Communications in Mathematics 27 (2019) 89–101
DOI: 10.2478/cm-2019-0009

Insertion of a Contra-Baire-1 (Baire-.5) Function

Majid Mirmiran

Abstract. Necessary and sufficient conditions in terms of lower cut sets are given for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that F_σ-kernel of sets are F_σ-sets.

1 Introduction

A generalized class of closed sets was considered by Maki in 1986 [17]. He investigated the sets that can be represented as union of closed sets and called them V-sets. Complements of V-sets, i.e., sets that are intersection of open sets are called Λ-sets [17].

Recall that a real-valued function f defined on a topological space X is called A-continuous [20] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of A-continuity. However, for unknown concepts the reader may refer to [4], [10]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [5] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1], [3], [7], [8], [9], [11], [12], [19].

Results of Katětov [13], [14] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient condition for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that F_σ-kernel of sets are F_σ-sets.

2010 MSC: 26A15, 54C30.
Key words: Insertion, strong binary relation, Baire-.5 function, kernel of sets, lower cut set.
This work was supported by University of Isfahan and Centre of Excellence for Mathematics (University of Isfahan).
Affiliation:
Majid Mirmiran – Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran
E-mail: mirmir@sci.ui.ac.ir

A real-valued function f defined on a topological space X is called *contra-Baire-.5 (Baire-.5)* if the preimage of every open subset of $\mathbb{R}$ is a G_δ-set in X [21].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [15].

A property P defined relative to a real-valued function on a topological space is a *B-.5-property* provided that any constant function has property P and provided that the sum of a function with property P and any Baire-.5 function also has property P. If P_1 and P_2 are B-.5-properties, the following terminology is used:

(i) A space X has the *weak B-.5-insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g \leq h \leq f$.

(ii) A space X has the *B-.5-insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g < h < f$.

In this paper, for a topological space that F_σ-kernel of sets are F_σ-sets, is given a sufficient condition for the weak B-.5-insertion property. Also for a space with the weak B-.5-insertion property, we give a necessary and sufficient condition for the space to have the B-.5-insertion property. Several insertion theorems are obtained as corollaries of these results.

2 The Main Result

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

Definition 1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:

$$A^\Lambda = \bigcap\{O : O \supseteq A, O \in (X, \tau)\}$$

and

$$A^V = \bigcup\{F : F \subseteq A, F^c \in (X, \tau)\}\,.$$

In [6], [16], [18], A^Λ is called the *kernel* of A.

Definition 2. We define the subsets $G_\delta(A)$ and $F_\sigma(A)$ as follows:

$$G_\delta(A) = \bigcup\{O : O \subseteq A, O \text{ is } G_\delta\text{-set}\}$$

and

$$F_\sigma(A) = \bigcap\{F : F \supseteq A, F \text{ is } F_\sigma\text{-set}\}\,.$$

$F_\sigma(A)$ is called the *F_σ-kernel* of A.

The following first two definitions are modifications of conditions considered in [13], [14].

Definition 3. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \, \bar{\rho} \, y$ if and only if $y \, \rho \, v$ implies $x \, \rho \, v$ and $u \, \rho \, x$ implies $u \, \rho \, y$ for any u and v in S.

Definition 4. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1. If $A_i \, \rho \, B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i \, \rho \, C$ and $C \, \rho \, B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.
2. If $A \subseteq B$, then $A \, \bar{\rho} \, B$.
3. If $A \, \rho \, B$, then $F_\sigma(A) \subseteq B$ and $A \subseteq G_\delta(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 5. If f is a real-valued function defined on a space X and if

$$\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$$

for a real number ℓ, then $A(f, \ell)$ is a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main results:

Theorem 1. *Let g and f be real-valued functions on the topological space X, that F_σ-kernel of sets in X are F_σ-sets, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \, \rho \, A(g, t_2)$, then there exists a Baire-.5 function h defined on X such that $g \leq h \leq f$.*

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \, \rho \, A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \, \bar{\rho} \, F(t_2), G(t_1) \, \bar{\rho} \, G(t_2)$, and $F(t_1) \, \rho \, G(t_2)$. By Lemmas 1 and 2 of [14] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1) \, \rho \, H(t_2), H(t_1) \, \rho \, H(t_2)$ and $H(t_1) \, \rho \, G(t_2)$.

