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On compatible linear connections of two-dimensional generalized Berwald manifolds: a classical approach

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In memoriam to V. Wagner on the 75th anniversary of publishing his pioneering work about generalized Berwald manifolds.

Abstract. In the paper we characterize the two-dimensional generalized Berwald manifolds in terms of the classical setting of Finsler surfaces (Berwald frame, main scalar etc.). As an application we prove that if a Landsberg surface is a generalized Berwald manifold then it must be a Berwald manifold. Especially, we reproduce Wagner's original result in honor of the 75th anniversary of publishing his pioneering work about generalized Berwald manifolds.

Introduction

The concept of generalized Berwald manifolds goes back to V. Wagner [17]. They are Finsler manifolds admitting linear connections such that the parallel transports preserve the Finslerian length of tangent vectors (compatibility condition). To express the compatible linear connection in terms of the canonical data of the Finsler manifold is the problem of the intrinsic characterization we are going to solve

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in case of two-dimensional generalized Berwald manifolds. The result is formulated in terms of linear inhomogeneous differential equations for the main scalar along the indicatrix curve (Subsection 2.1). As an application we prove that if a Landsberg surface is a generalized Berwald manifold then it must be a Berwald manifold (Subsection 2.2). Especially, we reproduce Wagner's original result in terms of the conventional setting of Finsler surfaces (Subsection 2.3) in honor of the 75th anniversary of publishing his pioneering work about generalized Berwald manifolds. Since Wagner's theorem (Subsection 2.3) does not contain information about the expression of the compatible linear connection we clarify these consequences in Section 3.

1 Notations and terminology

Let M be a connected differentiable manifold with local coordinates $u^1, \dots, u^n$. The induced coordinate system of the tangent manifold TM consists of the functions $x^1, \dots, x^n$ and $y^1, \dots, y^n$. For any $v \in T_pM$, $x^i(v) = u^i \circ \pi(v) = u^i(p)$ and $y^i(v) = v(u^i)$, where $\pi \colon TM \to M$ is the canonical projection, $i = 1, \dots, n$.

1.1 Finsler metrics

A Finsler metric is a continuous function $F \colon TM \to \mathbb{R}$ satisfying the following conditions:

(F1) F is smooth on the complement of the zero section (regularity),

(F2) $F(tv) = tF(v)$ for all $t > 0$ (positive homogeneity),

(F3) the Hessian $g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j}$, where $E = \frac{1}{2}F^2$ is positive definite at all nonzero elements $v \in T_pM$ (strong convexity).

The so-called *Riemann-Finsler metric* g is constituted by the components g_{ij}. It is defined on the complement of the zero section. The Riemann-Finsler metric makes each tangent space (except at the origin) a Riemannian manifold with standard canonical objects such as the *volume form* $d\mu = \sqrt{\det g_{ij}}\ dy^1 \wedge \ldots \wedge dy^n$, the *Liouville vector field* $C := y^1\partial/\partial y^1 + \ldots + y^n\partial/\partial y^n$ together with its *normalized dual form* $l_i = \partial F/\partial y^i$ with respect to the Riemann-Finsler metric and the *induced volume form*

$$\mu = \sqrt{\det g_{ij}} \sum_{i=1}^{n} (-1)^{i-1} \frac{y^i}{F} dy^1 \wedge \ldots \wedge dy^{i-1} \wedge dy^{i+1} \ldots \wedge dy^n$$

on the indicatrix hypersurface $\partial K_p := F^{-1}(1) \cap T_pM$ $(p \in M)$. In what follows we summarize some basic notations. As a general reference of Finsler geometry see [2] and [6]: $g^{ij} = (g_{ij})^{-1}$ denotes the inverse of the coefficient matrix of the Riemann-Finsler metric, the (lowered) first Cartan tensor is given by

$$C_{ijk} = \frac{1}{2}\partial g_{ij}/\partial y^k$$

and $\mathcal{C}^l_{ij} = g^{lk}\mathcal{C}_{ijk}$. The first Cartan tensor is totally symmetric and $y^k\mathcal{C}_{ijk} = 0$. Its semibasic trace is given by the quantities $\mathcal{C}_i = g^{jk}\mathcal{C}_{ijk}$ $(i, j, k = 1, \ldots, n)$. Differentiating $\det g_{ij}$ as a composite function we have that

$$\begin{aligned}\frac{\partial \det g_{rs}}{\partial y^i} &= \frac{\partial D}{\partial m_{jk}}(M)\frac{\partial g_{jk}}{\partial y^i}\\ &= (-1)^{j+k}\det(M \text{ without its } j^{\text{th}} \text{ row and } k^{\text{th}} \text{ column})\frac{\partial g_{jk}}{\partial y^i}\\ &= (\det g_{rs})g^{jk}\frac{\partial g_{jk}}{\partial y^i}, \quad \text{where } M := g_{ij}.\end{aligned}$$

Therefore

$$\frac{\partial \ln\sqrt{\det g_{rs}}}{\partial y^i} = \frac{1}{2}g^{jk}\frac{\partial g_{jk}}{\partial y^i} = g^{jk}\mathcal{C}_{ijk} = \mathcal{C}_i\,. \tag{1}$$

The geodesic spray coefficients and the horizontal sections are

$$G^l = \frac{1}{2}g^{lm}\left(y^k\frac{\partial^2 E}{\partial y^m\partial x^k} - \frac{\partial E}{\partial x^m}\right) \quad \text{and} \quad X_i^h = \frac{\partial}{\partial x^i} - G^l_i\frac{\partial}{\partial y^l}, \quad \text{where } G^l_i = \frac{\partial G^l}{\partial y^i}.$$

The second Cartan tensor (Landsberg tensor) and the mixed curvature are given by

$$P^l_{ij} = \frac{1}{2}g^{lm}(X^h_i g_{jm} - G^k_{ij}g_{km} - G^k_{im}g_{jk}), \quad \text{where } G^l_{ij} = \frac{\partial G^l_i}{\partial y^j}$$

and $P^l_{ijk} = -G^l_{ijk}$, where $G^l_{ijk} = \frac{\partial G^l_{ij}}{\partial y^k}$.

Lemma 1. [7], section 6.2.

$$P^l_{ij} = -\frac{F}{2}l_m g^{kl}P^m_{ijk} \tag{2}$$

1.2 Generalized Berwald manifolds

Definition 1. A linear connection ∇ on the base manifold M is called *compatible* to the Finslerian metric if the parallel transports with respect to ∇ preserve the Finslerian length of tangent vectors. Finsler manifolds admitting compatible linear connections are called generalized Berwald manifolds.

