if $J_1 \colon \mathcal{H}TM \to \mathcal{H}TM$ and $J_2 \colon \mathcal{V}TM \to \mathcal{V}TM$ are multiples of identity bundle maps $\mathrm{id} \colon \mathcal{H}TM \to \mathcal{H}TM$ and $\mathrm{id} \colon \mathcal{V}TM \to \mathcal{V}TM$, respectively, then $\bar{J}$ will be an isotropic almost complex structure.

We know that the necessary condition for integrability of the isotropic almost complex structures is that the base manifold is a space form, i.e., the curvature operator of the base manifold is a constant multiple of the identity operator. But the necessary condition for the integrability of a generalized one is that the curvature operator is diagonalizable and has three square blocks; two of them are arbitrary multiple of the identity matrix and one of them is the zero matrix.

By supposing $K\colon TTM \to TM$ the connection map with respect to the metric g and π_* the differentiate of $\pi\colon TM \to M$, the following theorem states the necessary and sufficient conditions for G to be a Riemannian metric.

Theorem 1. *Let (M,g) be a Riemannian manifold and $\bar{J}\colon TTM \longrightarrow TTM$ be an almost complex structure on TM given by (1). Then G is a Riemannian metric on TM if and only if $\pi_* J_1 X^h = -KJ_2X^v$ and $\alpha\delta - 1 \geq 0$ and J_1 is symmetric with respect to G, i.e., $G(J_1X^h, Y^h) = G(X^h, J_1Y^h)$, and has at most two eigen-values $-\sqrt{\alpha\delta-1} = -\sigma$, $\sqrt{\alpha\delta-1} = \sigma$.*

Let $(x^1, \dots, x^n)$ be a locally coordinate system on (M,g) then we may assume $g = g_{ij}dx^i \otimes dx^j$ in this coordinate. We suppose that (g^{ij}) is the inverse matrix of (g_{ij}). One can extend metric g on every tensor bundle $\otimes_l^k M$ of M for sections

$$F = F_{i_1,\dots,i_k}^{j_1,\dots,j_l} dx^{i_1} \otimes \dots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}},$$

and

$$G = G_{r_1,\dots,r_k}^{s_1,\dots,s_l} dx^{r_1} \otimes \dots \otimes dx^{r_k} \otimes \frac{\partial}{\partial x^{s_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{s_l}},$$

with the following definition

$$g(F,G) = g^{i_1 r_1} \dots g^{i_k r_k} g_{j_1 s_1} \dots g_{j_l s_l} G_{r_1,\dots,r_k}^{s_1,\dots,s_l} F_{i_1,\dots,i_k}^{j_1,\dots,j_l},$$

where we denoted the new metric by g, again. Since the bivector bundle $\wedge^2 M$ is a sub-bundle of $\otimes_2 M$, we can restrict the extended metric g to this sub-bundle and denote it by g, again. It is notable that an orthonormal frame on a tensor bundle contains tensorial product of orthonormal vectors.

Definition 1. (see [7]) If we suppose that R is the curvature tensor of manifold (M,g) then its curvature operator $\mathcal{R}\colon \wedge^2 M \to \wedge^2 M$ is defined by

$$g(\mathcal{R}(X \wedge Y), V \wedge W) = R(X,Y,W,V).$$

If $\mathcal{H}_\sigma$ and $\mathcal{H}_{-\sigma}$ denote the sub-distributions of $\mathcal{H}TM$ including the eigen-vectors of $J_1\colon \mathcal{H}TM \to \mathcal{H}TM$ associated to σ and $-\sigma$, respectively, then we will prove the following theorem.

Theorem 2. *Suppose $\{X_1, \dots, X_n, X_{n+1}, \dots, X_{n+m}\}$ be a set of locally orthonormal frame field on M such that $\mathcal{H}_\sigma$ is spanned by $\{X_1^h, \dots, X_n^h\}$ and $\mathcal{H}_{-\sigma}$ is spanned by $\{X_{n+1}^h, \dots, X_{n+m}^h\}$. If α, δ are functions of $E(u) = g(u,u)$ then the curvature operator $\mathcal{R}$ is diagonalizable with respect to this frame and we have*

$$\mathcal{R}(X_i \wedge X_j) = f X_i \wedge X_j, \quad \text{for all } 1 \leq i,j \leq n,$$

and

$$\mathcal{R}(X_i \wedge X_j) = 0, \quad \text{for all } 1 \leq i \leq n \text{ and } n+1 \leq j \leq n+m,$$

and

$$\mathcal{R}(X_i \wedge X_j) = h X_i \wedge X_j, \quad \text{for all } n+1 \leq i,j \leq n+m,$$

for some mappings f, h on M.

