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Generalized reverse derivations and commutativity of prime rings

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Abstract. Let R be a prime ring with center $Z(R)$ and I a nonzero right ideal of R. Suppose that R admits a generalized reverse derivation (F, d) such that $d(Z(R)) \neq 0$. In the present paper, we shall prove that if one of the following conditions holds:
(i) $F(xy) \pm xy \in Z(R)$
(ii) $F([x, y]) \pm [F(x), y] \in Z(R)$
(iii) $F([x, y]) \pm [F(x), F(y)] \in Z(R)$
(iv) $F(x \circ y) \pm F(x) \circ F(y) \in Z(R)$
(v) $[F(x), y] \pm [x, F(y)] \in Z(R)$
(vi) $F(x) \circ y \pm x \circ F(y) \in Z(R)$
for all $x, y \in I$, then R is commutative.

1 Introduction

Throughout this paper, unless otherwise mentioned, R will be a prime ring with center $Z(R)$. For all $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for Lie product $xy - yx$ and Jordan product $xy + yx$, respectively. Recall that a ring R is prime if for all $x, y \in R$, $xRy = 0$ implies $x = 0$ or $y = 0$ and is semiprime if for all $x \in R$, $xRx = 0$ implies $x = 0$. By a derivation on R we mean an additive mapping $d\colon R \to R$ such that $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F\colon R \to R$ is called a generalized derivation if there exists a derivation $d\colon R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Moreover, a Jordan derivation of a ring R is an additive mapping $J\colon R \to R$ that satisfies $J(x^2) = J(x)x + xJ(x)$ for all $x \in R$. A mapping $f\colon R \to R$ is called centralizing on a subset S of R if $[f(x), x] \in Z(R)$.

Many results in literature shows that the global structure of a ring R is often lightly connected to the behavior of additive mappings defined on R. The first

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result in this direction is the classical Posner's second theorem. In [14], it is shown that if a prime ring R admits a nonzero derivation d which is centralizing on R, then R is commutative. In the last few years, number of authors have investigated the relationship between the commutativity of ring R and certain specific types of derivations of R (for instance, see [3], [7] and [11] where further references can be found). In [6], Ashraf and Rehman proved that if a prime ring R admits a derivation d satisfying $d(xy) \pm xy \in Z(R)$ for all $x, y \in I$, a nonzero ideal of R, then R must be commutative. In [5], the authors extended the above results to generalized derivations. In [2], Albas proved that if R is a prime ring with characteristic different from two and (F, d) is a generalized derivation such that $F([x, y]) \pm [F(x), F(y)] = 0$ for all $x, y \in R$, then R is commutative or $d = 0$ or $d = -I_{\mathrm{id}}$, where I_{id} is the identical mapping on R. Hence, it should be interesting to study the commutativity of prime and semiprime rings admitting suitably constrained derivations, generalized derivations and so on.

On the other hand, the notion of reverse derivation was introduced by Herstein in [9], when he studied Jordan derivations in prime rings. It is proved that if R is a prime ring and d a nonzero reverse derivation of R, then R is a commutative integral domain and d is just a derivation. The reverse derivation on semiprime rings have been studied in [15]. The concept of reverse derivation has relations with some generalizations of derivations. We call an additive mapping d from a ring R into itself satisfying $d(yx) = d(x)y + xd(y)$ holds for all $x, y \in R$. It is clear that each reverse derivation is a Jordan derivation but the converse is not true in general. In the anticommutative case, the notions of reverse derivation and an antiderivation are equivalent. Moreover, Filippov [8] and Hopkins [10] studied the reverse derivations of prime Lie algebras and prime Malcev algebras. In particular, Filippov [8] proved that if a prime Lie algebra A admits a nonzero reverse derivation d, then A is a PI-algebra. In [12], Ibraheem obtained several commutativity theorems by studying reverse derivations and generalized reverse derivations. Following [12], an additive mapping $F : R \to R$ is called a generalized reverse derivation if there exists a reverse derivation d such that $F(xy) = F(y)x + yd(x)$ holds for all $x, y \in R$. For completeness of notation, in the present paper, a generalized reverse derivation will be denoted by (F, d). The following example demonstrates that generalized reverse derivations in rings do exist. Let S be any ring and

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| a, b, c \in S \right\}$$

Define maps $F, d \colon R \to R$ by

$$F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a-c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to check that F is a generalized reverse derivation associated with reverse derivation d. For more results concerning reverse derivations and generalized reverse derivations in prime and seiprime rings, we refer the reader to [1] and [16].