Proposition 1. *A linear connection ∇ on the base manifold M is* compatible *to the Finslerian metric function if and only if the induced horizontal distribution is conservative, i.e. the derivatives of the fundamental function F vanish along the horizontal directions with respect to ∇.*

Proof. Suppose that the parallel transports with respect to ∇ (a linear connection on the base manifold) preserve the Finslerian length of tangent vectors and let X be a parallel vector field along the curve $c\colon [0,1] \to M$:

$$(x^k \circ X)' = c^{k'} \quad \text{and} \quad (y^k \circ X)' = X^{k'} = -c^{i'}X^j\Gamma^k_{ij} \circ c \tag{3}$$

because of the differential equation for parallel vector fields. If F is the Finslerian fundamental function then

$$(F \circ X)' = (x^k \circ X)' \frac{\partial F}{\partial x^k} \circ X + (y^k \circ X)' \frac{\partial F}{\partial y^k} \circ X \tag{4}$$

and, by formula (3),

$$(F \circ X)' = c^{i\prime} \Big(\frac{\partial F}{\partial x^i} - y^j \Gamma_{ij}^k \circ \pi \frac{\partial F}{\partial y^k} \Big) \circ X. \tag{5}$$

This means that the parallel transports with respect to ∇ preserve the Finslerian length of tangent vectors (compatibility condition) if and only if

$$\frac{\partial F}{\partial x^i} - y^j \Gamma_{ij}^k \circ \pi \frac{\partial F}{\partial y^k} = 0 \quad (i = 1, \ldots, n), \tag{6}$$

where the vector fields of type

$$\frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \circ \pi \frac{\partial}{\partial y^k} \tag{7}$$

span the associated horizontal distribution belonging to ∇. □

Theorem 1. *[10] If a linear connection on the base manifold is compatible with the Finslerian metric function then it must be metrical with respect to the averaged Riemannian metric*

$$\gamma_p(v, w) := \int_{\partial K_p} g(v, w)\, \mu = v^i w^j \int_{\partial K_p} g_{ij}\, \mu \quad (v, w \in T_pM,\ p \in U). \tag{8}$$

Remark 1. The technic of averaging is an alternative way to solve the problem of the characterization of compatible linear connections. By the fundamental result of the theory [10] such a linear connection must be metrical with respect to the averaged Riemannian metric given by integration of the Riemann-Finsler metric on the indicatrix hypersurfaces (see Theorem 1). Therefore the linear connection is uniquely determined by its torsion tensor. The torsion tensor has a special decomposition in 2D because of

$$T(X, Y) = (X^1 Y^2 - X^2 Y^1) \Big(T_{12}^1 \frac{\partial}{\partial u^1} + T_{12}^2 \frac{\partial}{\partial u^2} \Big) = \rho(X) Y - \rho(Y) X, \tag{9}$$

where $\rho_1 = T_{12}^2$ and $\rho_2 = -T_{12}^1 = T_{21}^1$. In higher dimensional spaces such a linear connection is called semi-symmetric. Using some previous results [11], [12], [13] and [14], the torsion tensor of a semi-symmetric compatible linear connection can be expressed in terms of metrics and differential forms given by averaging independently of the dimension of the space. The basic idea is the comparison of ∇ with the Lévi-Civita connection of the averaged metric (cf. subsection 2.1.)

Especially, the compatible linear connection must be of zero curvature in 2D unless the manifold is Riemannian, see [15] and [16]. Therefore we can conclude some topological obstructions as well due to the divergence representation of the Gauss curvature [16]: any compact generalized Berwald surface without boundary must have zero Euler characteristic. Therefore the Euclidean sphere does not carry such a geometric structure. Using the theory of closed Wagner manifolds, this means that the local conformal flatness of the Riemannian surfaces is taking to fail in the differential geometry of non-Riemannian Finsler surfaces [16].

1.3 Finsler surfaces

In case of Finsler surfaces it is typical to introduce the vector field

$$V := \frac{\partial F}{\partial y^1}\frac{\partial}{\partial y^2} - \frac{\partial F}{\partial y^2}\frac{\partial}{\partial y^1}.$$

It is tangential to the indicatrix curve because of $VF = 0$. Since three vertical vector fields must be linearly dependent in 2D,

$$\begin{aligned}
0 &= \det\begin{pmatrix} g\left(\frac{\partial}{\partial y^1},\frac{\partial}{\partial y^1}\right) & g\left(\frac{\partial}{\partial y^1},\frac{\partial}{\partial y^2}\right) & g\left(\frac{\partial}{\partial y^1},C\right) \\ g\left(\frac{\partial}{\partial y^2},\frac{\partial}{\partial y^1}\right) & g\left(\frac{\partial}{\partial y^2},\frac{\partial}{\partial y^2}\right) & g\left(\frac{\partial}{\partial y^2},C\right) \\ g\left(C,\frac{\partial}{\partial y^1}\right) & g\left(C,\frac{\partial}{\partial y^2}\right) & g(C,C) \end{pmatrix} = \det\begin{pmatrix} g_{11} & g_{12} & \frac{\partial E}{\partial y^1} \\ g_{12} & g_{22} & \frac{\partial E}{\partial y^2} \\ \frac{\partial E}{\partial y^1} & \frac{\partial E}{\partial y^2} & 2E \end{pmatrix} \\
&= F^2 \det g_{ij} + 2g_{12}\frac{\partial E}{\partial y^1}\frac{\partial E}{\partial y^2} - \left(\frac{\partial E}{\partial y^1}\right)^2 g_{22} - \left(\frac{\partial E}{\partial y^2}\right)^2 g_{11} \\
&= F^2(\det g_{ij} - g(V,V)).
\end{aligned}$$

This means that $0 \neq \det g_{ij} = g(V,V)$ and, consequently,

$$V_0 := \frac{1}{\sqrt{g(V,V)}}V, \qquad C_0 := \frac{1}{F}C,$$

$$V_0^h := V_0^i X_i^h = V_0^i\left(\frac{\partial}{\partial x^i} - G_i^l\frac{\partial}{\partial y^l}\right), \qquad S_0 := \frac{1}{F}S = \frac{y^i}{F}X_i^h$$

form a local frame on the complement of the zero section in $\pi^{-1}(U)$. Such a collection of vector fields is called a *Berwald frame*.

Definition 2. The main scalar of a Finsler surface is defined as $\lambda := V_0^j V_0^k V_0^l \mathcal{C}_{jkl}$, where $V_0 = V/\sqrt{g(V,V)}$ is the unit tangential vector field to the indicatrix curve.

