Communications in Mathematics 26 (2018) 153–167

DOI: 10.2478/cm-2018-0012

New stability results for spheres and Wulff shapes

Julien Roth

Abstract. We prove that a closed convex hypersurface of the Euclidean space with almost constant anisotropic first and second mean curvatures in the L^p-sense is $W^{2,p}$-close (up to rescaling and translations) to the Wulff shape. We also obtain characterizations of geodesic hyperspheres of space forms improving those of [10] and [11].

1 Introduction

Let $F\colon \mathbb{S}^n \to \mathbb{R}$ be a smooth function satisfying the following convexity assumption

$$A_F = (\nabla dF + F \operatorname{Id}_{|T_x\mathbb{S}^n})_x > 0, \tag{1}$$

for all $x \in \mathbb{S}^n$, where ∇dF is the Hessian of F. Here, > 0 means positive definite in the sense of quadratic forms. Now, we consider the following map

$$\begin{aligned} \phi\colon \mathbb{S}^n &\to \mathbb{R}^{n+1} \\ x &\mapsto F(x)x + (\operatorname{grad}_{|\mathbb{S}^n} F)_x \end{aligned}$$

The image $\mathcal{W}_F = \phi(\mathbb{S}^n)$ is called the Wulff shape of F and is a smooth convex hypersurface of $\mathbb{R}^{n+1}$ due to condition (1). Note that if $F = 1$, then the Wulff shape is the sphere $\mathbb{S}^n$.

Now, let (M^n, g) be an n-dimensional closed, connected and oriented Riemannian manifold, isometrically immersed into by X into $\mathbb{R}^{n+1}$. We denote by ν a normal unit vector field globally defined on M, that is, we have $\nu\colon M \to \mathbb{S}^n$. We set $S_F = A_F \circ \mathrm{d}\nu$, where A_F is defined in (1). The operator S_F is called

2010 MSC: 53C42, 53A10
Key words: Hypersurfaces, Anisotropic mean curvatures, Wulff Shape, Almost umibilcal
Affiliation:
Julien Roth – Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris-Est, 5, boulevard Descartes, Cité Descartes, 77454 Marne-la-Vallée cedex 2, France
E-mail: julien.roth@u-pem.fr

the F-Weingarten operator or anisotropic shape operator, and we can define in this anisotrpic setting all the corresponding extrinsic quantities like anisotropic principal curvatures and anisotropic mean curvature and higher order mean curvatures (see the preliminaries section for the precise definitions).

In the isotropic context, geodesic hyperspheres in Euclidean spaces can be characterized among closed hypersurfaces by various properties. In particular, it is well known that geodesic hyperspheres are the only totally umbilical closed connected hypersurfaces in Euclidean spaces. The question of the stability of this characterization has been intensively studied in the last years by many authors (see [1], [3], [8], [10], [11], [12], [13] and references therein). In the anisotropic setting, the so-called Wulf shape plays the role of geodesic spheres and can be characterized by similar results (see [4], [7] for instance). Analogously to spheres, for a given F, the Wulff shape $\mathcal{W}_F$ is, up to homotheties and translations, the only closed convex hypersurface with vanishing traceless anisotropic second fundamental form. Very recently, De Rosa and Gioffrè [2] studied the stability of this characterization. Namely, they proved that if the traceless part of the anisotropic second fundamental form is sufficiently small, then the hypersurface is close to the Wulff shape. The aim of the present note is first to obtain a new stability result concerning the Wulff shape. Namely, we prove the following stability result.

Theorem 1. *Let $n \geqslant 2$ an integer, $F\colon \mathbb{S}^n \to \mathbb{R}$ be a smooth function satisfying the convexity assumption* (1)*, $h > 0$ and $p > n$. Let M be a closed, connected and oriented hypersurface of $\mathbb{R}^{n+1}$ bounding a convex domain. Assume that*

$$\mathrm{Vol}(M) = \mathrm{Vol}(\mathcal{W}_F).$$

Then there exists $\varepsilon_0 > 0$ depending only on n, p, h and F such that if for $\varepsilon \leqslant \varepsilon_0$, we have

- $\|H^F - h\|_p < \varepsilon h$ *and*
- $\|H_2^F - h_2\|_p < \varepsilon h^2$ *for a constant h_2,*

then M is close to the Wulff shape in the following sense: there exists a smooth parametrisation $\psi\colon \mathcal{W}_F \to M$, a vector $c_0 \in \mathbb{R}^{n+1}$ and a constant K depending on n, p, h and F so that

$$\|\psi - \mathrm{Id} - c_0\|_{W^{2,p}(\mathcal{W})} \leqslant K\varepsilon^{\frac{p}{2}}\,.$$

Remark 1.

- Here $\mathrm{Vol}(M)$ is the volume of M for the induced metric g.
- We recall that the extrinsic radius of M is the radius of the smallest closed ball in $\mathbb{R}^{n+1}$ containing M.
- Note that the right-hand sides in both pinching conditions of the theorem are respectively $h\varepsilon$ and $h^2\varepsilon$ for some homogeneity reasons, since for the Wulff shape, we have $H_2^F = (H^F)^2$.
- As we will see in the proof of Lemma 1, the constant h_2 will be necessarily close to h^2.

This result is a generalization in the anisotropic context of the main result of [10], but not only since the hypotheses are that both first and second anisotropic mean curvatures are close to constants for the L^p-norm. We can also improve the results of [10] for space forms in the same way and obtain new characterizations of geodesic hyperspheres under weaker assumptions. For this, we denote by $\mathbb{M}_\delta^{n+1}$ the simply connected real space form of constant curvature δ. We prove the following result.