Motivated by above observations, we shall investigate the commutativity of a prime ring R admitting a generalized reverse derivation (F, d) satisfying any one the following properties:

- $F(xy) \pm xy \in Z(R)$,
- $F([x, y]) \pm [F(x), y] \in Z(R)$,
- $F([x, y]) \pm [F(x), F(y)] \in Z(R)$,
- $F(x \circ y) \pm F(x) \circ F(y) \in Z(R)$,
- $[F(x), y] \pm [x, F(y)] \in Z(R)$,
- $F(x) \circ y \pm x \circ F(y) \in Z(R)$

for all x, y in some appropriate subset of R.

2 Some preliminaries

The following basic identities are useful in the sequel.

$$[xy, z] = x[y, z] + [x, z]y \quad \text{and} \quad [x, yz] = y[x, z] + [x, y]z$$
$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$
$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].$$

We begin our discussion with several lemmas which will be widely used in our results.

Lemma 1. *Let R be a prime ring with center $Z(R)$. If d is a reverse derivation of R, then $d(Z(R)) \subseteq Z(R)$.*

Proof. For all $\alpha \in Z(R)$ and $r \in R$, we have $\alpha r = r\alpha$. Hence, $d(\alpha r) = d(r\alpha)$. Since d is a reverse derivation, then $d(r)\alpha + rd(\alpha) = d(\alpha)r + \alpha d(r)$. This implies that $rd(\alpha) = d(\alpha)r$, proving the lemma. □

Lemma 2 ([4]). *Let R be a prime ring with center $Z(R)$. If $a, ab \in Z(R)$ for some $a, b \in R$, then either $a = 0$ or $b \in Z(R)$.*

Lemma 3 ([13]). *If a prime R contains a nonzero commutative right ideal, then R is commutative.*

3 Main results

Theorem 1. *Let R be a prime ring with center $Z(R)$ and I a nonzero right ideal of R. Suppose that R admits a generalized reverse derivation (F, d) satisfying $d(Z(R)) \neq 0$. If $F(xy) \pm xy \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof. Using the fact that $d(Z(R)) \neq 0$, there exist some element $c \in Z(R)$ such that $d(c) \neq 0$. Therefore, it follows from Lemma 2.1 that $d(c) \in Z(R)$. By the hypothesis we have $F(xy) - xy \in Z(R)$ for all $x, y \in I$. By the definition of F, the above relation can be written as

$$F(y)x + yd(x) - xy \in Z(R) \quad \text{for all } x, y \in I. \tag{1}$$

Substituting $yc = cy$ for y in equation (3.1), we obtain that

$$c\big(F(y)x + yd(x) - xy\big) + d(c)yx \in Z(R) \quad \text{for all } x, y \in I. \tag{2}$$

Combining (1) and (2), we arrive at $d(c)yx \in Z(R)$ for all $x, y \in I$. In the light of Lemma 2, we are forced to conclude that $yx \in Z(R)$ for all $x, y \in I$. If we interchange x and y, then we find that $xy \in Z(R)$ for all $x, y \in I$. The last two relations yields that $[x, y] \in Z(R)$ for all $x, y \in I$. In particular, $\big[[x, y], r\big] = 0$ for all $x, y \in I$ and $r \in R$. Replace x by yx to get

$$0 = [x, y][x, r] + \big[[x, y], r\big]x = [x, y][x, r]$$

for all $x, y \in I$ and $r \in R$. In the last equation, we take ry instead of r to get

$$0 = [x, y][x, ry] = [x, y]r[x, y] + [x, y][x, r]y = [x, y]r[x, y]$$

for all $x, y \in I$ and $r \in R$. That is $[x, y]R[x, y] = 0$ for all $x, y \in I$. The primeness of R forces that $[x, y] = 0$ for all $x, y \in I$. By Lemma 3, I is a commutative right ideal and so is R. Further, if (F, d) is a generalized reverse derivation satisfying $F(xy)+xy \in Z(R)$ for all $x, y \in I$, then the generalized reverse derivation $(-F, -d)$ satisfies that $(-F)(xy) - xy \in Z(R)$. By the above arguments, we get the required result. □

Theorem 2. *Let R be a prime ring with center $Z(R)$ and I a nonzero right ideal of R. Suppose that R admits a generalized reverse derivation (F, d) satisfying $d(Z(R)) \neq 0$. If $F([x, y]) \pm [F(x), y] \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof. For the sake of clearness, we only prove the case $F\big([x, y]\big) - [F(x), y] \in Z(R)$. By our hypothesis, we have

$$F\big([x, y]\big) - \big[F(x), y\big] \in Z(R) \quad \text{for all } x, y \in I. \tag{3}$$