2 Generalization and integrability discussion

Let (N, J) be an almost complex manifold and $T^C(N)$ be the complexfication of TN. For $x \in N$, define the spaces $T_x^{(0,1)}(N)$ and $T_x^{(1,0)}(N)$ of $T_x^C(N)$ as following

$$T_x^{(0,1)}(N) = \{X_x + \sqrt{-1}J_x X_x | X(x) \in T_x(N)\},$$

and

$$T_x^{(1,0)}(N) = \{X_x - \sqrt{-1}J_x X_x | X(x) \in T_x(N)\}.$$

Now, let $T^{(0,1)}(N) = \bigcup_x T_x^{(0,1)}(N)$ and $T^{(1,0)}(N) = \bigcup_x T_x^{(1,0)}(N)$. It is a well known fact that J is an integrable structure if and only if for all sections $A, B \in \Gamma(T^{(0,1)}(N))$ we have $[A, B] \in \Gamma(T^{(0,1)}(N))$; equivalently, for an arbitrary 1-form ζ of the dual space of $T^{(1,0)}(N)$ (where denoted by $T^{(1,0)}(N)^*$) we have

$$\mathrm{d}\zeta \in \wedge^2 T^{(1,0)}(N)^*.$$

Definition 2. Let $\eta, \eta^1, \ldots, \eta^n$ be 1-forms on a differentiable manifold N. We say that $\mathrm{d}\eta \equiv 0 \mod \{\eta^1, \ldots, \eta^n\}$, if and only if $\mathrm{d}\eta = \sum_{i,j} f_{ij}\eta^i \wedge \eta^j$, for some functions f_{ij} on N.

So, if $\zeta^1, \ldots, \zeta^n$ are locally $(1,0)$-forms generating $\Gamma(T^{(1,0)}(N)^*)$, then J is integrable if and only if $\mathrm{d}\zeta \equiv 0 \mod \{\zeta^1, \ldots, \zeta^n\}, \forall \zeta \in \Gamma(T^{(1,0)}(N)^*)$.

When we work with Θ, it is convenient to work with a locally orthonormal frame field on (M^n, g) like $X_1, \ldots, X_n$. Because, if we suppose that π, K are the natural projection from TM to M and the connection map, respectively and if we suppose θ^i is the dual 1-forms of X_i then

$$\mathrm{d}\Theta = \sum_{i=1}^{n} (\theta^i \circ K) \wedge (\pi^* \theta^i),$$

where $\{\theta^i \circ K, \pi^*\theta^i\}$ is the dual basis of $\{X_i^v, X_i^h\}$.

Proof of Theorem 1. Since G is a Riemannian metric, using the symmetric property of G, i.e., $G(X_l^h, X_s^v) = G(X_s^v, X_l^h)$ for all l, s, gives us

$$\mathrm{d}\Theta(\bar{J}X_l^h, X_s^v) = \mathrm{d}\Theta(\bar{J}X_s^v, X_l^h),$$

then using the equation $\mathrm{d}\Theta = \sum_{i=1}^{n} (\theta^i \circ K) \wedge (\pi^*\theta^i)$ gives

$$\pi_* J_1 X^h = -K J_2 X^v.$$

Now, from $G(\bar{J}X^h, \bar{J}Y^v) = G(\bar{J}Y^v, \bar{J}X^h)$ and $\pi_* J_1 X^h = -K J_2 X^v$ one can get that J_1 is a symmetric linear bundle map. Finally, $\bar{J}^2 = -\mathrm{id}$ gives us the equation

$$J_1^2 = (\alpha\delta - 1)\mathrm{id},$$

and since J_1 is symmetric it has at most two real eigenvalues, i.e., $\alpha\delta - 1 \geq 0$, and so $\sqrt{\alpha\delta - 1}$ and $-\sqrt{\alpha\delta - 1}$ are the eigenvalues of J_1 and the proof is completed. $\square$

The following lemma shows that the dimension of $\mathcal{H}_\sigma$ and $\mathcal{H}_{-\sigma}$ are constant along TM.