Theorem 2. *Let $n \geqslant 2$ an integer, $h > 0$ and $p > n$. Assume that (M^n, g) is a closed, connected and oriented hypersurface of $\mathbb{M}_\delta^{n+1}$ so that $\mathrm{Vol}(M) = \mathrm{Vol}(\mathbb{S}^n)$. If $\delta > 0$, assume moreover that M is contained in an open ball of radius $\frac{\pi}{4\sqrt{\delta}}$. Then, there exist $\varepsilon_0(n,p,h) > 0$, $K(n,p)$ and $\beta(n,p) \leqslant 1$ such that if for $\varepsilon \leqslant \varepsilon_0$, we have*

- *$\|H - h\|_p < \varepsilon h$ and*
- *$\|H_2 - h_2\|_p < \varepsilon h^2$ for a constant h_2,*

then M is diffeomorphic and $K\varepsilon^\beta$-close to a geodesic hypersphere of radius $\frac{1}{\|H\|_2}$ in the following sense: there exists a diffeomorphism F from M to $\mathbb{S}^n\left(\frac{1}{\|H\|_2}\right)$ so that

$$\left||d_xF(u)|^2 - 1\right| \leqslant K\varepsilon^\beta$$

for any $x \in M$ and any unit vector $u \in T_xM$.

Remark 2.
- For more convenience, we write the above theorem with H_2, but due to the twice traced Gauss formula, we have $n(n-1)H_2 + \delta = \mathrm{Scal}$, we can reformulate equivalently the theorem with almost constant scalar curvature.
- This result is an improvement of a previous result of [10] since we assume L^p-norms instead of pointwise almost proximity to constant. Moreover, in the case where $\delta > 0$, we assume that M is contained in an open geodesic ball of radius $\frac{\pi}{4\sqrt{\delta}}$. We can remove the assumption with as counterpart, the fact that C and ε_0 depend also on the extrinsic radius of M. The same remark holds the following two corollaries.

From the following theorem, we can obtain new characterizations of geodesic hyperspheres.

Corollary 1. *Let (M^n, g) be a closed, connected and oriented Riemannian manifold, isometrically immersed into $\mathbb{M}_\delta^{n+1}$ and $p > n$. If $\delta > 0$, we assume that M is contained in an open ball of radius $\frac{\pi}{4\sqrt{\delta}}$. Let $h > 0$. Then there exists $\varepsilon(n,h,\delta) > 0$ such that if M has constant mean curvature $H = h$, and $\|\mathrm{Scal} - s\|_p < \varepsilon$ for a constant s, then M is a geodesic sphere of radius $t_\delta^{-1}\left(\frac{1}{h}\right)$.*

Corollary 2. *Let (M^n, g) be a closed and oriented Riemannian manifold, isometrically immersed into $\mathbb{M}_\delta^{n+1}$ and $p > n$. If $\delta > 0$, we assume that M is contained in an open ball of radius $\frac{\pi}{4\sqrt{\delta}}$. Let $s > 0$ Then, there exists $\varepsilon(n,\delta) > 0$ such that if M has constant scalar curvature $\mathrm{Scal} = s$, and $\|H - h\|_p < \varepsilon$ for a constant h, then M is a geodesic sphere of radius $s_\delta^{-1}\left(\sqrt{\frac{s}{n(n-1)}}\right)$.*

Remark 3. • Note that the hypersurfaces are not supposed to be embbeded, but only immersed.

- In Corollary 1, as a consequence, the scalar curature is constant as so the constant s is close to this constant scalar curvature. The same remark holds for h and the mean curvature in Corollary 2.

When $p \in (1, n]$, one can not obtain similar result, since we use a pichning result for almost umbilical hypersurfaces for the L^p-norm with $p > n$. Nevertheless, we can obtain for the Euclidean space a stability result comparable to Theorem 1, with the assumption that the hypersurface is convex using a result by Gioffrè [3]. Namely, for $p > 1$, we have the following.

Theorem 3. *Let $n \geqslant 2$ an integer, $h > 0$, $p > 1$ and $R > 0$. Let M be a closed and oriented hypersurface of $\mathbb{R}^{n+1}$ bounding a convex domain. Assume that*

$$\mathrm{Vol}(M) = \mathrm{Vol}(\mathbb{S}^n)$$

and that the extrinsic radius of M is smaller than R. Then, there exists

$$\varepsilon_0(n, p, h, R) > 0$$

such that if for $\varepsilon \leqslant \varepsilon_0$, we have

- $\|H - h\|_p < \varepsilon h$ *and*
- $\|H_2 - h_2\|_p < \varepsilon h^2$ *for a constant h_2,*

then M is close to the unit sphere in the following sense: there exists a smooth parametrisation $\psi \colon \mathbb{S}^n \to M$, a vector $c_0 \in \mathbb{R}^{n+1}$ and a constant K depending on n, p, h and R so that

$$\|\psi - \mathrm{Id} - c_0\|_{W^{2,p}(\mathcal{W})} \leqslant K \varepsilon^{\frac{p}{2}}.$$

Moreover, if $p \geqslant n - 1$, then ε_0 does not depend on R.

Remark 4. Note that there is no interest here to obtain corollaries comparable to Corollaries 1 and 2. Indeed, if the hypersurface (which is supposed to bound a domain) has constant mean curvature, the Alexandrov theorem gives that M is a sphere without need of the almost constancy of the scalar curvature.

Remark 5. In all the statements, we assume a normalization of the volume for a sake of simplicity, but by scaling, we can obtain statements with constants depending also on the volume.

2 Preliminaries

Let (M^n, g) be an n-dimensional closed, connected and oriented Riemannian manifold isometrically immersed into the $(n + 1)$-dimensional simply connected real space form $\mathbb{M}_\delta^{n+1}$ of constant curvature δ. The (real-valued) second fundamental form II of the immersion is the bilinear symmetric form on $\Gamma(TM)$ defined for two vector fields X, Y by

$$II(X, Y) = -g\left(\overline{\nabla}_X \nu, Y\right),$$

where $\overline{\nabla}$ is the Riemannian connection on $\mathbb{M}_\delta^{n+1}$ and ν a normal unit vector field on M. When M is embedded, we choose ν as the inner normal field.

