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Lightlike hypersurfaces of an indefinite Kaehler manifold of a quasi-constant curvature

Dae Ho Jin, Jae Won Lee

Abstract. We study lightlike hypersurfaces M of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature subject to the condition that the characteristic vector field ζ of $\bar{M}$ is tangent to M. First, we provide a new result for such a lightlike hypersurface. Next, we investigate such a lightlike hypersurface M of $\bar{M}$ such that
(1) the screen distribution $S(TM)$ is totally umbilical or
(2) M is screen conformal.

1 Introduction

In the classical theory of Riemannian geometry, Chen-Yano [2] introduced the notion of a *Riemannian manifold of a quasi-constant curvature* as a Riemannian manifold $(\bar{M}, \bar{g})$ endowed with a curvature tensor $\bar{R}$ satisfying

$$\begin{aligned}\bar{R}(\bar{X}, \bar{Y})\bar{Z} = f_1\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}\} \\ + f_2\{\theta(\bar{Y})\theta(\bar{Z})\bar{X} - \theta(\bar{X})\theta(\bar{Z})\bar{Y} + \bar{g}(\bar{Y}, \bar{Z})\theta(\bar{X})\zeta - \bar{g}(\bar{X}, \bar{Z})\theta(\bar{Y})\zeta\}, \quad (1)\end{aligned}$$

where f_1 and f_2 are smooth functions which are called the *curvature functions*, ζ is a vector field which is called the *characteristic vector field* of $\bar{M}$, and θ is a 1-form associated with ζ by $\theta(X) = \bar{g}(X, \zeta)$. In the followings, we denote by $\bar{X}, \bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$. If $f_2 = 0$, then $\bar{M}$ is reduced to a space of constant curvature.

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* Corresponding author
Affiliation:
Dae Ho Jin – Department of Mathematics Dongguk University Kyongju 780-714, Korea
E-mail: jindh@dongguk.ac.kr
Jae Won Lee (Corresponding author) – Department of Mathematics Education and RINS, Gyeongsang National University of Education, Jinju 52828, Republic of Korea
E-mail: leejaew@gnu.ac.kr

In this paper, we study lightlike hypersurfaces M of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature subject such that ζ is tangent to M. After then, under the condition that ζ is tangent to M, we investigate lightlike hypersurfaces M of $\bar{M}$ such that

(1) the screen distribution $S(TM)$ of M is totally umbilical in M or

(2) M is screen conformal.

2 Preliminaries

Let (M, g) be a lightlike hypersurface, with a screen distribution $S(TM)$, of a semi-Riemannian manifold $\bar{M}$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E. Also denote by $(8)_i$ the i-th equation of (8). We use same notations for any others. We follow Duggal-Bejancu [3] for notations and structure equations used in this article. It is well known that

$$TM = TM^{\perp} \oplus_{\rm orth} S(TM),$$

where $\oplus_{\rm orth}$ denotes the orthogonal direct sum. For any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique lightlike vector bundle ${\rm tr}(TM)$ of rank 1 in the orthogonal complement $S(TM)^{\perp}$ of $S(TM)$ in $\bar{M}$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall X \in \Gamma(S(TM)).$$

Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow

$$T\bar{M} = TM \oplus {\rm tr}(TM) = \{TM^{\perp} \oplus {\rm tr}(TM)\} \oplus_{\rm orth} S(TM).$$

We call ${\rm tr}(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to $S(TM)$, respectively.

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$. In the sequel, denote by X, Y, Z and W the smooth vector fields on M, unless otherwise specified. The local Gauss and Weingartan formulae for M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{3}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{4}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{5}$$

where ∇ and ∇^* are the liner connections on TM and $S(TM)$, respectively, B and C are the local second fundamental forms on TM and $S(TM)$, respectively, A_N and A^*_ξ are the shape operators and τ is a 1-form on TM.

Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. As $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, B is independent of the choice of $S(TM)$ and

$$B(X, \xi) = 0. \tag{6}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{7}$$

where η is a 1-form such that $\eta(X) = \bar{g}(X,N)$. But ∇^* is metric. The above local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{8}$$

$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0. \tag{9}$$

From (8), A^*_ξ is $S(TM)$-valued and self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{10}$$

Denote by $\bar{R}, R$ and R^* the curvature tensors of the connections $\bar{\nabla}, \nabla$ and ∇^*, respectively. Using (2)–(5), we obtain the Gauss-Codazzi equations:

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ &\quad + \big\{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\big\}N,\end{aligned} \tag{11}$$

$$\begin{aligned}\bar{R}(X,Y)N &= -\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X,Y] \\ &+ \tau(X)A_N Y - \tau(Y)A_N X + \big\{B(Y,A_N X) - B(X,A_N Y) + 2d\tau(X,Y)\big\}N,\end{aligned} \tag{12}$$

$$\begin{aligned}R(X,Y)PZ &= R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X \\ &+ \big\{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + \tau(Y)C(X,PZ) - \tau(X)C(Y,PZ)\big\}\xi,\end{aligned} \tag{13}$$

$$\begin{aligned}R(X,Y)\xi &= -\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y] - \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X \\ &\quad + \big\{C(Y,A^*_\xi X) - C(X,A^*_\xi Y) - 2d\tau(X,Y)\big\}\xi.\end{aligned} \tag{14}$$

In the case $R = 0$, we say that M is *flat*.

The *Ricci tensor*, denoted by $\overline{\text{Ric}}$, of $\bar{M}$ is defined by

$$\overline{\text{Ric}}(\bar{X},\bar{Y}) = \text{trace}\{\bar{Z} \to \bar{R}(\bar{X},\bar{Z})\bar{Y}\}.$$

Let $\dim \bar{M} = n+2$. Locally, $\overline{\text{Ric}}$ is given by

$$\overline{\text{Ric}}(\bar{X},\bar{Y}) = \sum_{i=1}^{n+2} \epsilon_i \bar{g}\big(\bar{R}(E_i,\bar{X})\bar{Y}, E_i\big),$$

where $\{E_1,\ldots,E_{n+2}\}$ is an orthonormal basis of $T\bar{M}$.

Let $R^{(0,2)}$ denote the induced tensor of type $(0,2)$ on M given by

$$R^{(0,2)}(X,Y) = \text{trace}\{Z \to R(X,Z)Y\}. \tag{15}$$

Due to [4], using (8), (9) and the Gauss equation (11), we get

$$R^{(0,2)}(X,Y) = \overline{\mathrm{Ric}}(X,Y) + B(X,Y)\,\mathrm{tr}\,A_N - g(A_N X, A^*_\xi Y) - \bar{g}(\bar{R}(\xi,Y)X,N). \quad (16)$$

Using the transversal part of (12) and the first Bianchi's identity, we obtain

$$R^{(0,2)}(X,Y) - R^{(0,2)}(Y,X) = 2d\tau(X,Y).$$

This shows that $R^{(0,2)}$ is not symmetric. A tensor field $R^{(0,2)}$ of M, given by (15), is called the *induced Ricci tensor*, denoted by Ric, of M if it is symmetric. In this case, M is said to be *Ricci flat* if Ric $= 0$. M is called an *Einstein manifold* if there exist a smooth function κ such that

$$\mathrm{Ric} = \kappa g. \quad (17)$$

Let $\nabla^{\perp}_X N = \pi_1(\bar{\nabla}_X N)$, where π_1 is the projection morphism of $T\bar{M}$ on $\mathrm{tr}(TM)$. Then $\nabla^{\perp}$ is a linear connection on the transversal vector bundle $\mathrm{tr}(TM)$ of M. We say that $\nabla^{\perp}$ is the *transversal connection* of M. We define the curvature tensor $R^{\perp}$ on $\mathrm{tr}(TM)$ by

$$R^{\perp}(X,Y)N = \nabla^{\perp}_X \nabla^{\perp}_Y N - \nabla^{\perp}_Y \nabla^{\perp}_X N - \nabla^{\perp}_{[X,Y]} N.$$

The transversal connection $\nabla^{\perp}$ of M is said to be *flat* [5] if $R^{\perp} = 0$.

We quote the following result due to Jin [5].

Theorem 1. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. The following assertions are equivalent:*

(1) *The transversal connection of M is flat, i.e., $R^{\perp} = 0$.*

(2) *The 1-form τ is closed, i.e., $d\tau = 0$, on any neighborhood $\mathcal{U} \subset M$.*

(3) *The Ricci type tensor $R^{(0,2)}$ is an induced Ricci tensor of M.*

Remark 1. Due to [3, Section 4.2–4.3], we shown the following results:

(1) $d\tau$ is independent to the choice of the section $\xi \in \Gamma(TM^{\perp})$, that is, suppose τ and $\bar{\tau}$ are 1-forms with respect to the sections ξ and $\bar{\xi}$, respectively, then $d\tau = d\bar{\tau}$.