For any x in X, let

$$h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\} .$$

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence

$g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have

$$h^{-1}(t_1, t_2) = G_\delta(H(t_2)) \setminus F_\sigma(H(t_1)).$$

Hence $h^{-1}(t_1, t_2)$ is a G_δ-set in X, i.e., h is a Baire-.5 function on X. □

The above proof used the technique of Theorem 1 of [13].

Theorem 2. *Let P_1 and P_2 be B-.5-property and X be a space that satisfies the weak B-.5-insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g < f, g$ has property P_1 and f has property P_2. The space X has the B-.5-insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each n, $X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by Baire-.5 functions.*

Proof. Assume that X has the weak B-.5-insertion property for (P_1, P_2). Let g and f be functions such that $g < f, g$ has property P_1 and f has property P_2. By hypothesis there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a sequence (D_n) such that $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and such that for each $n, X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by Baire-.5 functions. Let k_n be a Baire-.5 function such that $k_n = 0$ on $A(f - g, 3^{-n+1})$ and $k_n = 1$ on $X \setminus D_n$. Let a function k on X be defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} 3^{-n} k_n(x).$$

By the Cauchy condition and the B-.5-properties, the function k is a Baire-.5 function. Since $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and since $k_n = 1$ on $X \setminus D_n$, it follows that $0 < k$. Also $2k < f - g$: In order to see this, observe first that if x is in $A(f - g, 3^{-n+1})$, then $k(x) \leq 1/4(3^{-n})$. If x is any point in X, then $x \notin A(f - g, 1)$ or for some n,

$$x \in A(f - g, 3^{-n+1}) - A(f - g, 3^{-n});$$

in the former case $2k(x) < 1$, and in the latter $2k(x) \leq 1/2(3^{-n}) < f(x) - g(x)$. Thus if $f_1 = f - k$ and if $g_1 = g + k$, then $g < g_1 < f_1 < f$. Since P_1 and P_2 are B-.5-properties, then g_1 has property P_1 and f_1 has property P_2. Since X has the weak B-.5-insertion property for (P_1, P_2), then there exists a Baire-.5 function such that $g_1 \leq h \leq f_1$. Thus $g < h < f$, it follows that X satisfies the B-.5-insertion property for (P_1, P_2). (The technique of this proof is by Katětov [13].)

Conversely, let g and f be functions on X such that g has property P_1, f has property P_2 and $g < f$. By hypothesis, there exists a Baire-.5 function such that $g < h < f$. We follow an idea contained in Lane [15]. Since the constant function 0 has property P_1, since $f - h$ has property P_2, and since X has the B-.5-insertion property for (P_1, P_2), then there exists a Baire-.5 function k such that $0 < k < f - h$. Let $A(f - g, 3^{-n+1})$ be any lower cut set for $f - g$ and let

$$D_n = \{x \in X : k(x) < 3^{-n+2}\}.$$

Since $k > 0$ it follows that $\bigcap_{n=1}^{\infty} D_n = \emptyset$. Since

$$A(f-g, 3^{-n+1}) \subseteq \{x \in X : (f-g)(x) \leq 3^{-n+1}\} \subseteq \{x \in X : k(x) \leq 3^{-n+1}\}$$

and since

$$\{x \in X : k(x) \leq 3^{-n+1}\}$$

and

$$\{x \in X : k(x) \geq 3^{-n+2}\} = X \setminus D_n$$

are completely separated by Baire-.5 function $\sup\{3^{-n+1}, \inf\{k, 3^{-n+2}\}\}$, it follows that for each $n, A(f-g, 3^{-n+1})$ and $X \setminus D_n$ are completely separated by Baire-.5 functions. □

3 Applications

Definition 6. A real-valued function f defined on a space X is called *contra-upper semi-Baire-.5* (resp. *contra-lower semi-Baire-.5*) if $f^{-1}(-\infty, t)$ (resp. $f^{-1}(t, +\infty)$) is a G_δ-set for any real number t.

