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A sequence adapted from the movement of the center of mass of two planets in solar system

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Abstract. In this paper we derive a sequence from a movement of center of mass of arbitrary two planets in some solar system, where the planets circle on concentric circles in a same plane. A trajectory of center of mass of the planets is discussed. A sequence of points on the trajectory is chosen. Distances of the points to the origin are calculated and a distribution function of a sequence of the distances is found.

1 Introduction

Let us suppose, that we have two planets, for example the Earth and the Venus circling about the Sun. Of course, they could be arbitrary planets, so let us name them as the outer and the inner planet. Suppose, that the trajectories of these two planets are circles in the same plane with the Sun as the common center. In our simplification, the weight of the Sun is so huge, that the weights of the planets have no influence to their trajectories. The question is, how would be the trajectory of the center of mass of these two planets?

The question has some variable parameters, like radiuses of the trajectories, weights of the planets and turnaround times. Then the trajectory of the center of mass of these two planets may look for example like in the Figure 1 or Figure 2 depending on the concrete values of the parameters.

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Key words: distribution function, g-discrepancy, sequence of points, center of mass, trajectory of two planets, solar system

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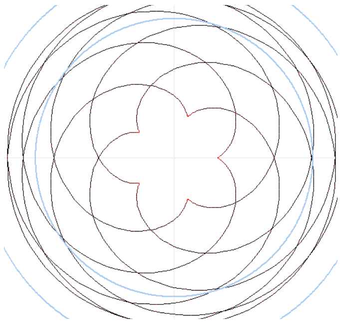

Figure 1: $T = 8/13$

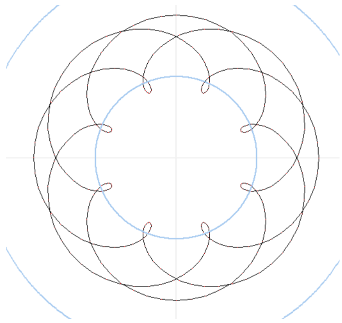

Figure 2: $T = 3/11$

2 Parameters

For the sake of simplicity, let the outer planet (the Earth) be a unity of weight, turnaround time and also of radius of its trajectory. For the inner planet (the Venus) let $R \in (0, 1)$ be radius of trajectory and $T > 0$ be its turnaround time. According to the third Keppler's law it must holds that $R^3 = T^2$. Therefore also $T \in (0, 1)$.

The first very important variable is then $T \in (0, 1)$. And we have $R = T^{2/3}$.

The second variable, which do not has so important influence, is the weight of the inner planet m. Variable m influences the location of the center of mass on the segment between the outer and the inner planet.

With these notations we have coordinates of center of mass of the two planets in the time t, where center of the whole system (the Sun) has coordinates $[0, 0]$. [2]

$$S_t = \left[\frac{\cos 2\pi t + mR \cos \frac{2\pi t}{T}}{1+m} ; \frac{\sin 2\pi t + mR \sin \frac{2\pi t}{T}}{1+m} \right]$$

In the Figures 1 and 2 we have $m = 0.815$ which is real value for the Earth and the Venus.

The shape of trajectory was at most influenced by the parameter T in such a way, that if T is a rational number, then the trajectory is periodic and has a symmetry characterized by the difference between denominator and numerator of the T in the irreducible fraction form.

Futhermore if $m < 1/R$ the numerator characterizes the number of lines crossing a ray $y = ax; a, x \in (0, \infty)$. And if m is sufficiently larger than $1/R$ the number of intersections is exactly the denominator of T. (see Fig. 3 and 4)

If T is an irrational number, then the trajectory tile whole annular area between two circles with the origin [0,0].

In the case $m < 1/R$ radiuses of the circles are $\frac{1-mR}{1+m}$ and $\frac{1+mR}{1+m}$.

In the case $m > 1/R$ radiuses of the circles are $\frac{mR-1}{1+m}$ and the same $\frac{1+mR}{1+m}$.

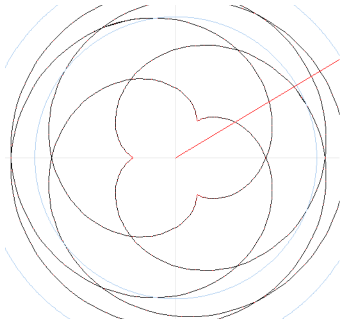

Figure 3: $T = 5/8, m = 0.815$

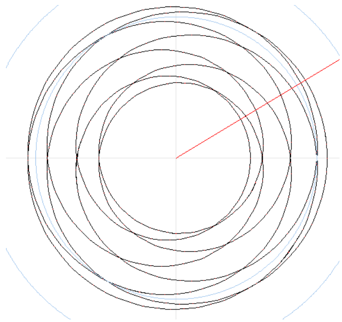

Figure 4: $T = 5/8, m = 4$

3 The sequence

For our study we need some sequence of points, so we take the distances of the center of mass S_t from the center (the Sun) in the specific times:

$$t_n = n; \quad n = 0, 1, 2, \dots$$

We get the sequence:

$$|OS_{t_n}| = \left(\frac{1}{1+m}\right) \cdot \sqrt{1 + m^2R^2 + 2mR\cos\left(2\pi n \frac{1-T}{T}\right)}$$

Evidently, if T is a rational number, then the sequence $|OS_{t_n}|$ is periodic.