(2) If $d\tau = 0$, then we can take a 1-form τ such that $\tau = 0$.

3 Quasi-constant curvature

Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a real $2m$-dimensional indefinite Kaeler manifold, where $\bar{g}$ is a semi-Riemannian metric of index $q = 2v$, $0 < v < m$, and J is an almost complex metric structure on $\bar{M}$ satisfying

$$J^2 = -I, \qquad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}), \qquad (\bar{\nabla}_{\bar{X}} J)\bar{Y} = 0. \quad (18)$$

Let (M, g) be a lightlike hypersurface of an indefinite Kaeler manifold $\bar{M}$, where g is a degenerate metric on M induced by $\bar{g}$. Due to [3, Section 6.2], we show that $J(TM^{\perp}) \oplus J(\mathrm{tr}(TM))$ is a subbundle of $S(TM)$ of rank 2. There exist two non--degenerate almost complex distributions D_o and D on M with respect to J, i.e., $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = \Big\{J(TM^{\perp}) \oplus J\big(\mathrm{tr}(TM)\big)\Big\} \oplus_{\mathrm{orth}} D_o,$$
$$D = \big\{TM^{\perp} \oplus_{\mathrm{orth}} J(TM^{\perp})\big\} \oplus_{\mathrm{orth}} D_o.$$

In this case, TM is decomposed as follow

$$TM = D \oplus J(\mathrm{tr}(TM)). \tag{19}$$

Consider lightlike vector fields U and V, and their 1-forms u and v such that

$$U = -JN, \qquad V = -J\xi, \qquad u(X) = g(X, V), \qquad v(X) = g(X, U). \tag{20}$$

Denote by S the projection morphism of TM on D with respect to (19). Then, for any vector field X on M, JX is expressed as follow

$$JX = FX + u(X)N, \tag{21}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$. Applying $\bar{\nabla}_X$ to $(20)_{1,2}$ and using (2)–(5) and (18)–(21), we have

$$B(X, U) = C(X, V), \tag{22}$$
$$\nabla_X U = F(A_N X) + \tau(X)U, \tag{23}$$
$$\nabla_X V = F(A^*_\xi X) - \tau(X)V. \tag{24}$$

From now and in the sequel, let $\bar{M}$ be an indefinite Kaeler manifold of a quasi--constant curvature. We shall assume that the characteristic vector field ζ of $\bar{M}$ is tangent to M and let $\alpha = \theta(N)$.

Theorem 2. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that ζ is tangent to M. Then the curvature functions f_1 and f_2, given by (1), are satisfied*

$$f_1 = 0, \qquad f_2\theta(V) = 0, \qquad \alpha f_2 = 0.$$

Proof. Comparing the tangent and transversal components of the two forms (1) and (11) of the curvature tensor $\bar{R}$ of $\bar{M}$, we get

$$R(X, Y)Z = B(Y, Z)A_N X - B(X, Z)A_N Y + f_1\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\} + f_2\Big\{\big[\theta(Y)X - \theta(X)Y\big]\theta(Z) + \big[g(Y, Z)\theta(X) - g(X, Z)\theta(Y)\big]\zeta\Big\}, \tag{25}$$

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) = 0\,. \tag{26}$$

Taking the product with N to (11) and using $(9)_2$ and (13), we get

$$\begin{aligned}(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ) \\ = f_1\{\eta(X)g(Y, PZ) - \eta(Y)g(X, PZ)\} + f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(PZ) \\ + \alpha f_2\{\theta(X)g(Y, PZ) - \theta(Y)g(X, PZ)\}. \quad (27)\end{aligned}$$

Applying ∇_Y to (22) and using (8), (9) and (22)–(24), we have

$$\begin{aligned}(\nabla_X B)(Y, U) = (\nabla_X C)(Y, V) - 2\tau(X)C(Y, V) \\ - g(A^*_\xi X, F(A_N Y)) - g(A^*_\xi Y, F(A_N X)).\end{aligned}$$

Substituting this equation into (26) with $Z = U$, we get

$$(\nabla_X C)(Y, V) - (\nabla_Y C)(X, V) - \tau(X)C(Y, V) + \tau(Y)C(X, V) = 0.$$