The vanishing of the main scalar implies that the surface is Riemannian and vice versa. The zero homogeneous version $I := F\lambda$ is also frequently used in the literature [3], [4], [5] and [6]. Consider the vector field $\mathcal{C}_{ij}^k \partial/\partial y^k$. Since it is also tangential to the indicatrix it follows that

$$\mathcal{C}_{ij}^k\frac{\partial}{\partial y^k} = \mathcal{C}_{ij}^l g\left(V_0, \frac{\partial}{\partial y^l}\right)V_0,$$

where $V_0 = V/\sqrt{g(V,V)}$ is the unit tangential vector field to the indicatrix curve. Therefore

$$\mathcal{C}_{ij}^k = \mathcal{C}_{ij}^l g_{lm} V_0^m V_0^k = V_0^m \mathcal{C}_{ijm} V_0^k \quad \Rightarrow \quad \mathcal{C}_{ijr} = V_0^m \mathcal{C}_{ijm} V_0^k g_{kr}.$$

Contracting by g^{jr}

$$\mathcal{C}_i = V_0^j V_0^m \mathcal{C}_{ijm}. \tag{10}$$

By formulas (1) and (10) we have that

$$\lambda := V_0^j V_0^k V_0^l \mathcal{C}_{jkl} = V_0^j \mathcal{C}_j = V_0\big(\ln\sqrt{\det g_{rs}}\big). \tag{11}$$

In what follows we summarize some of the general formulas to express the surviving components of the Landsberg tensor, the mixed curvature tensor and the pairwise Lie-brackets of a Berwald frame (Cartan's permutation formulas) [8]:

$$y^i V_0^j V_0^k P_{ijk} = y^i V_0^j V_0^k G^l_{ijk} g\Big(V_0, \frac{\partial}{\partial y^l}\Big) = 0, \tag{12}$$

$$V_0^i V_0^j V_0^k P_{ijk} = -S(\lambda), \quad V_0^i V_0^j V_0^k G^l_{ijk} g\Big(V_0, \frac{\partial}{\partial y^l}\Big) = V_0^h(\lambda) + V_0(S\lambda)$$

because of the homogeneity properties; see [8, Corollary 1.8] and [8, Formula (24a)]. E. Cartan's permutation formulas are

$$[V_0, V_0^h] = -\frac{1}{F}S_0 - \lambda V_0^h - S(\lambda)V_0, \quad [S_0, V_0] = -\frac{1}{F}V_0^h, \quad [V_0^h, S_0] = -\kappa V_0, \tag{13}$$

where κ is the only surviving coefficient of the curvature of the horizontal distribution [8, Theorem 1.10]. Let the indicatrix curve in T_pM be parameterized as the integral curve of V_0:

$$V_0 \circ c_p(\theta) = c_p'(\theta) \quad \Rightarrow \quad \lambda \circ c_p(\theta) = \big(\ln\sqrt{\det g_{rs}} \circ c_p\big)'(\theta).$$

It is called the *central affine arcwise parametrization of the indicatrix curve*. The parameter θ is "the central affine length of the arc of the indicatrix" and the main scalar can be interpreted as its "central affine curvature"; for the citations see [17].

2 Two-dimensional generalized Berwald manifolds

Let ∇ be a linear connection on the base manifold M and suppose that the parallel transports preserve the Finslerian length of tangent vectors (compatibility condition). By Proposition 1,

$$\frac{\partial E}{\partial x^i} - y^m \Gamma^l_{im} \circ \pi \frac{\partial E}{\partial y^l} = 0 \quad (i = 1, 2), \quad \text{where} \quad E = \frac{1}{2}F^2$$

is the Finslerian energy.

2.1 The comparison of ∇ with the canonical horizontal distribution of the Finsler manifold

Using the canonical horizontal sections we can write that

$$y^m \Gamma^l_{im} \circ \pi \frac{\partial E}{\partial y^l} - G^l_i \frac{\partial E}{\partial y^l} = 0.$$

Since the vertical vector fields are the linear combinations of V and C, it follows that

$$y^m\Gamma^l_{im}\circ\pi\frac{\partial}{\partial y^l}-G^l_i\frac{\partial}{\partial y^l}=f_iV+g_iC\quad(i=1,2);$$

the coefficients f_1, f_2 are positively homogeneous of degree one, g_1 and g_2 are positively homogeneous of degree zero. Taking into account that $VE=0$ and $CE=2E$, we have that $g_1=g_2=0$ and, consequently,

$$y^m\Gamma^l_{im}\circ\pi\frac{\partial}{\partial y^l}-G^l_i\frac{\partial}{\partial y^l}=f_iV \Rightarrow \quad y^m\Gamma^k_{im}\circ\pi\frac{\partial}{\partial y^k}=G^k_i\frac{\partial}{\partial y^k}+f_iV\quad(i=1,2). \tag{14}$$

To provide the linearity of the right hand side we should take the Lie brackets with the vertical coordinate vector fields two times:

$$\begin{aligned}0&=\left[\left[y^m\Gamma^l_{im}\circ\pi\frac{\partial}{\partial y^l},\frac{\partial}{\partial y^j}\right],\frac{\partial}{\partial y^k}\right]=\left[\left[G^l_i\frac{\partial}{\partial y^l},\frac{\partial}{\partial y^j}\right],\frac{\partial}{\partial y^k}\right]+\left[\left[f_iV,\frac{\partial}{\partial y^j}\right],\frac{\partial}{\partial y^k}\right]\\&=G^l_{ijk}\frac{\partial}{\partial y^l}+f_i\left[\left[V,\frac{\partial}{\partial y^j}\right],\frac{\partial}{\partial y^k}\right]-\frac{\partial f_i}{\partial y^j}\left[V,\frac{\partial}{\partial y^k}\right]-\frac{\partial f_i}{\partial y^k}\left[V,\frac{\partial}{\partial y^j}\right]+\frac{\partial^2 f_i}{\partial y^j\partial y^k}V\\&=:W_{ijk},\end{aligned}$$

where

$$\begin{aligned}\left[V,\frac{\partial}{\partial y^j}\right]&=\frac{\partial^2F}{\partial y^j\partial y^2}\frac{\partial}{\partial y^1}-\frac{\partial^2F}{\partial y^j\partial y^1}\frac{\partial}{\partial y^2},\\ \left[\left[V,\frac{\partial}{\partial y^j}\right],\frac{\partial}{\partial y^k}\right]&=-\frac{\partial^3F}{\partial y^j\partial y^k\partial y^2}\frac{\partial}{\partial y^1}+\frac{\partial^3F}{\partial y^j\partial y^k\partial y^1}\frac{\partial}{\partial y^2}.\end{aligned}$$

Since $y^jW_{ijk}=y^kW_{ijk}=0$ it is enough to investigate the quantity $W_i=V^jV^kW_{ijk}$. By some direct computations

$$V^j\frac{\partial^2F}{\partial y^j\partial y^2}=V\Big(\frac{\partial F}{\partial y^2}\Big)=\frac{1}{F}V\Big(F\frac{\partial F}{\partial y^2}\Big)=\frac{1}{F}g\Big(V,\frac{\partial}{\partial y^2}\Big)$$

because of $VF=0$. On the other hand

$$\begin{aligned}V^jV^k\frac{\partial^3F}{\partial y^j\partial y^k\partial y^2}&=\frac{1}{F}V^kV\Big(F\frac{\partial^2F}{\partial y^k\partial y^2}\Big)=\frac{1}{F}V^kV\Big(g_{k2}-\frac{\partial F}{\partial y^k}\frac{\partial F}{\partial y^2}\Big)\\&=\frac{1}{F}\Big(2V^jV^k\mathcal{C}_{jk2}-V^kV\Big(\frac{\partial F}{\partial y^k}\Big)\frac{\partial F}{\partial y^2}\Big)\\&=\frac{1}{F}\Big(2V^jV^k\mathcal{C}_{jk2}-\frac{1}{F}V^kV\Big(F\frac{\partial F}{\partial y^k}\Big)\frac{\partial F}{\partial y^2}\Big)\\&=\frac{1}{F}\Big(2V^jV^k\mathcal{C}_{jk2}-\frac{1}{F}g(V,V)\frac{\partial F}{\partial y^2}\Big)\end{aligned}$$