Lemma 1. *Let $\bar{J}$ be a generalized structure then the dimension of $\mathcal{H}_\sigma$ and $\mathcal{H}_{-\sigma}$ are constant along TM. Consequently, if M is connected and if there is a point $u \in TM$ such that $\sigma(u) \neq 0$ then $\sigma \neq 0$ every where.*

Proof. First, note that if $\sigma = 0$ on TM then the generalized structure $\bar{J}$ is an isotropic almost complex structure. So, we suppose that there exists a point $v \in TM$ such that $\sigma(v) \neq 0$. Let $r_v, s_v \in \mathbb{N}$ be the multiplicities of $\mathcal{H}_\sigma, \mathcal{H}_{-\sigma}$ at v, respectively. By using the continuity of $\operatorname{tr} J_1$ at v, there exist an open subset of TM like U containing v such that $|\operatorname{tr} J_1(v) - \operatorname{tr} J_1(v')| \leq \epsilon$ for a given $\epsilon \in \mathbb{R}^+$ and for any vector $v' \in U$. Let $r_{v'} = r_v + a$ and $s_{v'} = s_v - a$ for some $a \in \mathbb{N}^+$, then we have

$$\begin{aligned}|\operatorname{tr} J_1(v) - \operatorname{tr} J_1(v')| &= |(r_v - s_v)\sigma(v) - (r_{v'} - s_{v'})\sigma(v')| \\ &= |(r_v - s_v)(\sigma(v) - \sigma(v')) - 2a\sigma(v')| \leq \epsilon.\end{aligned}$$

When $v' \to v$ we get that $|2a\sigma(v)| \leq \epsilon$. Since $\sigma(v) \neq 0$ and ϵ was arbitrary, a must be zero. This shows that the dimensions of $\mathcal{H}_\sigma, \mathcal{H}_{-\sigma}$ are locally constants. So, if M is connected then the dimension of $\mathcal{H}_\sigma, \mathcal{H}_{-\sigma}$ are nonzero and the second part is proved. □

In general, the eigen-spaces $\mathcal{H}_\sigma, \mathcal{H}_{-\sigma}$ can not be spanned by the horizontal lifts of locally vector fields. The following proposition shows this matter in more details.

Proposition 1. *Suppose $\mathcal{H}_\sigma$ and $\mathcal{H}_{-\sigma}$ be the eigen-spaces of $\bar{J}$. Then there exist locally vector fields $\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+m}\}$ on M with the conditions $\operatorname{span}\{X_1^h, \ldots, X_n^h\} = \mathcal{H}_\sigma$ and $\operatorname{span}\{X_{n+1}^h, \ldots, X_{n+m}^h\} = \mathcal{H}_{-\sigma}$ if and only if*

$$\pi_*(\mathcal{H}_\sigma)_u = \pi_*(\mathcal{H}_\sigma)_v \quad \text{and} \quad \pi_*(\mathcal{H}_{-\sigma})_u = \pi_*(\mathcal{H}_{-\sigma})_v$$

for all $u, v \in T_pM$ and for all p where we mean by $(\mathcal{H}_\sigma)_u, (\mathcal{H}_{-\sigma})_u$ the eigen-spaces of $\bar{J}$ at $u \in T_pM$ with respect to σ and $-\sigma$.

Proof. $\Rightarrow$) The key of proof is that $\pi_* \colon \mathcal{H}TM \to TM$ is a vector bundle isomorphism.

$\Leftarrow$) We know that the dimension of $\mathcal{H}_\sigma$ and $\mathcal{H}_{-\sigma}$ are constants over TM. We will only prove that there exist locally vector fields $X_1, \ldots, X_n$ on M such that $\operatorname{span}\{X_1^h, \ldots, X_n^h\} = \mathcal{H}_\sigma$ and the other one is similar. Since, $\pi_*(\mathcal{H}_\sigma)_u = \pi_*(\mathcal{H}_\sigma)_v$ for all $u, v \in T_pM$ then $\pi_*\mathcal{H}_\sigma$ is a smooth distribution on M. So, there exist locally vector fields $X_1, \ldots, X_n$ on M such that $\operatorname{span}\{X_1^h, \ldots, X_n^h\} = \mathcal{H}_\sigma$. □