Comparing this equation and (27) such that $PZ = V$, we obtain

$$\begin{aligned}f_1\{\eta(X)u(Y) - \eta(Y)u(X)\} + f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(V) \\ + f_2\alpha\{\theta(X)u(Y) - \theta(Y)u(X)\} = 0. \quad (28)\end{aligned}$$

Replacing Y by ξ to this equation and using the fact that $\theta(\xi) = 0$, we have

$$f_1 u(X) + f_2\theta(X)\theta(V) = 0.$$

Taking $X = V$ and $X = U$ to this equation by turns, we get

$$f_2\theta(V) = 0, \qquad f_1 + f_2\theta(U)\theta(V) = 0.$$

From these two equations, we get $f_1 = 0$. Taking $Y = \zeta$ to (28) and using $f_1 = 0$ and $f_2\theta(V) = 0$, we have $\alpha f_2 u(X) = 0$. It follows that $\alpha f_2 = 0$. □

4 Totally umbilical screen distribution

Definition 1. A screen distribution $S(TM)$ is said to be *totally umbilical* [3], [6] in M if there exists a smooth function γ such that $A_N X = \gamma PX$, i.e.,

$$C(X, PY) = \gamma g(X, Y). \quad (29)$$

In case $\gamma = 0$, we say that $S(TM)$ is *totally geodesic* in M.

Theorem 3. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that ζ is tangent to M. If $S(TM)$ is totally umbilical, then*

(1) $S(TM)$ is totally geodesic and parallel distribution,

(2) $f_1 = f_2 = 0$, i.e., $\bar{M}$ is flat, and M is also flat,

(3) the transversal connection of M is flat, and

(4) M is locally a product manifold $\mathcal{C}_\xi \times M^*$, where $\mathcal{C}_\xi$ is a null geodesic tangent to $TM^\perp$, and M^* is a semi-Euclidean leaf of $S(TM)$.

Proof. Applying ∇_X to $C(Y, PZ) = \gamma g(Y, PZ)$ and using (7), we have

$$(\nabla_X C)(Y, PZ) = (X\gamma)g(Y, PZ) + \gamma B(X, PZ)\eta(Y).$$

Substituting this and (29) into (27) such that $f_1 = f_2\alpha = 0$, we obtain

$$\begin{aligned}\{X\gamma - \gamma\tau(X)\}g(Y, PZ) - \{Y\gamma - \gamma\tau(Y)\}g(X, PZ)\\ + \gamma\{B(X, PZ)\eta(Y) - B(Y, PZ)\eta(X)\}\\ = f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(PZ).\end{aligned}$$

Replacing Y by ξ to this and using (6) and the fact that $\theta(\xi) = 0$, we get

$$\gamma B(X, Y) = \{\xi\gamma - \gamma\tau(\xi)\}g(X, Y) - f_2\theta(X)\theta(Y). \tag{30}$$

Taking $Y = U$ to this equation and using (20), (22) and (29), we have

$$\gamma^2 u(X) = \{\xi\gamma - \gamma\tau(\xi)\}v(X) - f_2\theta(X)\theta(U).$$

Replacing X by V to this and using the fact that $f_2\theta(V) = 0$, we obtain

$$\xi\gamma - \gamma\tau(\xi) = 0, \qquad \gamma^2 u(X) = -f_2\theta(X)\theta(U). \tag{31}$$

Assume that $f_2 \neq 0$. Taking $X = \zeta$ to $(31)_2$, we have

$$\gamma^2\theta(V) = -f_2\theta(U).$$

Taking the product with f_2 to this and using $f_2\theta(V) = 0$, we get $f_2\theta(U) = 0$. Using this, from $(31)_2$, we see that $\gamma = 0$. Taking $X = Y = \zeta$ to (30), we have $f_2 = 0$. It is a contradiction. Thus $f_2 = 0$. We obtain $\gamma = 0$ by $(31)_2$.

(1) As $\gamma = 0$, $S(TM)$ is totally geodesic. Therefore, $S(TM)$ is a parallel distribution by (4) and the fact that $C = 0$.

(2) As $f_1 = f_2 = 0$, $\bar{M}$ is flat. As $f_1 = f_2 = A_N = 0$, from (27), we see that $R = 0$. Thus M is also flat.