and, consequently,

$$W_i = V^j V^k G^l_{ijk} \frac{\partial}{\partial y^l} - \frac{2V(f_i)}{F}\Big(g\Big(V, \frac{\partial}{\partial y^2}\Big)\frac{\partial}{\partial y^1} - g\Big(V, \frac{\partial}{\partial y^1}\Big)\frac{\partial}{\partial y^2}\Big)$$
$$- \frac{f_i}{F}\Big(\Big(2V^jV^k\mathcal{C}_{jk2} - \frac{1}{F}g(V,V)\frac{\partial F}{\partial y^2}\Big)\frac{\partial}{\partial y^1} - \Big(2V^jV^k\mathcal{C}_{jk1} - \frac{1}{F}g(V,V)\frac{\partial F}{\partial y^1}\Big)\frac{\partial}{\partial y^2}\Big)$$
$$+ V^jV^k\frac{\partial^2 f_i}{\partial y^j \partial y^k}V. \quad (15)$$

The vanishing of W_i is equivalent to

$$g(W_i, V_0) = 0 \quad \text{and} \quad g(W_i, C_0) = 0 \quad (i = 1, 2),$$

where $V_0 = V/\sqrt{g(V,V)}$ and $C_0 = C/F$ are the normalized vector fields of the vertical Berwald frame.

2.1.1 The vanishing of the orthogonal term to the indicatrix

It follows that

$$0 = g(W_i, C) = W_iE = FV^jV^kG^l_{ijk}\frac{\partial F}{\partial y^l} - 2V(f_i)g(V,V) - 2f_iV^jV^kV^l\mathcal{C}_{jkl}.$$

Therefore

$$\frac{\alpha_i}{\sqrt{g(V,V)}} = \lambda f_i + (V_0 f_i) \quad (i = 1, 2), \tag{16}$$

where $V_0 = V/\sqrt{g(V,V)}$ is the unit tangential vector field to the indicatrix curve, λ is the main scalar and

$$\alpha_i = \frac{1}{2}FV_0^jV_0^kG^l_{ijk}\frac{\partial F}{\partial y^l} \overset{(2)}{=} V_0^jV_0^kP_{ijk}.$$

Using that $\det g_{ij} = g(V,V)$, formula (11) says that

$$\alpha_i = V_0\big(f_i\sqrt{g(V,V)}\big) \quad (i = 1, 2). \tag{17}$$

Let the indicatrix curve c_p in T_pM be parameterized as the integral curve of V_0. Evaluating along c_p we have

$$\alpha_i \circ c_p(\theta) = (f_i \circ c_p\sqrt{g(V,V)} \circ c_p)'(\theta) \quad (i = 1, 2) \tag{18}$$

for any $p \in U$. Therefore

$$\beta_i \circ c_p(t) = f_i \circ c_p(t)\sqrt{g(V,V)} \circ c_p(t) - f_i \circ c_p(0)\sqrt{g(V,V)} \circ c_p(0), \tag{19}$$

where $\beta_i \colon \pi^{-1}(U) \to \mathbb{R}$ $(i = 1, 2)$ are the 1-homogeneous extensions of the functions defined by

$$\beta_i \circ c_p(t) = \int_0^t \alpha_i \circ c_p(\theta)\, d\theta \quad (i = 1, 2) \tag{20}$$

along the central affine arcwise parametrization of the indicatrix curve. We can write that

$$f_i \circ c_p(t) = \frac{1}{\sqrt{g(V,V) \circ c_p(t)}}(\beta_i \circ c_p(t) + k_i(p)) \quad (i = 1, 2) \tag{21}$$

for some constants $k_i(p)$ $(i = 1, 2)$ depending only on the position.

2.1.2 The vanishing of the tangential term to the indicatrix

It follows that

$$\begin{aligned}
0 &= g(W_i, V)\\
&= V^jV^kG^l_{ijk}g\Big(V, \frac{\partial}{\partial y^l}\Big) - \frac{2f_i}{F}\Big(V^jV^k\mathcal{C}_{jk2}g\Big(V, \frac{\partial}{\partial y^1}\Big) - V^jV^k\mathcal{C}_{jk1}g\Big(V, \frac{\partial}{\partial y^2}\Big)\Big)\\
&\quad + \frac{f_i}{F^2}g(V,V)\Big(\frac{\partial F}{\partial y^2}g\Big(V, \frac{\partial}{\partial y^1}\Big) - \frac{\partial F}{\partial y^1}g\Big(V, \frac{\partial}{\partial y^2}\Big)\Big) + V^jV^k\frac{\partial^2 f_i}{\partial y^j\partial y^k}g(V,V)\\
&= V^jV^kG^l_{ijk}g\Big(V, \frac{\partial}{\partial y^l}\Big) + V^jV^k\frac{\partial^2 f_i}{\partial y^j\partial y^k}g(V,V) - \frac{f_i}{F^2}g^2(V,V)\\
&\quad - \frac{2f_i}{F}\Big(V^jV^k\mathcal{C}_{jk2}g\Big(V, \frac{\partial}{\partial y^1}\Big) - V^jV^k\mathcal{C}_{jk1}g\Big(V, \frac{\partial}{\partial y^2}\Big)\Big),
\end{aligned}$$

where

$$V^jV^k\mathcal{C}_{jk2}g\Big(V, \frac{\partial}{\partial y^1}\Big) - V^jV^k\mathcal{C}_{jk1}g\Big(V, \frac{\partial}{\partial y^2}\Big) = 0$$

because the vector field

$$Z := g\Big(V, \frac{\partial}{\partial y^1}\Big)\frac{\partial}{\partial y^2} - g\Big(V, \frac{\partial}{\partial y^2}\Big)\frac{\partial}{\partial y^1}$$

is parallel to C, i.e. $g(V,Z) = 0$. Therefore

$$0 = V^jV^kG^l_{ijk}g\Big(V, \frac{\partial}{\partial y^l}\Big) + V^jV^k\frac{\partial^2 f_i}{\partial y^j\partial y^k}g(V,V) - \frac{f_i}{F^2}g^2(V,V)$$

and, consequently,

$$0 = V_0^jV_0^kG^l_{ijk}g\Big(V_0, \frac{\partial}{\partial y^l}\Big) + V_0^jV_0^k\frac{\partial^2 f_i}{\partial y^j\partial y^k}\sqrt{g(V,V)} - \frac{f_i}{F^2}\sqrt{g(V,V)}. \tag{22}$$

Lemma 2. *If g is a positively homogeneous function of degree k, then*

$$V_0(V_0^k)\frac{\partial g}{\partial y^k} = -\lambda V_0(g) - k\frac{g}{F^2}. \tag{23}$$

Especially,

$$V_0^jV_0^k\frac{\partial^2 f_i}{\partial y^j\partial y^k} = V_0(V_0 f_i) + \lambda V_0(f_i) + \frac{f_i}{F^2}. \tag{24}$$