From II, we can define the mean curvature,

$$H = \frac{1}{n}\operatorname{tr}(II).$$

Now, we recall the Gauss formula. For $X, Y, Z, W \in \Gamma(TM)$,

$$R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + \langle SX, Z\rangle \langle SY, W\rangle - \langle SY, Z\rangle \langle SX, W\rangle \quad (2)$$

where R and $\overline{R}$ are respectively the curvature tensor of M and $\mathbb{M}_\delta^{n+1}$, and S is the Weingarten operator defined by $SX = -\overline{\nabla}_X \nu$.

By taking the trace and for $W = Y$, we get

$$\operatorname{Ric}(Y) = \overline{\operatorname{Ric}}(Y) - \overline{R}(\nu, Y, \nu, Y) + nH\langle SY, Y\rangle - \langle S^2Y, Y\rangle \quad (3)$$

Since, the ambient space is of constant sectional curvature δ, by taking the trace a second time, we have

$$\operatorname{Scal} = n(n-1)\delta + n^2H^2 - \|S\|^2, \quad (4)$$

or equivalently

$$\operatorname{Scal} = n(n-1)\left(H^2 + \delta\right) - \|\tau\|^2, \quad (5)$$

where $\tau = S - H\,\mathrm{Id}$ is the umbilicity tensor.

Now, we define the higher order mean curvatures, for $k \in \{1, \dots, n\}$, by

$$H_k = \frac{1}{\binom{n}{k}}\sigma_k(\kappa_1, \dots, \kappa_n),$$

where σ_k is the k-th elementary symmetric polynomial and $\kappa_1, \dots, \kappa_n$ are the principal curvatures of the immersion.

From the definition, it is obvious that H_1 is the mean curvature H. We also remark from the Gauss formula (2) that

$$H_2 = \frac{1}{n(n-1)}\operatorname{Scal} - \delta. \quad (6)$$

On the other hand, we have the well-known Hsiung-Minkowski formula

$$\int_M \Big(H_{k+1}\langle Z, \nu\rangle + c_\delta(r)H_k\Big) = 0, \quad (7)$$

where $r(x) = d(p_0, x)$ is the distance function to a base point p_0, Z is the position vector defined by $Z = s_\delta(r)\overline{\nabla} r$, and the functions c_δ and s_δ are defined by

$$c_\delta(t) = \begin{cases} \cos(\sqrt{\delta}t) & \text{if } \delta > 0, \\ 1 & \text{if } \delta = 0, \\ \cosh(\sqrt{-\delta}t) & \text{if } \delta < 0 \end{cases} \quad \text{and} \quad s_\delta(t) = \begin{cases} \frac{1}{\sqrt{\delta}}\sin(\sqrt{\delta}t) & \text{if } \delta > 0, \\ t & \text{if } \delta = 0, \\ \frac{1}{\sqrt{-\delta}}\sinh(\sqrt{-\delta}t) & \text{if } \delta < 0. \end{cases}$$

Finally, we define the function $t_\delta = \frac{s_\delta}{c_\delta}$.

On the other hand, let $F\colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ be a smooth function satisfying the following convexity assumption (1):

$$A_F = (\nabla dF + F\,\mathrm{Id}_{|T_x\mathbb{S}^n})_x > 0,$$

for all $x \in \mathbb{S}^n$, where ∇dF is the Hessian of F. The Wulff shape is defined by $\mathcal{W}_F = \phi(\mathbb{S}^n)$ with

$$\begin{aligned} \phi\colon \mathbb{S}^n &\to \mathbb{R}^{n+1} \\ x &\mapsto F(x)x + (\mathrm{grad}_{|\mathbb{S}^n} F)_x \end{aligned}$$

Now, let (M^n, g) be an n-dimensional compact, connected, oriented manifold without boundary, isometrically immersed into by X into $\mathbb{R}^{n+1}$. We denote by ν a normal unit vector field globally defined on M and the F-Weingarten operator $S_F = A_F \circ S$, where A_F is defined in (1). The eigenvalues of A_F are the anisotropic principal curvatures that we will denote $\kappa_1^F, \kappa_2^F, \dots, \kappa_n^F$. Finally, for $r \in \{1, \dots, n\}$, the r-th anisotropic mean curvature is defined by

$$H_r^F = \frac{1}{\binom{n}{r}} \sum_{i_1 < \cdots < i_r} \kappa_{i_1}^F \dots \kappa_{i_r}^F.$$

We also set $H_0^F = 1$ for convenience. Note that the Wulff shape is F-umbilical, that is $S_F = H^F\,\mathrm{Id}$ and all its anisotropic principal curvatures are equal to 1 and therefore, for any $r \in \{1, \dots, n\}$, we have $H_r^F = 1$.

We finally recall the following integral forumlas proved by He and Li in [4] and which generalize the classical Hsiung-Minkowski formulas (7) in the anisotropic setting.

$$\int_M \big(F(\nu)H_{r-1}^F + H_r^F \langle X, \nu\rangle\big)\, dv_g = 0. \tag{8}$$

We finish this section of preliminaries by the following results which give an upper bound of the diameter of a hypersurface in $\mathbb{M}_\delta^{n+1}$ in terms of its mean curvature and their consequence on the extrinsic radius. We have the following.