(3) As $R = 0$, from (15), M is Ricci flat and $d\tau = 0$ by Theorem 2.1. Thus the transversal connection of M is flat.

(4) From (5) and (10), we see that $TM^\perp$ is an auto-parallel distribution. As $S(TM)$ is a parallel distribution and $TM = TM^\perp \oplus S(TM)$, by the decomposition theorem [7], M is locally a product manifold $\mathcal{C}_\xi \times M^*$, where $\mathcal{C}_\xi$ is a null geodesic tangent to $TM^\perp$ and M^* is a leaf of $S(TM)$. As $R = 0$ and $C = 0$, from (13) we see that $R^* = 0$. Thus M^* is semi-Euclidean. □

Denote by $\mathcal{G} = J(TM^\perp) \oplus_{\rm orth} D_o$. Then $\mathcal{G}$ is a complementary vector subbundle to $J(\mathrm{tr}(TM))$ in $S(TM)$ and we have the decomposition:

$$S(TM) = J(\mathrm{tr}(TM)) \oplus \mathcal{G}.$$

Theorem 4. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature such that ζ is tangent to M. If $S(TM)$ is totally umbilical, then M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_\xi$ and $\mathcal{C}_U$ are null geodesics tangent to $TM^\perp$ and $J(\mathrm{tr}(TM))$ respectively and $M^\sharp$ is a semi-Euclidean leaf of $\mathcal{G}$.*

Proof. By Theorem 4.1, we show that $d\tau = 0$ and $A_N = C = 0$. As $d\tau = 0$, we can take $\tau = 0$ by Remark 2.2, without loss generality. As $C = 0$, from (22) we see that $B(X, U) = 0$. Also, since $A_N = 0$, from (23) we have

$$\nabla_X U = 0. \tag{32}$$

Thus $J(\mathrm{tr}(TM))$ is a parallel distribution on M. From (5) and (10), $TM^\perp$ is also a parallel distribution on M. Using (32), we derive

$$g(\nabla_X Y, U) = 0, \quad g(\nabla_X V, U) = 0, \quad \forall X \in \Gamma(\mathcal{G}), \forall Y \in \Gamma(D_o).$$

Thus $\mathcal{G}$ is also a parallel distribution. By the decomposition theorem [7], M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_\xi$ and $\mathcal{C}_U$ are null geodesics tangent to $TM^\perp$ and $J(\mathrm{tr}(TM))$ respectively and $M^\sharp$ is a leaf of $\mathcal{G}$. Let π_2 be the projection morphism of $S(TM)$ on $\mathcal{G}$. Then $\pi_2 \circ R^*$ is the curvature tensor of $\mathcal{G}$. As $R = 0$ and $C = 0$, we have $R^* = 0$. Therefore, $\pi_2 \circ R^* = 0$ and $M^\sharp$ is a semi-Euclidean space. □

5 Screen conformal lightlike hypersurfaces

Definition 2. A lightlike hypersurface M is called *screen conformal* [1], [4] if there exists a non-vanishing smooth function φ such that $A_N = \varphi A^*_\xi$, i.e.,

$$C(X, PY) = \varphi B(X, Y).$$

If φ is a non-zero constant, then we say that M is *screen homothetic.*

Remark 2. If M is screen conformal, then, using (1) and the fact $f_1 = 0$,

$$\bar{g}(R(\xi, X)Y, N) = f_2 \theta(X)\theta(Y)$$

and

$$\overline{\mathrm{Ric}}(X, Y) = f_2\big\{g(X, Y) + n\theta(X)\theta(Y)\big\}.$$

Thus the form (16) of the Ricci type tensor $R^{(0,2)}$ is reduced to

$$R^{(0,2)}(X, Y) = f_2\big\{g(X, Y) + (n-1)\theta(X)\theta(Y)\big\} + B(X, Y)\,\mathrm{tr}\,A_N - \varphi g(A^*_\xi X, A^*_\xi Y). \tag{33}$$

Thus $R^{(0,2)}$ is symmetric. Thus $d\tau = 0$ and the transversal connection is flat by Theorem 2.1. As $d\tau = 0$, we can take $\tau = 0$ by Remark 2.2.