Since $d(Z(R)) \neq 0$, there exist some element $c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. Write $yc = cy$ instead of y in (3) to get

$$F\big(c[x, y]\big) - c\big[F(x), y\big] \in Z(R) \quad \text{for all } x, y \in I.$$

Since F is a generalized reverse derivation, then

$$c\Big(F\big([x, y]\big) - \big[F(x), y\big]\Big) + d(c)[x, y] \in Z(R) \quad \text{for all } x, y \in I. \tag{4}$$

Combining (3) and (4), we obtain that $d(c)[x, y] \in Z(R)$ for all $x, y \in I$. In view of Lemma 2, $[x, y] \in Z(R)$ for all $x, y \in I$. Using the same arguments as used in the proof of Theorem 1, we get the required result. In the same manner, we can prove that the conclusion holds for the case $F\big([x, y]\big) + [F(x), y] \in Z(R)$ for all $x, y \in I$. □

Theorem 3. *Let R be a prime ring with center $Z(R)$ and I a nonzero right ideal of R. Suppose that R admits a generalized reverse derivation (F,d) satisfying $d(Z(R)) \neq 0$. If $F([x,y]) \pm [F(x),F(y)] \in Z(R)$ for all $x,y \in I$, then R is commutative.*

Proof. We are given that

$$F\big([x,y]\big) - \big[F(x),F(y)\big] \in Z(R) \quad \text{for all } x,y \in I. \tag{5}$$

Following the same procedure as used in Theorem 1, we choose $c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. Replacing y by $yc = cy$ in (5), we get the formula

$$c\Big(F\big([x,y]\big) - \big[F(x),F(y)\big]\Big) + d(c)\big([x,y] - [F(x),y]\big) \in Z(R) \quad \text{for all } x,y \in I. \tag{6}$$

Comparing (5) and (6), we have $d(c)([x,y]-[F(x),y]) \in Z(R)$ for all $x,y \in I$. Now application of Lemma 2 gives that

$$[x,y] - \big[F(x),y\big] \in Z(R) \quad \text{for all } x,y \in I. \tag{7}$$

Taking $xc = cx$ instead of x in (3.8), we find that

$$c\big([x,y] - [F(x),y]\big) + d(c)[x,y] \in Z(R) \quad \text{for all } x,y \in I. \tag{8}$$

Combining equation (7) and (8) we arrived at $d(c)[x,y] \in Z(R)$ for all $x,y \in I$. Then, by Lemma 2, we conclude that $[x,y] \in Z(R)$ for all $x,y \in I$. Using the same arguments as used in the proof of Theorem 3.1, we get the required result.

In the event $F\big([x,y]\big) + \big[F(x),F(y)\big] \in Z(R)$ for all $x,y \in I$, it is equally easy to establish that $[x,y] \in Z(R)$ for all $x,y \in I$, therefore our proof is complete. □

Theorem 4. *Let R be a prime ring with center $Z(R)$ and I a nonzero right ideal of R. Suppose that R admits a generalized reverse derivation (F,d) satisfying $d(Z(R)) \neq 0$. If $F(x \circ y) \pm F(x) \circ F(y) \in Z(R)$ for all $x,y \in I$, then R is commutative.*

Proof. First of all, we dispose the case

$$F(x \circ y) - F(x) \circ F(y) \in Z(R) \quad \text{for all } x,y \in I. \tag{9}$$

Following the same procedure as used in Theorem 1, we choose $c \in Z(R)$ satisfying $0 \neq d(c) \in Z(R)$. Replacing y by $yc = cy$ in (9), we get

$$c\big(F(x \circ y) - F(x) \circ F(y)\big) + d(c)(x \circ y - F(x) \circ y) \in Z(R) \quad \text{for all } x,y \in I. \tag{10}$$

Comparing (9) and (10), we find that $d(c)(x \circ y - F(x) \circ y) \in Z(R)$ for all $x,y \in I$. Hence by Lemma 2 we have

$$x \circ y - F(x) \circ y \in Z(R) \quad \text{for all } x,y \in I. \tag{11}$$

Replacing $xc = cx$ in (11) and using (11), we conclude that $d(c)(x \circ y) \in Z(R)$ for all $x,y \in I$. By virtue of Lemma 2, $x \circ y \in Z(R)$ for all $x,y \in I$. Thus,