Proof. Let c_p be the parametrization of the indicatrix curve in T_pM as the integral curve of V_0, i.e. $V_0 \circ c_p = c_p'$. Differentiating equation

$$1 = g_{c_p}(V_0\circ c_p, V_0\circ c_p) = g_{ij}\circ c_p(c_p^i)'(c_p^j)' \tag{25}$$

we have that $0 = 2g_{ij}\circ c_p(c_p^i)''(c_p^j)' + 2\mathcal{C}_{ijk}\circ c_p(c_p^i)'(c_p^j)'(c_p^k)'$ and, consequently,

$$\begin{aligned}
g_{c_p}(V_0\circ c_p, c_p'') &= g_{c_p}(c_p', c_p'') = -\mathcal{C}_{ijk}\circ c_p(c_p^i)'(c_p^j)'(c_p^k)'\\
&= -(V_0^iV_0^jV_0^k\mathcal{C}_{ijk})\circ c_p = -\lambda\circ c_p.
\end{aligned} \tag{26}$$

Differentiating equation

$$0 = g_{c_p}(C \circ c_p, V_0 \circ c_p) = g_{ij} \circ c_p (c_p^i)(c_p^j)' \tag{27}$$

we have that

$$0 = 2\mathcal{C}_{ijk} \circ c_p (c_p^i)(c_p^j)'(c_p^k)' + g_{ij} \circ c_p (c_p^i)'(c_p^j)' + g_{ij} \circ c_p (c_p)^i (c_p^j)''. \tag{28}$$

Taking into account that $\mathcal{C}_{ijk} \circ c_p (c_p^i)(c_p^j)'(c_p^k)' = \mathcal{C}_{ijk} \circ c_p (y^i \circ c_p)(c_p^j)'(c_p^k)' = 0$,

$$g_{ij} \circ c_p (c_p^i)'(c_p^j)' = g_{c_p}(c_p', c_p') = 1 \quad \text{and} \quad g_{ij} \circ c_p (c_p)^i (c_p^j)'' = g_{c_p}(C \circ c_p, c_p''),$$

it follows that

$$g_{c_p}(C_0 \circ c_p, c_p'') = -\frac{1}{F \circ c_p}, \tag{29}$$

where $C_0 := C/F$ is the normalized Liouville vector field. From (26) and (29)

$$c_p'' = -(\lambda V_0) \circ c_p - \frac{1}{F \circ c_p} C_0 \circ c_p. \tag{30}$$

This means that

$$\begin{aligned}
\Big(V_0(V_0^k)\frac{\partial g}{\partial y^k}\Big) \circ c_p &= (V_0^k \circ c_p)' \frac{\partial g}{\partial y^k} \circ c_p = (c_p^k)'' \frac{\partial g}{\partial y^k} \circ c_p \\
&\overset{(30)}{=} -\Big((\lambda V_0^k) \circ c_p + \frac{1}{F \circ c_p} C_0^k \circ c_p\Big) \frac{\partial g}{\partial y^k} \circ c_p \\
&= -(\lambda V_0 g) \circ c_p - \frac{1}{F^2 \circ c_p} (Cg) \circ c_p,
\end{aligned}$$

where $Cg = kg$ because of the homogeneity. Note that the terms $V_0(V_0^k)\partial g/\partial y^k$, $\lambda V_0 g$ and g/F^2 are of the same degree of homogeneity, i.e. they are homogeneous of degree $k-2$. Therefore the equality along the indicatrix curve implies (23). Especially,

$$V_0(V_0 f_i) = V_0^j V_0^k \frac{\partial^2 f_i}{\partial y^j \partial y^k} + V_0(V_0^k) \frac{\partial f_i}{\partial y^k} = V_0^j V_0^k \frac{\partial^2 f_i}{\partial y^j \partial y^k} - \lambda V_0(f_i) - \frac{f_i}{F^2}$$

as was to be proved. □

Using Lemma 2 we can write formula (22) into the form

$$0 = \omega_i + (V_0(V_0 f_i))\sqrt{g(V,V)} + \lambda V_0(f_i)\sqrt{g(V,V)},$$

where

$$\omega_i = V_0^j V_0^k G_{ijk}^l g\Big(V_0, \frac{\partial}{\partial y^l}\Big) \quad (i = 1, 2).$$

By formula (11)

$$0 = \omega_i + (V_0(V_0 f_i))\sqrt{g(V,V)} + V_0(f_i) V_0\big(\sqrt{g(V,V)}\big) \quad (i = 1, 2)$$

because of $\det g_{ij} = g(V,V)$. Therefore

$$0 = \omega_i + V_0\big((V_0 f_i)\sqrt{g(V,V)}\big),$$

$$0 = \omega_i + V_0\Big(V_0\big(f_i\sqrt{g(V,V)}\big) - f_i V_0\big(\sqrt{g(V,V)}\big)\Big),$$

$$0 = \omega_i + V_0\Big(V_0\big(f_i\sqrt{g(V,V)}\big) - \lambda f_i\sqrt{g(V,V)}\Big),$$

$$0 = \omega_i + V_0\Big(V_0\big(f_i\sqrt{g(V,V)}\big)\Big) - V_0(\lambda) f_i\sqrt{g(V,V)} - \lambda V_0\big(f_i\sqrt{g(V,V)}\big) \quad (i = 1,2).$$

By formula (17)

$$0 = \omega_i + V_0(\alpha_i) - V_0(\lambda) f_i\sqrt{g(V,V)} - \lambda\alpha_i \quad (i = 1,2). \tag{31}$$

Evaluating formula (31) along c_p

$$\omega_i \circ c_p(t) + (\alpha_i \circ c_p)'(t) \\ = (\beta_i \circ c_p(t) + k_i(p))(\lambda \circ c_p)'(t) + \lambda \circ c_p(t)\alpha_i \circ c_p(t) \quad (i = 1,2) \tag{32}$$

because of (20) and (21). The constants $k_1(p)$ and $k_2(p)$ of integration can be expressed by (32) provided that $\lambda \circ c_p$ is not a constant function:

$$k_i(p) = \frac{\gamma_i \circ c_p(s) - \beta_i \circ c_p(s)\lambda \circ c_p(s) + \alpha_i \circ c_p(s) - \alpha_i \circ c_p(0)}{\lambda \circ c_p(s) - \lambda \circ c_p(0)},$$

where

$$\gamma_i \circ c_p(s) = \int_0^s \omega_i \circ c_p(t)\, dt \quad (i = 1,2)$$

and the parameter $s \in \mathbb{R}$ is choosen such that $\lambda \circ c_p(s) - \lambda \circ c_p(0) \neq 0$. Otherwise the function $\lambda \circ c_p$ is constant. Since $\det g_{ij}$ attains its extremals along the indicatrix curve, formula (11) shows that $\lambda \circ c_p$ is identically zero and the indicatrix is a quadratic curve in T_pM. The quadratic indicatrix curve of a (connected) generalized Berwald manifold at a single point implies that the indicatrices are quadratic curves at any point and we have a Riemannian surface. Indeed, the parallel transports induced by the compatible linear connection take a quadratic curve into quadratic curves.[1]