The abbreviations usc, lsc, cusB-.5 and clsB-.5 are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

Remark 1. [13], [14]. A space X has the weak c-insertion property for (usc, lsc) if and only if X is normal.

Before stating the consequences of Theorem 1, and Theorem 2 we suppose that X is a topological space that F_σ-kernel of sets are F_σ-sets.

Corollary 1. *For each pair of disjoint F_σ-sets F_1, F_2, there are two G_δ-sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ if and only if X has the weak B-.5-insertion property for* (cusB-.5, clsB-.5).

Proof. Let g and f be real-valued functions defined on the X, such that f is $\operatorname{lsc} B_1$, g is $\operatorname{usc} B_1$, and $g \leq f$. If a binary relation ρ is defined by $A \;\rho\; B$ in case $F_\sigma(A) \subseteq G_\delta(B)$, then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a F_σ-set and since $\{x \in X : g(x) < t_2\}$ is a G_δ-set, it follows that

$$F_\sigma(A(f, t_1)) \subseteq G_\delta(A(g, t_2)).$$

Hence $t_1 < t_2$ implies that $A(f, t_1) \;\rho\; A(g, t_2)$. The proof follows from Theorem 1.

On the other hand, let F_1 and F_2 are disjoint F_σ-sets. Set $f = \chi_{F_1^c}$ and $g = \chi_{F_2}$, then f is clsB-.5, g is cusB-.5, and $g \leq f$. Thus there exists Baire-.5 function h such that $g \leq h \leq f$. Set

$$G_1 = \left\{x \in X : h(x) < \frac{1}{2}\right\}$$

and

$$G_2 = \left\{x \in X : h(x) > \frac{1}{2}\right\},$$

then G_1 and G_2 are disjoint G_δ-sets such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. □

Remark 2. [22] A space X has the weak c-insertion property for (lsc, usc) if and only if X is extremally disconnected.

Corollary 2. *For every G of G_δ-set, $F_\sigma(G)$ is a G_δ-set if and only if X has the weak B-.5-insertion property for* (clsB-.5, cusB-.5).

Before giving the proof of this corollary, the necessary lemma is stated.

Lemma 1. *The following conditions on the space X are equivalent:*

(i) For every G of G_δ-set we have $F_\sigma(G)$ is a G_δ-set.

(ii) For each pair of disjoint G_δ-sets as G_1 and G_2 we have $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$.

The proof of Lemma 1 is a direct consequence of the definition F_σ-kernel of sets.

We now give the proof of Corollary 2.

Proof of Corollary 2. Let g and f be real-valued functions defined on the X, such that f is clsB-.5, g is cusB-.5, and $f \leq g$. If a binary relation ρ is defined by $A \; \rho \; B$ in case

$$F_\sigma(A) \subseteq G \subseteq F_\sigma(G) \subseteq G_\delta(B)$$

for some G_δ-set g in X, then by hypothesis and Lemma 1 ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(g, t_1) = \{x \in X : g(x) < t_1\} \subseteq \{x \in X : f(x) \leq t_2\} = A(f, t_2)\,;$$

since $\{x \in X : g(x) < t_1\}$ is a G_δ-set and since $\{x \in X : f(x) \leq t_2\}$ is a F_σ-set, by hypothesis it follows that $A(g, t_1) \; \rho \; A(f, t_2)$. The proof follows from Theorem 1.

On the other hand, Let G_1 and G_2 are disjoint G_δ-sets. Set $f = \chi_{G_2}$ and $g = \chi_{G_1^c}$, then f is clsB-.5, g is cusB-.5, and $f \leq g$.