Theorem 4. *(Topping [14], Wu-Zheng [15]) Let $n \geqslant 1$ and (M^n, g) be a closed connected Riemannian manifold isometrically immersed into a complete Riemannian manifold (N^{n+p}, h) of curvature K_N satisfying $K_N \leqslant b^2$ with b a real or purely imaginary number. For any $0 < \alpha < 1$, if*

$$b^2(1-\alpha)^{-2/n}\big(\omega_n^{-1}\,\mathrm{Vol}(M)\big)^{2/n} \leqslant 1, \tag{9}$$

$$2\rho_0 \leqslant \mathrm{inj}_M(N), \tag{10}$$

where $\mathrm{inj}_M(N)$ is the injectivity radius of N restricted to M, $\omega_n = \mathrm{Vol}(\mathbb{S}^n)$ and ρ_0 is given by

$$\rho_0 = \begin{cases} b^{-1}\sin^{-1}\big(b(1-\alpha)^{-1/n}(\omega_n^{-1}\,\mathrm{Vol}(M))^{1/n}\big) & \text{if } b \text{ is real,} \\ (1-\alpha)^{-1/n}(\omega_n^{-1}\,\mathrm{Vol}(M))^{1/n} & \text{if } b \text{ is imaginary.} \end{cases}$$

then, we have the following

$$\operatorname{diam}(M) \leqslant C(n,\alpha) \int_M |H|^{n-1} dv_g,$$

where $\operatorname{diam}(M)$ *is the intrinsic diameter of* M, H *its mean curvature (for the immersion into* N*) and* $C(n,\alpha)$ *a constant depending only on* n *and* α*.*

This result have been first proved by Topping if N is the Euclidean space using the extrinsic Sobolev inequality of Michael and Simon [6]. Then, it has been generalized by Wu and Zheng for arbitrary manifold with bounded curvature by using the general extrinsic Sobolev inequality of Hoffmann and Spruck [5]. This is the reason why assumptions (9) and (10) are neeeded. Note also that if N is the Euclidean or hyperbolic space, then both conditions (9) and (10) are trivially satisfied.

Finally, we recall that the extrinsic radius $R(M)$ of M is defined by

$$R(M) = \inf\{\rho > 0 \mid \exists x \in \mathbb{M}^{n+1}(\delta) \text{ s.t. } \phi(M) \subset B(x,r)\},$$

where ϕ is the immersion of M into $\mathbb{M}^{n+1}(\delta)$. By a slight abuse of notation, we denote it $R(M)$ but, this radius depends not only on M but also on the immersion ϕ. Since in this paper, the considered immersion will be fixed, this notation does not lead to any ambiguity.

The extrinsic radius is bounded from below by the mean curvature due to the following estimate (see [9])

$$t_\delta(R(M)) \geqslant \frac{1}{\|H\|_\infty},$$

with equality if and only if M is a geodesic sphere. On the other hand, even if this is not optimal at all, we remark obviously that $R(M) \leqslant \operatorname{diam}(M)$ and using Theorem 4, this implies that $R(M)$ is also bounded form above by in term of the mean curvature without any condition if $\delta \leqslant 0$. Now, we have the ingredients to prove the results.

3 Key lemmas

First, using the integral formula (8), we are able to prove the following technical lemma.

Lemma 1. *Let* (M^n, g) *be a closed Riemannian manifold, isometrically immersed into* $\mathbb{R}^{n+1}$ *and assume that the extrinsic radius of* M *is smaller than* R*. Let* $p > 1$, h *and* h_2 *be two positive constants and* $\varepsilon \in \left(0, \frac{1}{2}\right)$*. If the first and second anisotropic mean curvatures satisfy*

- $\|H^F - h\|_p < \varepsilon h$ *and*
- $\|H_2^F - h_2\|_p < \varepsilon h^2$,

for some positive ε, then

$$|h^2 - h_2| \leqslant A\varepsilon,$$

where A is an explicit positive constant depending on n, h, R and F.

Moreover, if $p \geqslant n-1$ and M is convex, then, the dependence on R can be replaced by a dependence on $\mathrm{Vol}(M)$.

Proof. The proof of this lemma is based on the Hisung-Minkowski formulas (8) for $r = 1$ and $k = 2$. Indeed, the Hisung-Minkowski formula for $r = 2$ is the following

$$\int_M \big(H_2^F \langle X, \nu\rangle + F(\nu)H^F\big)dv_g = 0. \tag{11}$$

Then, we get

$$\begin{aligned}
0 &= \int_M \big(H_2^F \langle X, \nu\rangle + F(\nu)H^F\big)dv_g \\
&= \int_M \big(h_2 \langle X, \nu\rangle + F(\nu)H^F\big)dv_g + \int_M (H_2^F - h_2)\langle X, \nu\rangle\, dv_g \\
&= \frac{h_2}{h}\int_M h\langle X, \nu\rangle + \int_M F(\nu)H^F dv_g + \int_M (H_2^F - h_2)\langle X, \nu\rangle\, dv_g \\
&= \frac{h_2}{h}\int_M H^F\langle X, \nu\rangle\, dv_g + \frac{h_2}{h}\int_M (h - H^F)\langle X, \nu\rangle\, dv_g + \int_M F(\nu)h dv_g \\
&\quad + \int_M F(\nu)(H^F - h)dv_g + \int_M (H_2^F - h_2)\langle X, \nu\rangle\, dv_g
\end{aligned}$$

Now, we use the Hsiung-Minkowski formula for $r = 1$, that is

$$\int_M \big(H^F \langle X, \nu\rangle + F(\nu)\big)dv_g = 0, \tag{12}$$

to get

$$\begin{aligned}
0 &= -\frac{h_2}{h}\int_M F(\nu)dv_g + \frac{h_2}{h}\int_M (h - H^F)\langle X, \nu\rangle\, dv_g + \int_M F(\nu)h dv_g \\
&\quad + \int_M F(\nu)(H^F - h)dv_g + \int_M (H_2^F - h_2)\langle X, \nu\rangle\, dv_g \\
&= \Big(h - \frac{h_2}{h}\Big)\int_M F(\nu)dv_g + \frac{h_2}{h}\int_M (h - H^F)\langle X, \nu\rangle\, dv_g \\
&\quad + \int_M F(\nu)(H^F - h)dv_g + \int_M (H_2^F - h_2)\langle X, \nu\rangle\, dv_g
\end{aligned}$$