Theorem 2. *The compatible linear connection of a non-Riemannian connected generalized Berwald surface must be of the local form*

$$\Gamma^1_{ij} \circ \pi = G^1_{ij} - \frac{\partial f_i}{\partial y^j}\frac{\partial F}{\partial y^2} - f_i\frac{\partial^2 F}{\partial y^j \partial y^2},$$

$$\Gamma^2_{ij} \circ \pi = G^2_{ij} + \frac{\partial f_i}{\partial y^j}\frac{\partial F}{\partial y^1} + f_i\frac{\partial^2 F}{\partial y^j \partial y^1} \qquad (i,j = 1,2),$$

[1] Non-Riemannian Finsler surfaces with main scalar depending only on the position must be singular; see Berwald's original list [3, Formulas 118 I-III], see also [4] and [9].

where the 1*-homogeneous functions* f_1*,* f_2 *are given by*

$$f_i \circ c_p(t) = \frac{1}{\sqrt{g(V,V) \circ c_p(t)}} \Big(\int_0^t \alpha_i \circ c_p(\theta)\, d\theta + k_i(p) \Big) \quad (i = 1, 2)$$

and the integration constants satisfy equations

$$\begin{aligned} &\omega_i \circ c_p(t) + (\alpha_i \circ c_p)'(t) \\ &\quad = \Big(\int_0^t \alpha_i \circ c_p(\theta)\, d\theta + k_i(p) \Big) (\lambda \circ c_p)'(t) + \lambda \circ c_p(t) \alpha_i \circ c_p(t) \quad (i = 1, 2) \end{aligned}$$

for any $p \in \pi^{-1}(U)$*.*

Proof. Equations for the functions f_1 and f_2 imply that $g(W_i, C) = 0$ because of subsection 2.1.1. Equations for the integration constants imply that $g(W_i, V_0) = 0$ because of subsection 2.1.2. Therefore $W_i = 0$ and we have a generalized Berwald surface. The explicit formulas for the coefficients of the linear connection preserving the Finslerian length of tangent vectors are

$$\begin{aligned} \Gamma^1_{ij} \circ \pi &= G^1_{ij} + \frac{\partial f_i}{\partial y^j} V^1 + f_i \frac{\partial V^1}{\partial y^j}, \\ \Gamma^2_{ij} \circ \pi &= G^2_{ij} + \frac{\partial f_i}{\partial y^j} V^2 + f_i \frac{\partial V^2}{\partial y^j} \qquad (i, j = 1, 2), \end{aligned}$$

because of formula (14). □

Note that the functions f_1 and f_2 are uniquely determined by their restrictions to the indicatrix because the 1-homogeneous extension is unique. In case of a Riemannian manifold, f_1 and f_2 are of the form $k_1(p)F$ and $k_2(p)F$, where $k_1(p)$ and $k_2(p)$ are arbitrary constants (cf. ρ_1 and ρ_2 in Remark 1).

Corollary 1. *The compatible linear connection of a non-Riemannian generalized Berwalds surface is uniquely determined.*

Proof. Recall that the constants $k_1(p)$ and $k_2(p)$ of integration can be expressed by (32) provided that $\lambda \circ c_p$ is not a constant function. □

2.2 An application: Landsberg and generalized Berwald surfaces

Definition 3. A Finsler manifold is called a Landsberg manifold if the Landsberg tensor of the canonical horizontal distribution vanishes. The Berwald manifolds are defined by the vanishing of the mixed curvature tensor of the canonical horizontal distribution.

Formula (2) implies that any Berwald manifold is a Landsberg manifold. The converse of this statement is the famous Unicorn problem in Finsler geometry [1].

Theorem 3. *A connected generalized Berwald surface is a Landsberg surface if and only if it is a Berwald surface.*

Proof. Suppose that we have a connected two-dimensional generalized Berwald manifold such that the Landsberg tensor vanishes, i.e. $\alpha_i = 0$ $(i = 1, 2)$. Then (21) implies that

$$f_i\sqrt{g(V,V)} = k_i(p)F.$$

On the other hand

$$\omega_i - V_0(\lambda)f_i\sqrt{g(V,V)} = 0$$

due to (31). Contracting by y^i, (12) says that

$$V_0(\lambda)y^i k_i(p) = 0. \tag{33}$$

If $k_1^2(p) + k_2^2(p) \neq 0$, then $y^1k_1(p) + y^2k_2(p) = 0$ is an equation of a line in T_pM. Therefore, there are at most two positions along ∂K_p such that

$$v^1k_1(p) + v^2k_2(p) = 0.$$

Otherwise $V_0(v)\lambda = 0$ because of (33). A continuity argument says that $V_0(v)\lambda = 0$ for any $v \in T_pM$, i.e. λ is constant along c_p. Since $\det g_{ij}$ attains its extremals along the indicatrix curve, formula (11) shows that $\lambda \circ c_p = 0$. This means that the indicatrix is a quadratic curve in T_pM. The quadratic indicatrix curve of a (connected) generalized Berwald surface at a single point implies that the indicatrices are quadratic curves at any point due to the compatible linear connection and the induced linear mapping between the tangent spaces. Therefore we have a Riemannian surface. Otherwise $k_1(p) = k_2(p) = 0$ for any $p \in M$, i.e. $f_i = 0$ $(i = 1, 2)$ and the compatible linear connection must be the canonical one. Therefore we have a Berwald manifold of dimension two. □

2.3 Wagner's equations

To present Wagner's equations in [17] we need the following simple observation:

$$H_i = 0 \quad (i = 1, 2) \quad \text{if and only if} \quad y^iH_i = 0 \quad \text{and} \quad V_0^iH_i = 0$$

because of

$$\det\begin{pmatrix} y^1 & y^2 \\ V_0^1 & V_0^2 \end{pmatrix} = y^1V_0^2 - y^2V_0^1 = \frac{F}{\sqrt{g(V,V)}} \neq 0.$$

Contracting (31) by y^i

$$0 = y^iV_0(\alpha_i) - V_0(\lambda)y^if_i\sqrt{g(V,V)},$$

where $y^iV_0(\alpha_i) = V_0(y^i\alpha_i) - V_0^i\alpha_i \overset{(12)}{=} S(\lambda)$ and, consequently,

$$S(\lambda) = V_0(\lambda)y^if_i\sqrt{g(V,V)}. \tag{34}$$

Contracting (31) by V_0^i

$$0 \overset{(12)}{=} V_0^h\lambda + V_0(S\lambda) + V_0^iV_0(\alpha_i) - V_0(\lambda)V_0^if_i\sqrt{g(V,V)} + \lambda S(\lambda),$$

where

$$V_0^iV_0(\alpha_i)=V_0(V_0^i\alpha_i)-V_0(V_0^i)\alpha_i\overset{(12)}{=}-V_0(S\lambda)-V_0(V_0^i)\alpha_i.$$