Thus there exists Baire-.5 function h such that $f \leq h \leq g$. Set

$$F_1 = \left\{x \in X : h(x) \leq \frac{1}{3}\right\}$$

and

$$F_2 = \left\{x \in X : h(x) \geq \frac{2}{3}\right\}$$

then F_1 and F_2 are disjoint F_σ-sets such that $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$. Hence

$$F_\sigma(F_1) \cap F_\sigma(F_2) = \emptyset\,.$$

□

Before stating the consequences of Theorem 2, we state and prove the necessary lemmas.

Lemma 2. *The following conditions on the space X are equivalent:*

(i) Every two disjoint F_σ-sets of X can be separated by G_δ-sets of X.

(ii) If F is a F_σ-set of X which is contained in a G_δ-set G, then there exists a G_δ-set H such that $F \subseteq H \subseteq F_\sigma(H) \subseteq G$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $F \subseteq G$, where F and G are F_σ-set and G_δ-set of X, respectively. Hence, G^c is a F_σ-set and $F \cap G^c = \emptyset$.

By (i) there exists two disjoint G_δ-sets G_1, G_2 such that $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since G_2^c is a F_σ-set containing G_1 we conclude that $F_\sigma(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq F_\sigma(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that F_1, F_2 are two disjoint F_σ-sets of X.

This implies that $F_1 \subseteq F_2^c$ and F_2^c is a G_δ-set. Hence by (ii) there exists a G_δ-set H such that $F_1 \subseteq H \subseteq F_\sigma(H) \subseteq F_2^c$.
But

$$H \subseteq F_\sigma(H) \Rightarrow H \cap (F_\sigma(H))^c = \emptyset$$

and

$$F_\sigma(H) \subseteq F_2^c \Rightarrow F_2 \subseteq (F_\sigma(H))^c.$$

Furthermore, $(F_\sigma(H))^c$ is a G_δ-set of X. Hence $F_1 \subseteq H$, $F_2 \subseteq (F_\sigma(H))^c$ and $H \cap (F_\sigma(H))^c = \emptyset$. This means that condition (i) holds. □

Lemma 3. *Suppose that X is the topological space such that we can separate every two disjoint F_σ-sets by G_δ-sets. If F_1 and F_2 are two disjoint F_σ-sets of X, then there exists a Baire-.5 function $h\colon X \to [0,1]$ such that*

$$h(F_1) = \{0\} \quad \textit{and} \quad h(F_2) = \{1\}.$$

Proof. Suppose F_1 and F_2 are two disjoint F_σ-sets of X. Since $F_1 \cap F_2 = \emptyset$, hence $F_1 \subseteq F_2^c$. In particular, since F_2^c is a G_δ-set of X containing F_1, by Lemma 2, there exists a G_δ-set $H_{1/2}$ such that,

$$F_1 \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq F_2^c.$$

Note that $H_{1/2}$ is a G_δ-set and contains F_1, and F_2^c is a G_δ-set and contains $F_\sigma(H_{1/2})$. Hence, by Lemma 2, there exists G_δ-sets $H_{1/4}$ and $H_{3/4}$ such that,

$$F_1 \subseteq H_{1/4} \subseteq F_\sigma(H_{1/4}) \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq H_{3/4} \subseteq F_\sigma(H_{3/4}) \subseteq F_2^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain G_δ-sets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by

$$h(x) = \inf\{t : x \in H_t\}$$

for $x \notin F_2$ and $h(x) = 1$ for $x \in F_2$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, F_1 \subseteq H_t$; hence $h(F_1) = \{0\}$. Furthermore, by definition, $h(F_2) = \{1\}$. It remains only to prove that h is a Baire-.5 function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \emptyset$ and if $0 < \alpha$ then

$$\{x \in X : h(x) < \alpha\} = \bigcup\{H_t : t < \alpha\},$$

hence, they are G_δ-sets of X. Similarly, if $\alpha < 0$ then

$$\{x \in X : h(x) > \alpha\} = X$$

and if $0 \leq \alpha$ then

$$\{x \in X : h(x) > \alpha\} = \bigcup\{(F_\sigma(H_t))^c : t > \alpha\}$$

hence, every of them is a G_δ-set. Consequently h is a Baire-.5 function. □

Lemma 4. *Suppose that X is the topological space such that every two disjoint F_σ-sets can be separated by G_δ-sets. The following conditions are equivalent:*

(i) Every countable convering of G_δ-sets of X has a refinement consisting of G_δ-sets such that, for every $x \in X$, there exists a G_δ-set containing x such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{F_n\}$ of F_σ-sets with empty intersection there exists a decreasing sequence $\{G_n\}$ of G_δ-sets such that, $\bigcap_{n=1}^\infty G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$.