$[x \circ y, r] = 0$ for all $x, y \in I$ and $r \in R$. Replacing y by yx in the last relation, we get $(x \circ y)[x, r] = 0$ for all $x, y \in I$ and $r \in R$. Substitute sr for r to get $(x \circ y)R[x, r] = 0$ for all $x, y \in I$ and $r \in R$. By the primeness of R, for each $x \in I$, it gives either $x \circ y = 0$ or $[x, r] = 0$. Let $I_1 = \{x \in I \mid x \circ y = 0\}$ and $I_2 = \{x \in I \mid [x, r] = 0\}$. Then, I_1 and I_2 are both additive subgroups of I such that $I = I_1 \cup I_2$. By Brauer's trick, either $I_1 = I$ or $I_2 = I$. On the one hand, if $I_1 = I$, then $x \circ y = 0$ for all $x, y \in I$. Replacing y by yz, we get

$$0 = x \circ (yz) = (x \circ y)z - y[x, z] = -y[x, z]\,,$$

that is $y[x, z] = 0$ for all $x, y, z \in I$. This implies that $yR[x, z] = 0$ for all $x, y, z \in I$. Since I is nonzero, the primeness of R forces that $[x, z] = 0$ for all $x, z \in I$. By Lemma 2.3, I is a commutative right ideal and so is R. On the other hand, if $I_2 = I$, then $[x, r] = 0$ for all $x \in I$ and $r \in R$. Hence, $I \subseteq Z(R)$ and it is clear that I is commutative and so is R. Next, if we have $F(x \circ y) + F(x) \circ F(y) \in Z(R)$ for all $x, y \in I$, then it is easy to see that $x \circ y \in Z(R)$ for all $x, y \in I$. This completes the proof. □

Theorem 5. *Let R be a prime ring with center $Z(R)$ and I a nonzero right ideal of R. Suppose that R admits a generalized reverse derivation (F, d) satisfying $d(Z(R)) \neq 0$. If $[F(x), y] \pm [x, F(y)] \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof. By the given hypothesis, we have

$$[F(x), y] - [x, F(y)] \in Z(R) \quad \text{for all } x, y \in I. \tag{12}$$

Choose $c \in Z(R)$ satisfying $0 \neq d(c) \in Z(R)$. Replacing y by $yc = cy$ in (12), we arrive at

$$c\Big(\big[F(x), y\big] - \big[x, F(y)\big]\Big) - d(c)[x, y] \in Z(R) \quad \text{for all } x, y \in I. \tag{13}$$

Comparing (12) and (13), we obtain that $d(c)[x, y] \in Z(R)$ for all $x, y \in I$. Thus, by Lemma 2, $[x, y] \in Z(R)$ for all $x, y \in I$. Using the same arguments as used in the proof of Theorem (1), we obtained the required result. Further, if $[F(x), y] + [x, F(y)] \in Z(R)$ for all $x, y \in I$, then using the same arguments as used above with necessary variations we get the required result. □

Replace the Lie product $[x, y]$ by Jordan product $x \circ y$ in the above theorem, we can prove the following.

Theorem 6. *Let R be a prime ring with center $Z(R)$ and I a nonzero right ideal of R. Suppose that R admits a generalized reverse derivation (F, d) satisfying $d(Z(R)) \neq 0$. If $F(x) \circ y \pm x \circ F(y) \in Z(R)$ for all $x, y \in I$, then R is commutative.*

The following example shows that the restrictions imposed on the hypotheses of the various results are not superfluous.

Example 1. Consider the ring

$$R = \left\{ \begin{pmatrix} 0 & x & y & z \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| x, y, z, m, n \in S \right\},$$

where S is the set of all integers. Define maps $F, d : R \to R$ by

$$F \begin{pmatrix} 0 & x & y & z \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & y+n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$d \begin{pmatrix} 0 & x & y & z \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & y-m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to check that F is a generalized reverse derivation associated with reverse derivation d. It is obvious that R is not a prime ring and $d\big(Z(R)\big) = 0$. Let

$$I = \left\{ \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| a, b \in S \right\}.$$

Then I is a nonzero right ideal of R and

- $F(XY) \pm XY \in Z(R)$,
- $F([X, Y]) \pm [F(X), Y] \in Z(R)$,
- $F([X, Y]) \pm [F(X), F(Y)] \in Z(R)$,
- $F(X \circ Y) \pm F(X) \circ F(Y) \in Z(R)$,
- $[F(X), Y] \pm [X, F(Y)] \in Z(R)$,
- $F(X) \circ Y \pm X \circ F(Y) \in Z(R)$

for all $X, Y \in I$. However, R is not commutative.

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