Since $V_0(V_0^i)\circ c_p=(c_p^i)''$ it follows, by formula (30), that

$$V_0(V_0^i)=-\lambda V_0^i-\frac{y^i}{F^2} \tag{35}$$

due to the -1-homogeneous extension. Therefore

$$V_0^iV_0(\alpha_i)=-V_0(S\lambda)-V_0(V_0^i)\alpha_i\overset{(12),(35)}{=}-V_0(S\lambda)-\lambda S(\lambda).$$

Finally we have

$$V_0^h(\lambda)=V_0(\lambda)V_0^if_i\sqrt{g(V,V)}. \tag{36}$$

Equations (34) and (36) are equivalent to (31). Differentiating (34) along the indicatrix curve

$$V_0(S\lambda)=[V_0,S](\lambda)+S(V_0\lambda)\overset{(13)}{=}V_0^h(\lambda)+S(V_0\lambda),$$

$$\begin{aligned}V_0\big(V_0(\lambda)y^if_i\sqrt{g(V,V)}\big)&\\ &\overset{(17)}{=}V_0(V_0\lambda)y^if_i\sqrt{g(V,V)}+V_0(\lambda)V_0^if_i\sqrt{g(V,V)}+V_0(\lambda)y^i\alpha_i\\ &\overset{(12)}{=}V_0(V_0\lambda)y^if_i\sqrt{g(V,V)}+V_0(\lambda)V_0^if_i\sqrt{g(V,V)}\end{aligned}$$

and, consequently,

$$\begin{aligned}V_0(\lambda)V_0^h(\lambda)+V_0(\lambda)S(V_0\lambda)&\\ &=V_0(V_0\lambda)V_0(\lambda)y^if_i\sqrt{g(V,V)}+V_0(\lambda)V_0(\lambda)V_0^if_i\sqrt{g(V,V)}\\ &\overset{(34),(36)}{=}V_0(V_0\lambda)S(\lambda)+V_0(\lambda)V_0^h(\lambda),\end{aligned}$$

i.e.

$$V_0(\lambda)S(V_0\lambda)=V_0(V_0\lambda)S(\lambda). \tag{37}$$

In a similar way, differentiating (36) along the indicatrix curve

$$V_0(V_0^h\lambda)=[V_0,V_0^h]\lambda+V_0^h(V_0\lambda)\overset{(13)}{=}-\frac{1}{F}S_0(\lambda)-\lambda V_0^h(\lambda)-S(\lambda)V_0(\lambda)+V_0^h(V_0\lambda),$$

$$\begin{aligned}V_0\big(V_0(\lambda)V_0^if_i\sqrt{g(V,V)}\big)&\\ &\overset{(17)}{=}V_0(V_0\lambda)V_0^if_i\sqrt{g(V,V)}+V_0(\lambda)V_0(V_0^i)f_i\sqrt{g(V,V)}+V_0(\lambda)V_0^i\alpha_i\\ &\overset{(12),(35)}{=}V_0(V_0\lambda)V_0^if_i\sqrt{g(V,V)}-\lambda V_0(\lambda)V_0^if_i\sqrt{g(V,V)}\\ &\qquad-V_0(\lambda)\frac{y^i}{F^2}f_i\sqrt{g(V,V)}-V_0(\lambda)S(\lambda)\end{aligned}$$

and, consequently,

$$
\begin{aligned}
-V_0(\lambda)\Big(\frac{1}{F}S_0(\lambda)+\lambda V_0^h(\lambda)+S(\lambda)V_0(\lambda)-V_0^h(V_0\lambda)\Big)&\\
=V_0(V_0\lambda)V_0(\lambda)V_0^i f_i\sqrt{g(V,V)}-\lambda V_0(\lambda)V_0(\lambda)V_0^i f_i\sqrt{g(V,V)}&\\
-V_0(\lambda)V_0(\lambda)\frac{y^i}{F^2}f_i\sqrt{g(V,V)}-V_0(\lambda)V_0(\lambda)S(\lambda)&\\
\overset{(34),(36)}{=}V_0(V_0\lambda)V_0^h(\lambda)-\lambda V_0(\lambda)V_0^h(\lambda)-\frac{1}{F}V_0(\lambda)S_0(\lambda)&\\
-V_0(\lambda)V_0(\lambda)S(\lambda),&
\end{aligned}
$$

i.e.

$$V_0(\lambda)V_0^h(V_0\lambda)=V_0(V_0\lambda)V_0^h(\lambda). \tag{38}$$

Since S and V_0^h span the horizontal subspaces we can write, by (37) and (38), that

$$V_0(\lambda)X_i^h(V_0\lambda)=V_0(V_0\lambda)X_i^h(\lambda)\quad (i=1,2). \tag{39}$$

Equations (39) are called Wagner's equations [17, Formula 18].

Wagner's notations [17]		
the evaluation of the main scalar along the central affine arcwise parametrization: $A=\lambda\circ c_p$	$\frac{\partial A}{\partial\theta}=(\lambda\circ c_p)'$ $=V_0(\lambda)\circ c_p$	the canonical horizontal sections: $\nabla_\beta=X_\beta^h,\ \beta=1,2$

Consider the indicatrix bundle $IM:=F^{-1}(1)$. Wagner's equations imply that

$$V_0(\lambda)d(V_0\lambda)=V_0(V_0\lambda)d\lambda \tag{40}$$

holds on the manifold IM because $V_0(\lambda)V_0(V_0\lambda)=V_0(V_0\lambda)V_0(\lambda)$ is automathic; note that

$$V_0(F)=X_i^h(F)=0\quad (i=1,2),$$

i.e. V_0, X_1^h and X_2^h form a local frame of the indicatrix bundle. Suppose that $F(v)=1$ and $V_0(v)\lambda\neq 0$. Equation (40) implies that $d(V_0\lambda)$ is the proportional of $d\lambda$ around v and, consequently,

$$d(V_0\lambda)\wedge d\lambda=0\quad\Leftrightarrow\quad d((V_0\lambda)d\lambda)=0.$$

This means that there is a (local) solution $\mu_{\rm loc}$ such that

$$(V_0\lambda)d\lambda=d\mu_{\rm loc}. \tag{41}$$

Taking a coordinate system $\varphi=(z^1,z^2,\lambda)$ of the indicatrix bundle around v, formula (41) says that $\partial\mu_{\rm loc}/\partial z^1=\partial\mu_{\rm loc}/\partial z^2=0$. This means that $\mu_{\rm loc}$ depends only on λ. If the function f is defined by $f(\lambda):=\mu_{\rm loc}'(\lambda)$, where $\mu_{\rm loc}$ is the local solution of (41), then $V_0(\lambda)=f(\lambda)$ as Wagner's theorem states; note that $f(s):=0$, where $s=\lambda(v)$ and $V_0(v)\lambda=0$.