Then, since $|\langle X, \nu\rangle| \leqslant R$, using the Hölder's inequality and both conditions

$$\|H^F - h\|_p < \varepsilon h \quad \text{and} \quad \|H_2^F - h_2\|_p < \varepsilon h^2\,,$$

we get

$$\Big|h - \frac{h_2}{h}\Big|\int_M F(\nu)dv_g \leqslant h_2\varepsilon R\,\mathrm{Vol}(M) + \varepsilon h \sup(F)\,\mathrm{Vol}(M) + \varepsilon h^2 R\,\mathrm{Vol}(M).$$

Using the fact that $|H_2^F| \leqslant \big(H^F\big)^2$, we deduce

$$|h_2| \leqslant h^2 + (H^F - h)^2 + 2h(H^F - h) + (h_2 - H_2^F)$$

and so with the assumptions $\|H^F - h\|_p < \varepsilon h$ and $\|H_2^F - h_2\|_p < \varepsilon h^2$, we get

$$|h_2| \leqslant 5h^2.$$

Thus, we have

$$\begin{aligned} |h^2 - h_2| \int_M F(\nu) dv_g &\leqslant \varepsilon h^2 \sup(F) \operatorname{Vol}(M) + (h^3 + hh_2) R \operatorname{Vol}(M) \varepsilon \\ &\leqslant \varepsilon h^2 \sup(F) \operatorname{Vol}(M) + 6h^3 R \operatorname{Vol}(M) \varepsilon \end{aligned}$$

and we obtain

$$\begin{aligned} |h^2 - h_2| &\leqslant \left(h^2 \frac{\sup(F)}{\inf(F)} + \frac{6h^3 R}{\inf(F)} \right) \varepsilon \\ &\leqslant h^2 A(h, R, F) \varepsilon, \end{aligned} \tag{13}$$

which gives the wanted assertion.

Now assume that $p \geqslant n-1$ and M is convex. We will show that R can be controlled from above by h. First, as we have already mentionned, $R \leqslant \operatorname{diam}(M)$ and so, by Theorem 4, we have

$$\begin{aligned} R &\leqslant C(n) \int_M |H|^{n-1} dv_g \\ &\leqslant C(n) \operatorname{Vol}(M) \|H\|_p^{n-1}. \end{aligned} \tag{14}$$

Now, let $\{e_1, \dots, e_n\}$ be an orthonormal basis diagonalizing S_F. Then, we have

$$\begin{aligned} H &= \sum_{i=1}^n \langle S e_i, e_i \rangle \\ &= \sum_{i=1}^n \langle A_F^{-1} \circ S_F e_i, e_i \rangle \\ &= \sum_{i=1}^n \kappa_i^F \langle A_F^{-1} e_i, e_i \rangle \\ &\leqslant \|A_F^{-1}\| \sum_{i=1}^n \kappa_i^F = \|A_F^{-1}\| H^F, \end{aligned} \tag{15}$$

since all κ_i^F are nonnegative by convexity of M. Moreover, from the assumption $\|H^F - h\|_p < \varepsilon h$ with $\varepsilon < \frac{1}{2}$, we get that $\|H^F\|_p \leqslant (1+\varepsilon) h \leqslant 2h$. Combining this with (14) and (15), we obtain

$$R \leqslant C(n) \operatorname{Vol}(M) \big(2h \|A_F^{-1}\|\big)^{n-1}.$$

Finally, reporting this upper bound of R into (13), we get that A can be choosen to be independent on R if $p \geqslant n-1$ and M is convex. This conludes the proof of the Lemma. □

Now, we give this second lemma also valid for hypersurfaces of spheres and hyperbolic spaces.

Lemma 2. *Let (M^n, g) be a closed Riemannian manifold, isometrically immersed into $\mathbb{M}_\delta^{n+1}$ and assume that the extrinsic radius of M is smaller than R. Let $p > 1$, h and h_2 be two positive constants and $\varepsilon \in (0,1)$. If the first and second mean curvatures satisfy*

- $\|H - h\|_p < \varepsilon h$ *and*
- $\|H_2 - h_2\|_p < \varepsilon h^2$,

for some positive ε, then

$$|h^2 - h_2| \leqslant Bh^2\varepsilon,$$

where B is an explicit positive constant depending on n, δ, h and R.

Moreover, if $\delta \leqslant 0$ and $p \geqslant n-1$, the dependence on R can be replaced by a dependence on $\mathrm{Vol}(M)$. If $\delta > 0$ and M is contained in a geodesic ball of radius $\frac{\pi}{4\sqrt{\delta}}$ then B does not depend on R.