From now on, we would like to investigate the integrability of the structures $\bar{J}$ which satisfies the proposition 1. So, suppose $\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+m}\}$ be an orthonormal frame field on M such that $\operatorname{span}\langle X_1^h, \ldots, X_n^h\rangle = \mathcal{H}_\sigma$, and $\operatorname{span}\langle X_{n+1}^h, \ldots, X_{n+m}^h\rangle = \mathcal{H}_{-\sigma}$. Therefore, we have

$$\bar{J}(X_i^h) = \sigma X_i^h + \alpha X_i^v, \quad \bar{J}(X_i^v) = -\delta X_i^h - \sigma X_i^v, \quad \text{for } i = 1, \ldots, n,$$

and

$$\bar{J}(X_i^h) = -\sigma X_i^h + \alpha X_i^v, \quad \bar{J}(X_i^v) = -\delta X_i^h + \sigma X_i^v, \quad \text{for } i = n+1, \ldots, n+m,$$

and the following 1-forms are a locally basis for $(1,0)$-forms.

$$u^i = (1 - \sqrt{-1}\sigma)\eta^i + \sqrt{-1}\delta\xi^i, \quad \text{for } i = 1, \ldots, n,$$

and

$$u^i = (1 + \sqrt{-1}\sigma)\eta^i + \sqrt{-1}\delta\xi^i, \quad \text{for } i = n+1, \ldots, n+m.$$

Furthermore, the following vector fields are the locally basis for $(0,1)$ vectors

$$V_k = (1 + \sqrt{-1}\sigma)X_k^h + \sqrt{-1}\alpha X_k^v, \quad \forall 1 \leq k \leq n,$$

and

$$V_k = (1 - \sqrt{-1}\sigma)X_k^h + \sqrt{-1}\alpha X_k^v, \quad \forall n+1 \leq k \leq n+m.$$

Let $\theta^1, \ldots, \theta^{n+m}$ be the dual 1-forms of $X_1, \ldots, X_{n+m}$. If we suppose that $\eta^i = \pi^*\theta^i$ and $\xi^i = \theta^i \, o \, K$ then we have

$$\mathrm{d}\eta^i = \sum_{k=1}^{n+m} \eta^k \wedge \pi^*\omega_k^i, \quad \text{and} \quad \mathrm{d}\xi^i = \sum_{k=1}^{n+m} (\xi^k \wedge \pi^*\omega_k^i + \natural\theta^k \pi^*\Omega_k^i), \tag{2}$$

where ω_k^i, Ω_k^i are the connection 1-forms and curvature 2-forms of the metric g, respectively. Moreover, suppose that $\natural\theta^s(u) = \theta^s(u)$ for $u \in TM$ is a locally mapping on the tangent bundle. Using the above notations, the following propositions can be concluded as the integrability conditions.

Proposition 2. *Let $\bar{J}$ be an integrable structure on the tangent bundle of (M^{n+m}, g) which satisfies the proposition 1. If $n \geq 3$ then we have*

$$R_{skl}^i = 0, \quad i \neq k, l, \quad 1 \leq i, k, l \leq n, \quad 1 \leq s \leq n+m, \tag{3}$$

and if $m \geq 3$ we have

$$R_{skl}^i = 0, \quad i \neq k, l, \quad n+1 \leq i, k, l \leq n+m, \quad 1 \leq s \leq n+m. \tag{4}$$

Proof. Using the equations (2) we get

$$\begin{aligned} \mathrm{d}u^i = & -\sqrt{-1}\mathrm{d}\sigma \wedge \eta^i + (1 - \sqrt{-1}\sigma)\eta^r \wedge \omega_r^i + \sqrt{-1}\mathrm{d}\delta \wedge \xi^i \\ & + \sqrt{-1}\delta(\xi^r \wedge \pi^*\omega_r^i + \natural\theta^r \pi^*\Omega_r^i), \quad \text{for } i = 1, \ldots, n, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \mathrm{d}u^i = & \sqrt{-1}\mathrm{d}\sigma \wedge \eta^i + (1 + \sqrt{-1}\sigma)\eta^r \wedge \omega_r^i + \sqrt{-1}\mathrm{d}\delta \wedge \xi^i \\ & + \sqrt{-1}\delta(\xi^r \wedge \pi^*\omega_r^i + \natural\theta^r \pi^*\Omega_r^i), \quad \text{for } i = n+1, \ldots, n+m. \end{aligned} \tag{6}$$

Let $1 \leq i, k, l \leq n$ and $i \neq k, l$. If $\bar{J}$ is integrable then

$$\mathrm{d}u^i \equiv 0 \mod u^1, \ldots, u^{n+m},$$

so from (5) we will have

$$0 = \mathrm{d}u^i(V_k, V_l) = \delta\sqrt{-1}(1+\sqrt{-1}\sigma)\natural\theta^s R^i_{skl}, \tag{7}$$

so we have $R^i_{skl} = 0$ for $i \neq k, l$ and $1 \leq i, k, l \leq n$. The same argument can be done for prove the equation 4. □

Now, one can get the following proposition as an integrability condition for $\bar{J}$.