Since a distribution function of the sequence will be studied, the sequence should be transformed by the linear function

$$f(x) = \frac{1+m}{2mR} \cdot x - \frac{1}{2mR} + \frac{1}{2} \tag{1}$$

respectivelly for $m > 1/R$:

$$f(x) = \frac{1+m}{2} \cdot x - \frac{mR}{2} + \frac{1}{2} \tag{2}$$

to the sequence (x_n) in a unit interval, where:

$$x_n = \frac{1}{2mR} \cdot \sqrt{1 + m^2R^2 + 2mR\cos\left(2\pi n \frac{1-T}{T}\right)} - \frac{1}{2mR} + \frac{1}{2} \tag{3}$$

respectivelly for $m > 1/R$:

$$x_n = \frac{1}{2} \cdot \sqrt{1 + m^2R^2 + 2mR\cos\left(2\pi n \frac{1-T}{T}\right)} - \frac{mR}{2} + \frac{1}{2} \tag{4}$$

4 Distribution function

Theorem 1. *If T is an irrational number and $m < 1/R$ then the sequence (x_n) (3) has the asymptotic distribution function*

$$g(x) = 1 - \frac{1}{\pi}\arccos(2mRx^2 - 2mRx + 2x - 1)$$

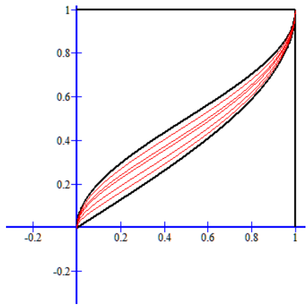

Figure 5: Distribution functions for all mR

Proof. The sequence (x_n) is said to have the asymptotic distribution function $g(x)$ if the relation

$$g(x) = \lim_{N\to\infty} \frac{A([0,x);N;x_n)}{N}$$

holds at every point $x \in [0,1]$ of continuity of $g(x)$.[4]

Solving the inequality $x_n < x$, where x_n is in the form (3) we get

$$\cos\left(2\pi n\frac{1-T}{T}\right) < 2mRx^2 - 2mRx + 2x - 1\,.$$

As $2mRx^2 - 2mRx + 2x - 1$ can be labelled as $u(x)$ we have for n:

$$n \in \left[\frac{T}{1-T}\left(i + \frac{\arccos u(x)}{2\pi}\right); \frac{T}{1-T}\left(i + 1 - \frac{\arccos u(x)}{2\pi}\right)\right], \quad i = 0,1,2,\dots \quad (5)$$

Now each interval and each number n can be multiplicated by $\frac{1-T}{T}$. So distances between nearby intervals will be 1 and from the numbers n we get $\frac{1-T}{T}n$. The question how many n we have in intervals (5) work down to the question: How many of the numbers $\frac{1-T}{T}n \mod 1; n = 0,1,2,\dots$ lie in one of the intervals

$$\left[\left(i + \frac{\arccos u(x)}{2\pi}\right); \left(i + 1 - \frac{\arccos u(x)}{2\pi}\right)\right], \quad i = 0,1,\dots$$

So

$$\lim_{N\to\infty} \frac{A([0,x);N;x_n)}{N} = \lim_{N\to\infty} \frac{A\left(\left[\frac{\arccos u(x)}{2\pi}, 1 - \frac{\arccos u(x)}{2\pi}\right); N; \frac{1-T}{T}n \mod 1\right)}{N}$$

Since (qn) is an u.d. sequence if q is an irrational number [1], [3], it holds:

$$\lim_{N\to\infty} \frac{A([a,b);N;qn \mod 1)}{N} = b - a$$

from which we directly have the desired result. □

Theorem 2. *If T is an irrational number and $m > 1/R$ then the sequence (x_n) (4) has the asymptotic distribution function*

$$g(x) = 1 - \frac{1}{\pi}\arccos\left(\frac{2x^2 - 2x}{mR} + 2x - 1\right)$$

A proof is very similar to the above one and a set of distribution functions of all such sequences is the same as in the previous case.

5 Open problems and discrepancy

Find the sequence (x_n) with the asymptotic distribution function

$$g(x) = 1 - \frac{1}{\pi}\arccos(\alpha x^2 - \alpha x + 2x - 1), \quad \alpha \in (0,1).$$

Since $2mR = \alpha$ and T could be any number in $(0,1)$ there are infinitely many sequences (3) with the given distribution function.

The question is which would be the one with sufficiently small discrepancy? Or how to find number T for which the finite sequence is as near to the distribution function as possible?

The star discrepancy of the g-distributed sequences is given by

$$D_N^* = \sup_{x\in[0,1]} \left| \frac{A([0,x);N;x_n)}{N} - g(x) \right|$$

We have calculated some examples for special values of T. Onehundredpointed sequences and their star discrepancies are exactly calculated. In all cases we take $m = 0.815$. Notice that distribution functions are not the same. But it could be the same, because of the fact, that the parameter m does not influence discrepancy, so it could be changed in such a way that $2mT^{\frac{2}{3}} = \alpha$. So we could slightly change the sequence of points to get desire distribution function with the same discrepancy.

For example if $T = \sqrt{2}/2$ then $D_{100}^* = 0.0584$. For $T = \frac{\sqrt{7}-1}{2}$ the star discrepancy is 0.3395. After a lot of examples, we could assume, that a very good choice for T is the golden ratio. Then $D_{100}^* = 0.0155$. Nevertheless we found also some better values. If $T = \frac{\sqrt{6}-1}{2}$ then we have $D_{100}^* = 0.0118$. The question is how would it be for largerpointed sequences?

And the second very natural question for the further research is, how would it be for three, four, ... planets?

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