Proof. The proof is close to the proof of Lemma 1 with some slight differences. Proceeding as in the proof of Lemma 1 with the Hsiung-Minkowski (7) instead of the anisotropic one (8), we get

$$0 = \left(h - \frac{h_2}{h}\right)\int_M c_\delta(r)dv_g + \frac{h_2}{h}\int_M (h-H)\langle Z,\nu\rangle\, dv_g + \int_M c_\delta(r)(H-h)dv_g + \int_M (H_2 - h_2)\langle Z,\nu\rangle\, dv_g\,.$$

Then, since $|\langle Z,\nu\rangle| \leqslant s_\delta(R)$, using the Hölder inequality and both conditions $\|H-h\|_p < \varepsilon h$ and $\|H_2 - h_2\|_p < \varepsilon h^2$, we get

$$\left|h - \frac{h_2}{h}\right| \inf(c_\delta(r))\,\mathrm{Vol}(M) \leqslant h_2\varepsilon s_\delta(R)\,\mathrm{Vol}(M) + \varepsilon h \sup(c_\delta(r))\,\mathrm{Vol}(M) + \varepsilon h^2 s_\delta(R)\,\mathrm{Vol}(M).$$

Using the fact that $|H_2| \leqslant (H)^2$, we deduce

$$|h_2| \leqslant h^2 + (H-h)^2 + 2h(H-h) + (h_2 - H_2)$$

and so with the assumptions $\|H-h\|_p < \varepsilon h$ and $\|H_2 - h_2\|_p < \varepsilon h^2$, we get

$$|h_2| \leqslant 5h^2.$$

Thus, we have

$$\begin{aligned}|h^2 - h_2|\inf(c_\delta(r))\,\mathrm{Vol}(M) &\leqslant \varepsilon h^2 \sup(c_\delta(r))\,\mathrm{Vol}(M) + (h^3 + hh_2)s_\delta(R)\,\mathrm{Vol}(M)\varepsilon \\ &\leqslant \varepsilon h^2 \sup(c_\delta(r))\,\mathrm{Vol}(M) + 6h^3 s_\delta(R)\,\mathrm{Vol}(M)\varepsilon\end{aligned}$$

and we obtain

$$|h^2 - h_2| \leqslant \left(h^2 \frac{\sup(c_\delta(r))}{\inf(c_\delta(r))} + \frac{6h^3 s_\delta(R)}{\inf(c_\delta(r))} \right) \varepsilon.$$

If $\delta > 0$, then $c_\delta(t) = \cos(\sqrt{\delta}t)$, so we deduce immediately that $c_\delta(R) \leqslant c_\delta(r) \leqslant 1$ and then

$$h^2 \frac{\sup(c_\delta(r))}{\inf(c_\delta(r))} + \frac{6h^3 s_\delta(R)}{\inf(c_\delta(r))} \leqslant \frac{h^2}{c_\delta(R)} + \frac{6h^3 s_\delta(R)}{c_\delta(R)}.$$

If $\delta = 0$, then $c_\delta = 1$ and so

$$\left(h^2 \frac{\sup(c_\delta(r))}{\inf(c_\delta(r))} + \frac{6h^3 s_\delta(R)}{\inf(c_\delta(r))} \right) = h^2 + 6h^3 R.$$

If $\delta < 0$, then $c_\delta(t) = \cosh(\sqrt{-\delta}t)$ and thus $c_\delta(R) \geqslant c_\delta(r) \geqslant 1$ and then

$$h^2 \frac{\sup(c_\delta(r))}{\inf(c_\delta(r))} + \frac{6h^3 s_\delta(R)}{\inf(c_\delta(r))} \leqslant h^2 c_\delta(R) + 6h^3 s_\delta(R).$$

Then, in the three cases, we have $|h^2 - h_2| \leqslant Bh^2\varepsilon$, with B a positive constant depending only on δ, h, F and R.

As in the proof of Lemma 1, if $p \geqslant n-1$, from Theorem 4, we can bound from above R by $\|H\|_{n-1}$ and so therefore by h due to the pinching condition $|H - h| \leqslant \varepsilon h$. Hence if $\delta \leqslant 0$, form its expression obtained in (3), the constant B can be choosen independent on R. Note that this can also been done if $\delta > 0$ by with the two additional conditions (on $\mathrm{Vol}(M)$) needed to apply Theorem 4.

But, if we assume that M is contained in a geodesic ball of radius $\frac{\pi}{4\sqrt{\delta}}$, then, we get that

$$\frac{h^2}{c_\delta(R)} + \frac{6h^3 s_\delta(R)}{c_\delta(R)} \leqslant \sqrt{2}h^2 + \frac{6h^3}{\sqrt{\delta}},$$

and A does not depend on R. This concludes the proof of the lemma. □

4 Proofs of the Theorems

Now, using this lemma together with appropriate result for almost umbilical hypersurfaces, we can prove the different theorems of this note.

Proof of Theorem 1. We begin with the proof of Theorem 1. For this, we first recall the main result of [2]. We will use this result together with Lemma 1 to conclude.

Theorem 5. *(De Rosa-Gioffrè [2]) Let $n > 2$, $p \in (1, p)$ and $F\colon \mathbb{S}^n \to \mathbb{R}^+$ satisfying the convexity assumption* (1). *There exists a constant $\delta_0 = \delta_0(n, p, F) > 0$ such that if Σ is close convex hypersurface into $\mathbb{R}^{n+1}$ satisfying*

$$\mathrm{Vol}(M) = V(\mathcal{W}_F) \quad \textit{and} \quad \int_M \|S_F - H^F \mathrm{Id}\|^p dv_g \leqslant \delta$$

with $\delta \leqslant \delta_0$ then there exists a smooth parametrisation $\psi\colon \mathcal{W}_F \to M$, a vector $c_0 \in \mathbb{R}^{n+1}$ and a constant C depending on n, p and F so that

$$\|\psi - \mathrm{Id} - c_0\|_{W^{2,p}(\mathcal{W})} \leqslant C\delta.$$

Now, if $\|H^F - h\| < \varepsilon h$ and $\|H_2^F - h_2\| < \varepsilon h^2$, then from Lemma 1

$$|h^2 - h_2| \leqslant Ah^2\varepsilon,$$

with A a positive constant depending on n, h and F. It is important to note that due to Lemma 1, A depends on n, h, F and $\mathrm{Vol}(M)$, but since we assume that $\mathrm{Vol}(M) = \mathrm{Vol}(\mathcal{W}_F)$, thus A depends in fact only on n, h and F. Thus, we deduce that