Proposition 3. *Let $\bar{J}$ be an integrable structure on the tangent bundle of (M^{n+m}, g) which satisfies the proposition 1. If α, δ are functions of $E(u) = g(u, u)$ then*

$$R^i_{jkl} = 0, \quad 1 \leq i \leq n, \quad n+1 \leq k, l \leq n+m, \quad 1 \leq j \leq n+m, \tag{8}$$

and

$$R^i_{jkl} = 0, \quad n+1 \leq i \leq n+m, \quad 1 \leq k, l \leq n, \quad 1 \leq j \leq n+m. \tag{9}$$

Proof. We only prove the equation (8) and the proof of second one is similar. Let $1 \leq i \leq n$ and $n+1 \leq k, l \leq n+m$ then by considering the integrability of $\bar{J}$ we will have

$$\begin{aligned} 0 = \mathrm{d}u^i(V_k, V_l) &= (1-\sqrt{-1}\sigma)^3\omega^i_k(X_l) \\ &+ \sqrt{-1}\delta[\sqrt{-1}\alpha(1-\sqrt{-1}\sigma)\omega^i_k(X_l) + \natural\theta^s(1-\sqrt{-1}\sigma)^2 R^i_{skl}], \end{aligned} \tag{10}$$

The complex equation (10) results the following equations

$$\omega^i_k(X_l)(1-\sigma^2-\alpha\delta) + \sigma\delta\natural\theta^s R^i_{skl} = 0, \quad \text{and} \quad -2\sigma\omega^i_k(X_l) + \delta\natural\theta^s R^i_{skl} = 0. \tag{11}$$

Since, $\sigma \neq 0$ and $\alpha\delta - \sigma^2 = 1$, the above equations are the same. So,

$$\frac{-2\sigma}{\delta}\omega^i_k(X_l) + \natural\theta^s R^i_{skl} = 0. \tag{12}$$

If we suppose that there exist mappings $f, h\colon \mathbb{R} \to \mathbb{R}$ such that

$$\delta(u) = f(g(u,u)) = f\left(\sum_{i=1}^{n+m} \natural(\theta i)^2\right)$$

and

$$\sigma(u) = h(g(u,u)) = h(g(u,u)) = h\left(\sum_{i=1}^{n+m} \natural(\theta i)^2\right),$$

for $u = \sum_{i=1}^{n+m} \natural\theta^i X_i$ then from (12) we have $\frac{\partial^2\sigma/\delta}{\partial(\natural\theta^s)^2} = 0$. Letting

$$\left(\frac{\sigma}{\delta}\right)(u) = o\left(\sum_{i=1}^{n+m} (\natural\theta i)^2\right)$$

for $u = \sum_{i=1}^{n+m} \natural\theta^i X_i$ and for some mapping $o\colon \mathbb{R} \to \mathbb{R}$ then

$$\frac{\partial\sigma/\delta}{\partial\natural\theta^s} = 2\natural\theta^s o'\left(\sum_{i=1}^{n+m} (\natural\theta^i)^2\right).$$

So, we have $0 = \frac{\partial^2\sigma/\delta}{\partial(\natural\theta^s)^2} = 2o' + (2\natural\theta^s)^2 o''$. If we consider $s = 1, \ldots, n+m$ the result is that $o' = 0$. So, the equation (12) gives us

$$R^i_{jkl} = 0, \tag{13}$$

for $1 \leq i \leq n$ and $n+1 \leq k, l \leq n+m$ and $1 \leq j \leq n+m$. □

Now, we can prove the main Theorem 2.

Proof of Theorem 2. The proof will be done in two parts; for $n, m \geq 3$ and the cases that $n, m < 3$ and the way is that we prove the curvature operator is diagonalizable.