Proof. (i) ⇒ (ii). suppose that $\{F_n\}$ be a decreasing sequence of F_σ-sets with empty intersection. Then $\{F_n^c : n \in \mathbb{N}\}$ is a countable covering of G_δ-sets. By hypothesis (i) and Lemma 2, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every V_n is a G_δ-set and $F_\sigma(V_n) \subseteq F_n^c$. By setting $F_n = (F_\sigma(V_n))^c$, we obtain a decreasing sequence of G_δ-sets with the required properties.

(ii) ⇒ (i). Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of G_δ-sets, we set for $n \in \mathbb{N}$, $F_n = (\bigcup_{i=1}^n H_i)^c$. Then $\{F_n\}$ is a decreasing sequence of F_σ-sets with empty intersection. By (ii) there exists a decreasing sequence $\{G_n\}$ consisting of G_δ-sets such that, $\bigcap_{n=1}^\infty G_n = \emptyset$ and for every $n \in \mathbb{N}$, $F_n \subseteq G_n$. Now we define the subsets W_n of X in the following manner:

W_1 is a G_δ-set of X such that $G_1^c \subseteq W_1$ and $F_\sigma(W_1) \cap F_1 = \emptyset$.

W_2 is a G_δ-set of X such that $F_\sigma(W_1) \cup G_2^c \subseteq W_2$ and $F_\sigma(W_2) \cap F_2 = \emptyset$, and so on. (By Lemma 2, W_n exists).

Then since $\{G_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of G_δ-sets. Moreover, we have

(i) $F_\sigma(W_n) \subseteq W_{n+1}$

(ii) $G_n^c \subseteq W_n$

(iii) $W_n \subseteq \bigcup_{i=1}^n H_i$.

Now suppose that $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus F_\sigma(W_{n-1})$.

Then since $F_\sigma(W_{n-1}) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of G_δ-sets and covers X. Furthermore, $S_i \cap S_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Finally, consider the following sets:

$$
\begin{array}{llll}
S_1 \cap H_1, & S_1 \cap H_2 & & \\
S_2 \cap H_1, & S_2 \cap H_2, & S_2 \cap H_3 & \\
S_3 \cap H_1, & S_3 \cap H_2, & S_3 \cap H_3, & S_3 \cap H_4
\end{array}
$$

and continue ad infinitum. These sets are G_δ-sets, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a G_δ-set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently,

$$\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i+1\}$$

refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are G_δ-sets, and for every point in X we can find a G_δ-set containing the point that intersects only finitely many elements of that refinement. □

Remark 3. [13], [14] A space X has the c-insertion property for $(\mathrm{usc}, \mathrm{lsc})$ if and only if X is normal and countably paracompact.

Corollary 3. *X has the B-.5-insertion property for* $(\mathrm{cusB\text{-}.5}, \mathrm{clsB\text{-}.5})$ *if and only if every two disjoint F_σ-sets of X can be separated by G_δ-sets, and in addition, every countable covering of G_δ-sets has a refinement that consists of G_δ-sets such that, for every point of X we can find a G_δ-set containing that point such that, it intersects only a finite number of refining members.*

Proof. Suppose that F_1 and F_2 are disjoint F_σ-sets . Since $F_1 \cap F_2 = \emptyset$, it follows that $F_2 \subseteq F_1^c$. We set $f(x) = 2$ for $x \in F_1^c$, $f(x) = \frac{1}{2}$ for $x \notin F_1^c$, and $g = \chi_{F_2}$.