$$(H^F)^2 - H_2^F \leqslant (H^F - h)^2 + 2h(H^F - h) + |h^2 - h_2| + |h_2 - H_2^F|$$

and so

$$\|(H^F)^2 - H_2^F\|_p \leqslant (4h^2 + Ah^2)\varepsilon = A'\varepsilon$$

where A' is a positive constant depending only on n, h and F. On the other hand, we have

$$(H^F)^2 - H_2^F = \frac{1}{n^2(n-1)} \sum_{i,j=1}^{n} (\kappa_i - \kappa_j)^2,$$

so we get

$$\left\| \sum_{i,j=1}^{n} (\kappa_i - \kappa_j)^2 \right\|_p \leqslant A''\varepsilon.$$

where $A'' = n^2(n-1)A'$ is also a positive constant depending only on h, n and F. Hence, M has almost vanishing anisotropic second fundamental form. Indeed, we have at a point $x \in M$,

$$\begin{aligned}
\|S_F - H^F \mathrm{Id}\|^2 &= \sum_{i=1}^{n} (k_i - H^F)^2 \\
&= \sum_{i=1}^{n} \left(\kappa_i - \frac{1}{n} \sum_{j=1}^{n} \kappa_j \right)^2 \\
&= \frac{1}{n} \sum_{i,j=1}^{n} (\kappa_i - \kappa_i)^2
\end{aligned}$$

which give after integration

$$\|S_F - H^F \mathrm{Id}\|_p^2 \leqslant \frac{1}{n} A'' h^2 \varepsilon.$$

Finally, we fix $p > n$ and set $\varepsilon_0 = \inf\left\{ 1, \dfrac{n(\delta_0 \mathrm{Vol}(\mathcal{W}_F))^{\frac{2}{p}}}{A''} \right\}$ where A'' is the constant defined above and δ_0 comes from Theorem 2. Note that ε_0 depends on n, p, h and F. Now, let $\varepsilon \leqslant \varepsilon_0$. We set $\delta = \dfrac{(A''\varepsilon)^{\frac{p}{2}}}{n^{\frac{p}{2}} V(\mathcal{W}_F)}$. Since $\varepsilon \leqslant \varepsilon_0$ and from the definiton of δ, we have $\delta \leqslant \delta_0$ and

$$\int_M \|S_F - H^F \mathrm{Id}\|^p dv_g \leqslant \delta.$$

Thus, since by assumption, we also have $\mathrm{Vol}(M) = \mathrm{Vol}(\mathcal{W}_F)$, we can apply Theorem 2 to obtain that there exists a smooth parametrisation $\psi\colon \mathcal{W}_F \to M$ a vector $c_0 \in \mathbb{R}^{n+1}$ and a constant C depending on n, p, h and F so that

$$\|\psi - \mathrm{Id} - c_0\|_{W^{2,p}(\mathcal{W})} \leqslant C\delta = K\varepsilon^{\frac{p}{2}},$$

where $K = \dfrac{(nA'')^{\frac{p}{2}}C}{\mathrm{Vol}(\mathcal{W}_F)}$ is a positive constant depending only on n, p, h and F since A'' depends on n, h and F, $\mathrm{Vol}(\mathcal{W}_F)$ depends on n and F and C depends on n, p and F. This concludes the proof of Theorem 1. □

Proof of Theorem 2. The proof of Theorem 2 is a combination of Lemma 2 and the main Theorem of [13]. We recall this result

Theorem 6. *(Roth-Scheuer [13]) Let $M \to \mathbb{R}^{n+1}$ be a closed, connected, oriented and isometrically immersed hypersurface with* $\mathrm{Vol}(M) = 1$. *Let $p > n \geq 2$. Then, there exist $\eta_0(n, p, \|A\|_p) > 0$, $C(n, p, \|A\|_p) > 0$ and $\alpha(n, p) \leqslant 1$ such that if for $\eta \leqslant \eta_0$,*

$$\|A - H\,\mathrm{Id}\,\|_p \leq \|H\|_p\eta$$

holds, then M is diffeomorphic and $C\eta^{\alpha}$-close to a geodesic hypersphere (of radius $\frac{1}{\|H\|_2}$).

First, if $\|H - h\| < \varepsilon h$ and $\|H_2 - h_2\| < \varepsilon h^2$, then from Lemma 2

$$\left|h^2 - h_2\right| \leqslant B\varepsilon,$$

with B a positive constant depending on n,h and δ. As in the proof of Theorem 1, if $\delta \leqslant 0$, then B does not depend on R but on $\mathrm{Vol}(M)$ which is assumed to be equal to $\mathrm{Vol}(\mathbb{S}^n)$, so B depending on n,h and δ and no more. So, we deduce that

$$H^2 - H_2 \leqslant (H - h)^2 + 2h(H - h) + |h^2 - h_2| + |h_2 - H_2|$$

and so after integration, we get immediately

$$\|H^2 - H_2\|_p \leqslant (4h + B)\varepsilon = B'\varepsilon$$

with B' a positive constant depending only on n, h and δ. But, since

$$\|H^2 - H_2\|_p = n(n-1)\|A - H\,\mathrm{Id}\,\|_p^2,$$

we deduce that

$$\|A - H\,\mathrm{Id}\,\|_p \leqslant \left(\frac{B'\varepsilon}{n(n-1)}\right)^{\frac{1}{2}} = B''\varepsilon^{\frac{1}{2}},$$

with B depending on n, h and δ. Second, from the assumption $\|H - h\|_p < \varepsilon h$, we get immediately

$$\frac{h}{2} \leqslant (1 - \varepsilon)h \leqslant \|H\|_p \leqslant (1 + \varepsilon)h \leqslant 2h,$$