First note that when we say that for example $R^i_{jkl} = 0$ for $1 \leq i \leq n$ and $n+1 \leq k, l \leq n+m$ and for all $1 \leq j \leq n+m$, it means $R(X_j, X_k, X_l, X_i) = 0$. When $n, m \geq 3$ then using the propositions 2, 3 the theorem will be easily proved. If $n = 2$ and $m = 3$, using the propositions 2 and 3 we will get that

$$\begin{aligned}
\mathcal{R}(X_3 \wedge X_4) &= f_{34} X_3 \wedge X_4,\\
\mathcal{R}(X_3 \wedge X_5) &= f_{35} X_3 \wedge X_5,\\
\mathcal{R}(X_4 \wedge X_5) &= f_{45} X_4 \wedge X_5,
\end{aligned}$$

for some functions f_{ij} on M. On the other hand, the proposition 3 says that

$$\mathcal{R}(X_1 \wedge X_i) = 0, \quad \mathcal{R}(X_2 \wedge X_i) = 0, \quad \forall i = 3, 4, 5,$$

and so $\mathcal{R}(X_1 \wedge X_2) = f_{12} X_1 \wedge X_2$ for some mapping f_{12} on M. Now, let $n = m = 2$ then again using the proposition 3 and the same argument we will get

$$\mathcal{R}(X_1 \wedge X_2) = f_{12} X_1 \wedge X_2, \quad \mathcal{R}(X_3 \wedge X_4) = f_{34} X_3 \wedge X_4,$$

for some functions f_{12}, f_{34} on M. If $n = 1$ and $m = 2$, we get the same result and finally if $n = m = 1$, then using the proposition 3 we get that the base manifold must be flat and the theorem is proved. □

The following is an example of generalized structures.

Example 1. Let (M, g) be the Euclidean space $(\mathbb{R}^4, \langle , \cdot, \rangle)$ and $(x^1, \ldots, x^4)$ be its standard coordinate system and $(x^1, \ldots, x^4, y^1, \ldots, y^4)$ be the coordinate system on its tangent bundle. Then the almost complex structure $\bar{J}$ defined by

$$\bar{J}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}, \quad \bar{J}\left(\frac{\partial}{\partial y^i}\right) = -2\frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^i}, \quad \text{for } i = 1, 2$$

and

$$\bar{J}\left(\frac{\partial}{\partial x^i}\right) = -\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}, \quad \bar{J}\left(\frac{\partial}{\partial y^i}\right) = -2\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}, \quad \text{for } i = 3, 4$$

is an integrable structure.

References

[1] M.T.K. Abbassi, G. Calvaruso, D. Perrone: Harmonic sections of tangent bundles equipped with Riemannian g-natural metrics. *Q. J. Math.* 62 (2) (2011) 259–288.

[2] R. M. Aguilar: Isotropic almost complex structures on tangent bundles. *Manuscripta Math.* 90 (4) (1996) 429–436.

[3] I. Biswas, J. Loftin, M. Stemmler: Flat bundles on affine manifolds. *Arabian Journal of Mathematics* 2 (2) (2013) 159–175.

[4] J. Choi, A. P. Mullhaupt: Kählerian information geometry for signal processing. *Entropy* 17 (2015) 1581–1605.

[5] R. M. Friswell, C. M. Wood: Harmonic vector fields on pseudo-Riemannian manifolds. *Journal of Geometry and Physics* 112 (2017) 45–58.

[6] S.T. Lisi: Applications of Symplectic Geometry to Hamiltonian Mechanics. PhD thesis, New York University (2006)

[7] P. Petersen: *Riemannian Geometry*. Springer (2006).

[8] E. Peyghan, A. Heydari, L. Nourmohammadi Far: On the geometry of tangent bundles with a class of metrics. *Annales Polonici Mathematici* 103 (2012) 229–246.

[9] E. Peyghan, H. Nasrabadi, A. Tayebi: The homogenous lift to the $(1,1)$-tensor bundle of a Riemannian metric. *Int. J. Geom Meth. Modern Phys.* 10 (4) (2013) 18p.

[10] A. A. Salimov, A. Gezer: On the geometry of the $(1,1)$-tensor bundle with Sasaki type metric. *Chinese Ann. Math. Ser. B* 32 (3) (2011) 1–18.

[11] J. Zhang, F. Li: Symplectic and Kähler structures on statistical manifolds induced from divergence functions. In: *Conference paper in Geometric Science of Information*. Springer (2013) 595–603.

Received: 18 June, 2017
Accepted for publication: 21 September, 2017
Communicated by: Olga Rossi