Proposition 1. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that ζ is tangent to M. If M is screen conformal, then the curvature function f_2 is satisfied $f_2\theta(U) = 0$.*

Proof. Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this equation into (26) and using (25), we obtain

$$(X\varphi)B(Y, PZ) - (Y\varphi)B(X, PZ) = f_2\big\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\big\}\theta(PZ). \tag{34}$$

Taking $Y = \xi$ to (34) and using (6) and the fact that $\theta(\xi) = 0$, we get

$$(\xi\varphi)B(X, Y) = f_2\theta(X)\theta(Y). \tag{35}$$

Replacing Y by V to (35) and using the fact that $f_2\theta(V) = 0$, we have

$$(\xi\varphi)B(X, V) = 0.$$

Taking $Y = U$ to (35) and using the fact $B(X, U) = C(X, V) = \varphi B(X, V)$, we obtain $f_2\theta(X)\theta(U) = 0$. Replacing X by ζ, we have $f_2\theta(U) = 0$. □

Corollary 1. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that ζ is tangent to M. If M is screen homothetic, then $f_1 = f_2 = 0$, i.e., $\bar{M}$ is flat.*

Proof. As M is screen homothetic, we get $\xi\varphi = 0$. Taking $X = Y = \zeta$ to (35) such that $\xi\varphi = 0$, we obtain $f_2 = 0$. As $f_1 = f_2 = 0$, $\bar{M}$ is flat. □

As $\{U, V\}$ is a null basis of $J(TM^\perp) \oplus J(\mathrm{tr}(TM))$, let

$$\mu = U - \varphi V, \qquad \nu = U + \varphi V,$$

then $\{\mu, \nu\}$ is an orthogonal basis of $J(TM^\perp) \oplus J(\mathrm{tr}(TM))$ and satisfies

$$B(X, \mu) = 0, \qquad A^*_\xi \mu = 0, \tag{36}$$

due to (22). Thus μ is an eigenvector field of A^*_ξ on $S(TM)$ corresponding to the eigenvalue 0. As $f_2\theta(V) = 0$ and $f_2\theta(U) = 0$, we also have

$$f_2\theta(\mu) = 0, \qquad f_2\theta(\nu) = 0. \tag{37}$$

Let $\mathcal{H}' = \mathrm{Span}\{\mu\}$. Then $\mathcal{H} = D_o \oplus_{\mathrm{orth}} \mathrm{Span}\{\nu\}$ is a complementary vector subbundle to $\mathcal{H}'$ in $S(TM)$ and we have the following decomposition

$$S(TM) = \mathcal{H}' \oplus_{\mathrm{orth}} \mathcal{H}. \tag{38}$$

Theorem 5. *Let M be a screen homothetic lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature such that ζ is tangent to M. Then M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\natural$, where $\mathcal{C}_\xi$ and $\mathcal{C}_\mu$ are null and non-null geodesics tangent to $TM^\perp$ and $\mathcal{H}'$, respectively and $M^\natural$ is a leaf of a non-degenerate distribution $\mathcal{H}$.*

Proof. In general, from (23), (24) and the fact that F is linear, we have

$$\nabla_X \mu = -(X\varphi)V.$$

Therefore, if M is screen homothetic, then we have

$$\nabla_X \mu = 0. \tag{39}$$

This implies that $\mathcal{H}'$ is a parallel distribution on M. From (5) and (10), $TM^{\perp}$ is also a parallel distribution on M. Using (39), we derive

$$g(\nabla_X Y, \mu) = g(\bar{\nabla}_X Y, \mu) = -g(Y, \nabla_X \mu) = 0,$$
$$g(\nabla_X \nu, \mu) = -g(\nu, \nabla_X \mu) = X\varphi = 0,$$

for $X \in \Gamma(\mathcal{H})$ and $Y \in \Gamma(D_o)$. Thus $\mathcal{H}$ is also a parallel distribution. By the decomposition theorem of de Rham [7], M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\natural$, where $\mathcal{C}_\xi$ and $\mathcal{C}_\mu$ are null and non-null geodesics tangent to $TM^\perp$ and $\mathcal{H}'$ respectively and $M^\natural$ is a leaf of $\mathcal{H}$.