Since F_2 is a F_σ-set, and F_1^c is a G_δ-set, therefore g is cusB-.5,f is clsB-.5 and furthermore $g < f$. Hence by hypothesis there exists a Baire-.5 function h such that, $g < h < f$. Now by setting

$$G_1 = \{x \in X : h(x) < 1\}$$

and

$$G_2 = \{x \in X : h(x) > 1\}.$$

We can say that G_1 and G_2 are disjoint G_δ-sets that contain F_1 and F_2, respectively. Now suppose that $\{F_n\}$ is a decreasing sequence of F_σ-sets with empty intersection. Set $F_0 = X$ and define for every $x \in F_n \setminus F_{n+1}$,

$$f(x) = \frac{1}{n+1}.$$

Since

$$\bigcap_{n=0}^{\infty} F_n = \emptyset$$

and for every $x \in X$, there exists $n \in \mathbb{N}$, such that, $x \in F_n \setminus F_{n+1}$, f is well defined. Furthermore, for every $r \in \mathbb{R}$, if $r \leq 0$ then

$$\{x \in X : f(x) > r\} = X$$

is a G_δ-set and if $r > 0$ then by Archimedean property of $\mathbb{R}$, we can find $i \in \mathbb{N}$ such that

$$\frac{1}{i+1} \leq r.$$

Now suppose that k is the least natural number such that $\frac{1}{k+1} \leq r$. Hence $\frac{1}{k} > r$ and consequently,

$$\{x \in X : f(x) > r\} = X \setminus F_k$$

is a G_δ-set. Therefore, f is clsB-.5. By setting $g = 0$, we have g is cusB-.5 and $g < f$. Hence by hypothesis there exists a Baire-.5 function h on X such that, $g < h < f$.

By setting

$$G_n = \left\{x \in X : h(x) < \frac{1}{n+1}\right\},$$

we have G_n is a G_δ-set. But for every $x \in F_n$, we have

$$f(x) \leq \frac{1}{n+1}$$

and since $g < h < f$ therefore

$$0 < h(x) < \frac{1}{n+1},$$

i.e., $x \in G_n$ therefore $F_n \subseteq G_n$ and since $h > 0$ it follows that

$$\bigcap_{n=1}^{\infty} G_n = \emptyset.$$

Hence by Lemma 4, the conditions holds.

On the other hand, since every two disjoint F_σ-sets can be separated by G_δ-sets, therefore by Corollary 1, X has the weak B-.5-insertion property for (cusB-.5, clsB-.5). Now suppose that f and g are real-valued functions on X with $g < f$, such that, g is cusB-.5 and f is clsB-.5. For every $n \in \mathbb{N}$, set

$$A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) \leq 3^{-n+1}\}\,.$$

Since g is cusB-.5, and f is clsB-.5, therefore $f-g$ is clsB-.5. Hence $A(f-g, 3^{-n+1})$ is a F_σ-set of X. Consequently, $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of F_σ-sets and furthermore since $0 < f-g$, it follows that

$$\bigcap_{n=1}^{\infty} A(f-g, 3^{-n+1}) = \emptyset\,.$$

Now by Lemma 4, there exists a decreasing sequence $\{D_n\}$ of G_δ-sets such that

$$A(f-g, 3^{-n+1}) \subseteq D_n$$

and

$$\bigcap_{n=1}^{\infty} D_n = \emptyset\,.$$

But by Lemma 3, $A(f-g, 3^{-n+1})$ and $X \setminus D_n$ of F_σ-sets can be completely separated by Baire-.5 functions. Hence by Theorem 2, there exists a Baire-.5 function h defined on X such that, $g < h < f$, i.e., X has the B-.5-insertion property for (cusB-.5, clsB-.5). □

Remark 4. [15] A space X has the c-insertion property for (lsc, usc) iff X is extremally disconnected and if for any decreasing sequence $\{G_n\}$ of open subsets of X with empty intersection there exists a decreasing sequence $\{F_n\}$ of closed subsets of X with empty intersection such that $G_n \subseteq F_n$ for each n.