if we assume that $\varepsilon < \frac{1}{2}$. Hence, we deduce that

$$\|A\|_p \leqslant B'' + 2h\sqrt{n}.$$

So $\|A\|_p$ is bounded from above by a constant depending only on n, h, R and δ. Now, we set $\varepsilon_1 = \inf\left\{\frac{1}{2}, \left(\frac{2\eta_1}{B''h}\right)^2\right\}$. With this choice, if $\varepsilon < \varepsilon_1$, we get that

$$\eta = \frac{B''}{\|H\|_p}\varepsilon^{\frac{1}{2}} \leqslant \eta_1$$

and $\|A - H\,\mathrm{Id}\,\|_p^p \leqslant \|H\|_p\eta$, and we conclude that M is diffeomorphic and $C\eta^\alpha$-close to a geodesic sphere of radius $\frac{1}{\|H\|_2}$. But,

$$C\eta^\alpha = C\left(\frac{B''}{\|H\|_p}\right)^\alpha \varepsilon^{\frac{\alpha}{2}} \leqslant C\eta^\alpha = C\left(\frac{2B''}{h}\right)^\alpha \varepsilon^{\frac{\alpha}{2}} = K\varepsilon^\beta\,,$$

where K is a constant depending only on n, p, h and $\beta = \frac{\alpha}{2}$ depends only on n and p.
In the case where the ambient space is the space form of constant curvature δ, the proof is analogue using Theorem 3.1 of [13] for sphere and hyperbolic spaces obtain the Euclidean theorem with a conformal change of metric. In this case, the constants C, and so K too, depend also on δ. This concludes the proof. □

Proof of Corollaries 1 and 2. Assume that M has constant mean curvature $H = h$, and $\|\mathrm{Scal} - s\|_p < \varepsilon$ for a constant s. First, by the Gauss formula, we have clearly $\mathrm{Scal} = n(n-1)(H_2 + \delta)$ and so $\|\mathrm{Scal} - s\|_p < \varepsilon$ gives $\|H_2 - h_2\|_p\varepsilon$ with $h_2 = \frac{1}{n(n-1)}\mathrm{Scal} - \delta$ and we can apply Theorem 2 to conclude that M is diffeomorphic to a geodesic hypersphere of radius ρ. But this diffeomorphism is explicitely given (see [10], [11]) by $F = \rho\frac{X}{|X|}$ where X is the immersion of M into $\mathbb{M}^{n+1}(\delta)$. Hence, F is of the form $G \circ X$. Necessarily, X is injective and so the immersion of M is an embedding. By the Alexandrov theorem, we conclude that M is a geodesic hypersphere.

If Scal is constant and $\|H - h\|_p \leqslant \varepsilon$, the proof is the same and we conclude by the Alexandrov theorem for H_2. The radius of the geodesic sphere are thus necessarily those stated in both Corollaries. □

Proof of Theorem 3. The proof of Theorem 3 is analogous to the proof of Theorem 1 by taking $F = 1$. Since $p > 1$, the constants A, A', etc… depend also on R. The only other difference is that we use the result of Gioffrè [3] (which is isotropic version of Theorem 5). Here again if $p \in [n-1, n)$, then by Theorem 4, then there is no dependence on R. □

References

[1] C. De Lellis, S. Müller: Optimal rigidity estimates for nearly umbilical surfaces. *J. Diff. Geom.* 69 (1) (2005) 75–110.

[2] A. De Rosa, S. Gioffrè: Quantitative stability for anisotropic nearly umbilical hypersurfaces. arXiv:1705.09994 (2017)

[3] S. Gioffrè: A $W^{2,p}$-estimate for nearly umbilical hypersurfaces. arXiv:1612.08570 (2016)

[4] Y. He, H. Li: Integral formula of Minkowski type and new characterization of the Wulff shape. *Acta Math. Sinica* 24 (4) (2008) 697–704.

[5] D. Hoffman, D. Spruck: Sobolev and isoperimetric inequalities for Riemannian submanifolds. *Comm. Pure Appl. Math.* 27 (1974) 715–727.

[6] J. H. Michael, L. M. Simon: Sobolev and mean-value inequalities on generalized submanifolds of R^n. *Comm. Pure Appl. Math.* 26 (1973) 361–379.

[7] L. Onat: Some characterizations of the Wulff shape. *C. R. Math. Acad. Sci. Paris* 348 (17-18) (2010) 997–1000.

[8] D. Perez: On nearly umbilical hypersurfaces. Ph.D. thesis, Universität Zürich (2011)

[9] J. Roth: Extrinsic radius pinching for hypersurfaces of space forms. *Diff. Geom. Appl.* 25 (5) (2007) 485–499.

[10] J. Roth: Rigidity results for geodesic spheres in space forms. In: *Differential Geometry, Proceedings of the VIIIth International Colloquium in Differential Geometry, Santiago de Compostela*. World Scientific (2009) 156–163.

[11] J. Roth: Une nouvelle caractérisation des sphères géodésiques dans les espaces modèles. *Compte-Rendus – Mathématique* 347 (19-20) (2009) 1197–1200.

[12] J. Roth: A remark on almost umbilical hypersurfaces. *Arch. Math. (Brno)* 49 (1) (2013) 1–7.

[13] J. Roth, J. Scheuer: Explicit rigidity of almost-umbilical hypersurfaces. (2015). arXiv preprint arXiv:1504.05749

[14] P. Topping: Relating diameter and mean curvature for submanifolds of Euclidean space. *Comment. Math. Helv.* 83 (3) (2008) 539–546.

[15] J.Y. Wu, Y. Zheng: Relating diameter and mean curvature for Riemannian submanifolds. *Proc. Amer. Math. Soc.* 139 (11) (2011) 4097–4104.

Received: 07 March, 2018
Accepted for publication: 25 May, 2018
Communicated by: Haizhong Li