□

Theorem 6. *Let M be an Einstein lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that ζ is tangent to M. If M is screen conformal, then the function κ, given by (17), satisfies $\kappa = f_2$. If M is screen homothetic, then it is Ricci flat, i.e., $\kappa = 0$.*

Proof. Since M is Einstein manifold, (33) is reduced to

$$g(A^*_\xi X, A^*_\xi Y) - \ell g(A^*_\xi X, Y) - \varphi^{-1}\{(\kappa - f_2)g(X,Y) - f_2(n-1)\theta(X)\theta(Y)\} = 0, \tag{40}$$

where $\ell = \operatorname{tr} A^*_\xi$ is the trace of A^*_ξ. Put $X = Y = \mu$ in (40) and using $(36)_2$ and $(37)_1$, we have $\kappa = f_2$. If M is screen homothetic, then M is Ricci flat as $f_2 = 0$ by Corollary 5.3. □

Theorem 7. *Let M be a screen homothetic Einstein lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature such that $q = 2$ and ζ is tangent to M. Then M is locally a product manifold*

$$M = \mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\natural \quad \text{or} \quad M = \mathcal{C}_\xi \times \mathcal{C}_\mu \times \mathcal{C}_\ell \times M^\sharp,$$

where $\mathcal{C}_\xi$, $\mathcal{C}_\mu$ and $\mathcal{C}_\ell$ are null geodesic, timelike geodesic and spacelike geodesic respectively, and $M^\natural$ and $M^\sharp$ are Euclidean spaces.

Proof. In this proof, we set $\mu = \frac{1}{\sqrt{2\epsilon\varphi}}\{U - \varphi V\}$ where $\epsilon = \operatorname{sgn}\varphi$. Then μ is a unit timelike eigenvector of A^*_ξ corresponding to the eigenvalue 0 by (36) and $\mathcal{H}$ is a parallel Riemannian distribution by Theorem 5.4 due to $q = 2$. Since $g(A^*_\xi X, N) = 0$ and $g(A^*_\xi X, \mu) = 0$, A^*_ξ is $\mathcal{H}$-valued real self-adjoint operator. Thus A^*_ξ have $(n-1)$ real orthonormal eigenvectors in $\mathcal{H}$ and is diagonalizable. Consider a frame field of eigenvectors $\{\mu, e_1, \ldots, e_{n-1}\}$ of A^*_ξ on $S(TM)$ such that $\{e_1, \ldots, e_{n-1}\}$ is an orthonormal frame field of $\mathcal{H}$. Then $A^*_\xi e_i = \lambda_i e_i$ $(1 \le i \le n-1)$.

Put $X = Y = e_i$ in (40) such that $\kappa = f_2 = 0$, we show that each eigenvalue λ_i of A^*_ξ is a solution of

$$x(x - \ell) = 0. \tag{41}$$

The equation (41) has at most two distinct real solutions 0 and ℓ on $\mathcal{U}$. Assume that there exists $p \in \{1, \ldots, n-1\}$ such that $\lambda_1 = \cdots = \lambda_p = 0$ and $\lambda_{p+1} = \cdots = \lambda_{n-1} = \ell$, by renumbering if necessary. Then we have

$$\ell = \operatorname{tr} A^*_\xi = (n - p - 1)\ell.$$

If $\ell = 0$, then $A^*_\xi = 0$ and also $A_N = 0$. Thus M and $S(TM)$ are totally geodesic. From (11) and (13), we have $R^*(X, Y)Z = \bar{R}(X, Y)Z = 0$ for all $X, Y, Z \in \Gamma(S(TM))$. Thus M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\natural$, where $\mathcal{C}_\xi$ and $\mathcal{C}_\mu$ are null and timelike geodesic tangent to $TM^\perp$ and $\mathcal{H}'$ respectively and $M^\natural$ is a leaf of $\mathcal{H}$, where the leaf $M^*(= \mathcal{C}_\mu \times M^\natural)$ of $S(TM)$ is a Minkowski space. Since $\nabla_X \mu = 0$ and

$$g(\nabla^*_X Y, \mu) = -g(Y, \nabla^*_X \mu) = -g(Y, \nabla_X \mu) = 0,$$

for all $X, Y, Z \in \Gamma(S(TM))$, we have $\nabla^*_X Y \in \Gamma(\mathcal{H})$ and $R^*(X, Y)Z \in \Gamma(\mathcal{H})$. This imply $\nabla^*_X Y = Q(\nabla^*_X Y)$, i.e., $M^\natural$ is totally geodesic and $Q(R^*(X, Y)Z) = R^*(X, Y)Z = 0$, where Q is a projection morphism of $S(TM)$ on $\mathcal{H}$ with respect to (38). Thus $M^\natural$ is a Euclidean space.