Corollary 4. *For every G of G_δ-set, $F_\sigma(G)$ is a G_δ-set and in addition for every decreasing sequence $\{G_n\}$ of G_δ-sets with empty intersection, there exists a decreasing sequence $\{F_n\}$ of F_σ-sets with empty intersection such that for every $n \in \mathbb{N}$, $G_n \subseteq F_n$ if and only if X has the B-.5-insertion property for* (clsB-.5, cusB-.5).

Proof. Since for every G of G_δ-set, $F_\sigma(G)$ is a G_δ-set, therefore by Corollary 2, X has the weak B-.5-insertion property for (clsB-.5, cusB-.5). Now suppose that f and g are real-valued functions defined on X with $g < f, g$ is clsB-.5, and f is cusB-.5. Set

$$A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) < 3^{-n+1}\}\,.$$

Then since $f-g$ is cusB-.5, hence $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of G_δ-sets with empty intersection. By hypothesis, there exists a decreasing sequence $\{D_n\}$ of F_σ-sets with empty intersection such that, for every $n \in \mathbb{N}$, $A(f-g, 3^{-n+1}) \subseteq D_n$. Hence $X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are two disjoint G_δ-sets and therefore by Lemma 1, we have

$$F_\sigma(A(f-g, 3^{-n+1})) \cap F_\sigma((X \setminus D_n)) = \emptyset$$

and therefore by Lemma 3, $X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are completely separable by Baire-.5 functions. Therefore by Theorem 2, there exists a Baire-.5 function h on X such that, $g < h < f$, i.e., X has the B-.5-insertion property for $(\mathrm{clsB}\text{-}.5, \mathrm{cusB}\text{-}.5)$.

On the other hand, suppose that G_1 and G_2 be two disjoint G_δ-sets. Since $G_1 \cap G_2 = \emptyset$. We have $G_2 \subseteq G_1^c$. We set $f(x) = 2$ for $x \in G_1^c$, $f(x) = \frac{1}{2}$ for $x \notin G_1^c$ and $g = \chi_{G_2}$.

Then since G_2 is a G_δ-set and G_1^c is a F_σ-set, we conclude that g is clsB-.5 and f is cusB-.5 and furthermore $g < f$. By hypothesis, there exists a Baire-.5 function h on X such that, $g < h < f$. Now we set

$$F_1 = \left\{x \in X : h(x) \leq \frac{3}{4}\right\}$$

and

$$F_2 = \{x \in X : h(x) \geq 1\}.$$

Then F_1 and F_2 are two disjoint F_σ-sets contain G_1 and G_2, respectively. Hence $F_\sigma(G_1) \subseteq F_1$ and $F_\sigma(G_2) \subseteq F_2$ and consequently $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$. By Lemma 1, for every G of G_δ-set, the set $F_\sigma(G)$ is a G_δ-set.

Now suppose that $\{G_n\}$ is a decreasing sequence of G_δ-sets with empty intersection.

We set $G_0 = X$ and $f(x) = \frac{1}{n+1}$ for $x \in G_n \setminus G_{n+1}$. Since $\bigcap_{n=0}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}$ there exists $x \in G_n \setminus G_{n+1}$, f is well-defined. Furthermore, for every $r \in \mathbb{R}$, if $r \leq 0$ then

$$\{x \in X : f(x) < r\} = \emptyset$$

is a G_δ-set and if $r > 0$ then by Archimedean property of $\mathbb{R}$, there exists $i \in \mathbb{N}$ such that $\frac{1}{i+1} \leq r$. Suppose that k is the least natural number with this property. Hence $\frac{1}{k} > r$. Now if $\frac{1}{k+1} < r$ then

$$\{x \in X : f(x) < r\} = G_k$$

is a G_δ-set and if $\frac{1}{k+1} = r$ then

$$\{x \in X : f(x) < r\} = G_{k+1}$$

is a G_δ-set. Hence f is a cusB-.5 on X. By setting $g = 0$, we have conclude that g is clsB-.5 on X and in addition $g < f$. By hypothesis there exists a Baire-.5 function h on X such that, $g < h < f$.