Theorem 4 (Wagner's theorem). *[17] A necessary and sufficient condition that F_2 $\left(\frac{\partial A}{\partial\theta}\neq 0\right)$ be a generalized Berwald space is that $\frac{\partial A}{\partial\theta}$ be a function of A.*

3 Some remarks about the converse of Wagner's theorem

Consider a Finslerian unit vector $v \in TM$ such that $V_0(v)\lambda \neq 0$ and $c_p(0) = v$. If

$$V_0(\lambda) = f(\lambda) \tag{42}$$

then Wagner's equations (37) and (38) are automatically satisfied (it follows without the regularity condition $V_0(v)\lambda \neq 0$ as well). Using Cartan's permutation formulas (13) together with (37) and (38),

$$V_0\Big(\frac{S\lambda}{V_0\lambda}\Big) = \frac{V_0^h\lambda}{V_0\lambda}, \tag{43}$$

$$V_0\Big(V_0\Big(\frac{S\lambda}{V_0\lambda}\Big)\Big) = V_0\Big(\frac{V_0^h\lambda}{V_0\lambda}\Big) = -\frac{1}{F^2}\frac{S\lambda}{V_0\lambda} - \lambda V_0\Big(\frac{S\lambda}{V_0\lambda}\Big) - S\lambda. \tag{44}$$

Introducing the function

$$w_p(t) := c_p^i(t)\beta_i \circ c_p(t) \overset{(20)}{=} c_p^i(t)\int_0^t \alpha_i \circ c_p(\theta)\, d\theta$$

we also have that

$$(w_p)'(t) = (c_p^i)'(t)\int_0^t \alpha_i \circ c_p(\theta)\, d\theta + (y^i\alpha_i)\circ c_p(t) \overset{(12)}{=} (c_p^i)'(t)\int_0^t \alpha_i \circ c_p(\theta)\, d\theta,$$

$$\begin{aligned} w_p''(t) &= (c_p^i)''(t)\int_0^t \alpha_i \circ c_p(\theta)\, d\theta + (V_0^i\alpha_i)\circ c_p(t) \\ &\overset{(12),(30)}{=} -\Big(\lambda\circ c_p(t)(c_p^i)'(t) + \frac{y^i}{F^2}\circ c_p(t)\Big)\int_0^t \alpha_i \circ c_p(\theta)\, d\theta - (S\lambda)\circ c_p(t) \\ &= -\frac{1}{F^2}\circ c_p(t) w_p(t) - \lambda\circ c_p w_p'(t) - (S\lambda)\circ c_p(t). \end{aligned}$$

Comparing with (44), the existence and the unicity of the solution of a second order linear equation initial value problem and formula (30) imply that

$$\frac{S\lambda}{V_0\lambda}\circ c_p(t) = w_p(t) + (y^i k_i(p))\circ c_p(t)$$

for any parameter t sufficiently close to 0, where the integration constants are choosen such that

$$\frac{S\lambda}{V_0\lambda}\circ c_p(0) = c_p^1(0)k_1(p) + c_p^2(0)k_2(p),$$

$$\frac{V_0^h\lambda}{V_0\lambda}\circ c_p(0) = (c_p^1)'(0)k_1(p) + (c_p^2)'(0)k_2(p).$$

Recall that the determinant of the coefficient matrix is

$$c_p^1(0)(c_p^2)'(0) - c_p^2(0)(c_p^1)'(0) = \det\begin{pmatrix} y^1 & y^2 \\ V_0^1 & V_0^2\end{pmatrix}\circ c_p(0) = \frac{F}{\sqrt{g(V,V)}}\circ c_p(0) = 1.$$

Therefore (34) and (36) are locally satisfied by the uniquely determined functions

$$f_i \circ c_p(t) = \frac{1}{\sqrt{g(V,V)} \circ c_p(t)} (\beta_i \circ c_p(t) + k_i(p)) \quad (i = 1, 2)$$

(cf. formula (21)) and equation (32) also holds provided that t is sufficiently close to 0. Unfortunately, condition $\partial A/\partial \theta \neq 0$ can not be satisfied all along the central affine arcwise parametrization of the indicatrix curve of a non-singular Finsler metric because of the smooth periodicity. What about the case of $V_0(v)\lambda = 0$? If we can not use a continuity argument to conclude (34) and (36) then condition (42) must be completed by equations $S(v)\lambda = 0$ (cf. formula (34)) and $V_0^h(v)\lambda = 0$ (cf. formula (36)). Especially, this is the case along quadratic parts of the indicatrix curve. For explicit constructions of generalized Berwald surfaces, see [16].

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References

[1] D. Bao: On two curvature-driven problems in Riemann-Finsler geometry. *Advanced Studies in Pure Mathematics* 48 (2007) 19–71.

[2] D. Bao, S.-S. Chern, Z. Shen: *An Introduction to Riemann-Finsler geometry*. Springer-Verlag (2000).

[3] L. Berwald: Über zweidimensionale allgemeine metrische Räume. *Journal für die reine und angewandte Mathematik* 156 (1927) 191–222.

[4] L. Berwald: On Finsler and Cartan geometries. III: two-dimensional Finsler spaces with rectilinear extremals. *Annals of Mathematics* (1941) 84–112.

[5] M. Hashiguchi: On conformal transformations of Finsler metrics. *J. Math. Kyoto Univ.* 16 (1976) 25–50.

[6] M. Matsumoto: *Foundations of Finsler geometry and special Finsler spaces*. Kaiseisha press (1986).

[7] Z. Shen: *Differential Geometry of Spray and Finsler Spaces*. Kluwer Academic Publishers (2001).

[8] Sz. Vattamány, Cs. Vincze: Two-dimensional Landsberg manifolds with vanishing Douglas tensor. *Annales Univ. Sci. Budapest* 44 (2001) 11–26.

[9] Sz. Vattamány, Cs. Vincze: On a new geometrical derivation of two-dimensional Finsler manifolds with constant main scalar. *Period. Math. Hungar.* 48 (1–2) (2004) 61–67.

[10] Cs. Vincze: A new proof of Szabó's theorem on the Riemann-metrizability of Berwald manifolds. *Acta Math. Acad. Paedagog. Nyházi (NS)* 21 (2) (2005) 199–204.

[11] Cs. Vincze: On a scale function for testing the conformality of Finsler manifolds to a Berwald manifold. *Journal of Geometry and Physics* 54 (4) (2005) 454–475.

[12] Cs. Vincze: On Berwald and Wagner manifolds. *Acta Math. Acad. Paedagog. Nyházi.(NS)* 24 (2008) 169–178.

[13] Cs. Vincze: On generalized Berwald manifolds with semi-symmetric compatible linear connections. *Publ. Math. Debrecen* 83 (4) (2013) 741–755.

[14] Cs. Vincze: On a special type of generalized Berwald manifolds: semi-symmetric linear connections preserving the Finslerian length of tangent vectors. *European Journal of Mathematics* 3 (4) (2017) 1098–1171.

[15] Cs. Vincze: Lazy orbits: an optimization problem on the sphere. *Journal of Geometry and Physics* 124 (2018) 180–198.

[16] Cs. Vincze, M. Oláh, Layth M. Alabdulsada: On the divergence representation of the Gauss curvature of Riemannian surfaces and its applications. *Rendiconti del Circolo Matematico di Palermo Series 2* (2018) 1–13.

[17] V. Wagner: On generalized Berwald spaces. *CR (Doklady) Acad. Sci. URSS (NS)* 39 (1943) 3–5.

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