If $\ell \neq 0$, then $p = n - 2$. Consider the following two distributions on $\mathcal{H}$;

$$\Gamma(E_0) = \{X \in \Gamma(\mathcal{H}) | A^*_\xi X = 0\},$$
$$\Gamma(E_\ell) = \{X \in \Gamma(\mathcal{H}) | A^*_\xi X = \ell X\}.$$

Then we know that the distributions E_0 and E_ℓ are mutually orthogonal non--degenerate subbundle of $\mathcal{H}$, of rank $(n-2)$ and 1 respectively, satisfy $\mathcal{H} = E_0 \oplus_{\text{orth}} E_\ell$. From (40), we get $A^*_\xi(A^*_\xi - \ell Q) = 0$. Using this equation, we have

$$\operatorname{Im} A^*_\xi \subset \Gamma(E_\ell) \quad \text{and} \quad \operatorname{Im}(A^*_\xi - \ell Q) \subset \Gamma(E_0).$$

For any $X, Y \in \Gamma(E_0)$ and $Z \in \Gamma(\mathcal{H})$, we get

$$(\nabla_X B)(Y, Z) = -g(A^*_\xi \nabla_X Y, Z).$$

Using this and the fact that

$$(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z),$$

we have $g(A^*_\xi[X, Y], Z) = 0$. If we take $Z \in \Gamma(E_\ell)$, since $\operatorname{Im} A^*_\xi \subset \Gamma(E_\ell)$ and E_ℓ is non-degenerate, we have $A^*_\xi[X, Y] = 0$. Thus $[X, Y] \in \Gamma(E_0)$ and E_0 is integrable. From (11) and (13), we have

$$R^*(X, Y)Z = \bar{R}(X, Y)Z = 0$$

for all $X, Y, Z \in \Gamma(E_0)$.

Since $g(\nabla^*_X Y, \mu) = 0$ and $g(\nabla^*_X Y, e_{n-1}) = -g(Y, \nabla_X e_{n-1}) = 0$ for all $X, Y \in \Gamma(E_0)$ because $\nabla_X W \in \Gamma(E_\ell)$ for $X \in \Gamma(E_0)$ and $W \in \Gamma(E_\ell)$. In fact, from (26) such that $\tau = 0$, we get

$$g\Big(\big\{(A^*_\xi - \ell Q)\nabla_X W - A^*_\xi \nabla_W X\big\}, Z\Big) = 0,$$

for all $X \in \Gamma(E_0), W \in \Gamma(E_\ell)$ and $Z \in \Gamma(\mathcal{H})$. Using the fact that $\mathcal{H}$ is non-degenerate distribution, we have

$$(A^*_\xi - \ell Q)\nabla_X W = A^*_\xi \nabla_W X.$$

Since the left term of this equation is in $\Gamma(E_0)$ and the right term is in $\Gamma(E_\ell)$ and $E_0 \cap E_\ell = \{0\}$, we have

$$(A^*_\xi - \ell Q)\nabla_X W = 0 \quad \text{and} \quad A^*_\xi \nabla_W X = 0.$$

These imply that $\nabla_X W \in \Gamma(E_\ell)$. Thus $\nabla^*_X Y = \pi_3 \nabla^*_X Y$ for all $X, Y \in \Gamma(E_0)$, where π_3 is the projection morphism of $S(TM)$ on E_0 and $\pi_3 \nabla^*$ is the induced connection on E_0. These imply that the leaf $M^\sharp$ of E_0 is totally geodesic. Thus E_0 is a parallel distribution and M is locally a product manifold $\mathcal{C}_\xi \times M^* (= \mathcal{C}_\mu \times \mathcal{C}_\ell \times M^\sharp)$, where $\mathcal{C}_\ell$ is a spacelike curve and $M^\sharp$ is an $(n-2)$-dimensional Riemannian manifold satisfies $A^*_\xi = 0$. As

$$g(R^*(X,Y)Z, \mu) = 0 \quad \text{and} \quad g(R^*(X,Y)Z, e_{n-1}) = 0$$

for all $X, Y, Z \in \Gamma(E_0)$, we have

$$R^*(X,Y)Z = \pi_3 R^*(X,Y)Z \in \Gamma(E_0)$$

and the curvature tensor $\pi_3 R^*$ of E_0 is flat. Thus $M^\sharp$ is a Euclidean space. □

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