Set

$$F_n = \left\{x \in X : h(x) \leq \frac{1}{n+1}\right\}.$$

This set is a F_σ-set. But for every $x \in G_n$, we have $f(x) \leq \frac{1}{n+1}$ and since $g < h < f$ thus $h(x) < \frac{1}{n+1}$, this means that $x \in F_n$ and consequently $G_n \subseteq F_n$.

By definition of F_n, $\{F_n\}$ is a decreasing sequence of F_σ-sets and since $h > 0$, $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Thus the conditions holds. $\square$

References

[1] A. Al-Omari, M.S. Md Noorani: Some properties of contra-b-continuous and almost contra-b-continuous functions. *European J. Pure. Appl. Math.* 2 (2) (2009) 213–230.

[2] F. Brooks: Indefinite cut sets for real functions. *Amer. Math. Monthly* 78 (1971) 1007–1010.

[3] M. Caldas, S. Jafari: Some properties of contra-β-continuous functions. *Mem. Fac. Sci. Kochi. Univ.* 22 (2001) 19–28.

[4] J. Dontchev: The characterization of some peculiar topological space via α- and β-sets. *Acta Math. Hungar.* 69 (1–2) (1995) 67–71.

[5] J. Dontchev: Contra-continuous functions and strongly S-closed space. *Internat. J. Math. Math. Sci.* 19 (2) (1996) 303–310.

[6] J. Dontchev, H. Maki: On sg-closed sets and semi-λ-closed sets. *Questions Answers Gen. Topology* 15 (2) (1997) 259–266.

[7] E. Ekici: On contra-continuity. *Annales Univ. Sci. Bodapest* 47 (2004) 127–137.

[8] E. Ekici: New forms of contra-continuity. *Carpathian J. Math.* 24 (1) (2008) 37–45.

[9] A.I. El-Magbrabi: Some properties of contra-continuous mappings. *Int. J. General Topol.* 3 (1–2) (2010) 55–64.

[10] M. Ganster, I. Reilly: A decomposition of continuity. *Acta Math. Hungar.* 56 (3–4) (1990) 299–301.

[11] S. Jafari, T. Noiri: Contra-continuous function between topological spaces. *Iranian Int. J. Sci.* 2 (2001) 153–167.

[12] S. Jafari, T. Noiri: On contra-precontinuous functions. *Bull. Malaysian Math. Sc. Soc.* 25 (2002) 115–128.

[13] M. Katětov: On real-valued functions in topological spaces. *Fund. Math.* 38 (1951) 85–91.

[14] M. Katětov: Correction to "On real-valued functions in topological spaces". *Fund. Math.* 40 (1953) 203–205.

[15] E. Lane: Insertion of a continuous function. *Pacific J. Math.* 66 (1976) 181–190.

[16] S.N. Maheshwari, R. Prasad: On R_{Os}-spaces. *Portugal. Math.* 34 (1975) 213–217.

[17] H. Maki: Generalized Λ-sets and the associated closure operator. *The special Issue in commemoration of Prof. Kazuada IKEDA's Retirement* (1986) 139–146.

[18] M. Mrsevic: On pairwise R and pairwise R_1 bitopological spaces. *Bull. Math. Soc. Sci. Math. R. S. Roumanie* 30 (1986) 141–145.

[19] A.A. Nasef: Some properties of contra-continuous functions. *Chaos Solitons Fractals* 24 (2005) 471–477.

[20] M. Przemski: A decomposition of continuity and α-continuity. *Acta Math. Hungar.* 61 (1–2) (1993) 93–98.

[21] H. Rosen: Darboux Baire-.5 functions. *Proc. Amer. math. Soc.* 110 (1) (1990) 285–286.

[22] M.H. Stone: Boundedness properties in function-lattices. *Canad. J. Math* 1 (1949) 176–189.

Received: 15 May 2018
Accepted for publication: 7 July 2019
Communicated by: Karl Dilcher