Communications in Mathematics 27 (2019) 103–112
DOI: 10.2478/cm-2019-0010

A Deformed Quon Algebra

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Abstract. The quon algebra is an approach to particle statistics in order to provide a theory in which the Pauli exclusion principle and Bose statistics are violated by a small amount. The quons are particles whose annihilation and creation operators obey the quon algebra which interpolates between fermions and bosons. In this paper we generalize these models by introducing a deformation of the quon algebra generated by a collection of operators $a_{i,k}$, $(i,k) \in \mathbb{N}^* \times [m]$, on an infinite dimensional vector space satisfying the deformed q-mutator relations $a_{j,l}a_{i,k}^\dagger = qa_{i,k}^\dagger a_{j,l} + q^{\beta_{k,l}}\delta_{i,j}$. We prove the realizability of our model by showing that, for suitable values of q, the vector space generated by the particle states obtained by applying combinations of $a_{i,k}$'s and $a_{i,k}^\dagger$'s to a vacuum state $|0\rangle$ is a Hilbert space. The proof particularly needs the investigation of the new statistic cinv and representations of the colored permutation group.

1 Introduction

Let $\mathbb{R}(q)$ be the fraction field of the real polynomials with variable q. By a deformed quon algebra $\mathbf{A}$, we mean the free algebra

$$\mathbb{R}(q)\big[a_{i,k} \;\big|\; (i,k) \in \mathbb{N}^* \times [m]\big]$$

subject to the anti-involution † exchanging $a_{i,k}$ with $a_{i,k}^\dagger$, and to the commutation relation

$$a_{j,l}a_{i,k}^\dagger = qa_{i,k}^\dagger a_{j,l} + q^{\beta_{k,l}}\delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta and

$$\beta_{k,l} = \begin{cases} 0 & \text{if } l-k \equiv m \mod m \\ 1 & \text{otherwise} \end{cases}.$$

2010 MSC: 05E15, 81R10, 15A15
Key words: Quon Algebra, Infinite Statistics, Hilbert Space, Colored Permutation Group
This research was supported through the programme "Oberwolfach Leibniz Fellows" by the Mathematisches Forschungsinstitut Oberwolfach in 2017
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This algebra is a generalization of the quon algebra introduced by Greenberg [2], subject to the commutation relation $a_j a_i^\dagger = q a_i^\dagger a_j + \delta_{i,j}$ obeyed by the annihilation and creation operators of the quon particles, and generating a model of infinite statistics. Moreover, the quon algebra is a generalization of the classical Bose and Fermi algebras corresponding to the restrictions $q = 1$ and $q = -1$ respectively, as well as of the intermediate case $q = 0$ suggested by Hegstrom and investigated by Greenberg [1].

In a Fock-like representation, the generators of $\mathbf{A}$ are the linear operators $a_{i,k}, a_{i,k}^\dagger \colon \mathbf{V} \to \mathbf{V}$ on an infinite dimensional real vector space $\mathbf{V}$ satisfying the commutation relations

$$a_{j,l} a_{i,k}^\dagger - q a_{i,k}^\dagger a_{j,l} = q^{\beta_{k,l}} \delta_{i,j},$$

and the relations

$$a_{i,k}|0\rangle = 0,$$

where $a_{i,k}^\dagger$ is the adjoint of $a_{i,k}$, and $|0\rangle$ is a nonzero distinguished vector of $\mathbf{V}$. The $a_{i,k}$'s are the annihilation operators and the $a_{i,k}^\dagger$'s the creation operators.

Let $\mathbf{H}$ be the vector subspace of $\mathbf{V}$ generated by the particle states obtained by applying combinations of $a_{i,k}$'s and $a_{i,k}^\dagger$'s to $|0\rangle$, or

$$\mathbf{H} := \big\{ a|0\rangle \;\big|\; a \in \mathbf{A} \big\}.$$

The aim of this article is to prove the realizability of this model through the following theorem.

Theorem 1. *$\mathbf{H}$ is a Hilbert space for the bilinear form $(\cdot,\cdot)\colon \mathbf{H}\times\mathbf{H} \to \mathbb{R}(q)$ defined by*

$$\big(a|0\rangle, b|0\rangle\big) := \langle 0|a^\dagger\, b|0\rangle \quad \textit{with} \quad \langle 0|0\rangle = 1,$$

and for

$$-1 < q < 1 \textit{ if } m = 1 \quad \textit{and} \quad \frac{1}{1-m} < q < 1 \textit{ if } m > 1.$$

Theorem 1 is a generalization of the realizability of the quon algebra model in infinite statistics proved by Zagier [3, Theorem 1].

To prove Theorem 1, we begin by showing in Section 3 that

$$\mathcal{B} := \big\{ a_{i_1,k_1}^\dagger \dots a_{i_n,k_n}^\dagger |0\rangle \;\big|\; (i_u, k_u) \in \mathbb{N}^* \times [m],\, n \in \mathbb{N} \big\}$$

is a basis of $\mathbf{H}$, so that we can assume that

$$\mathbf{H} = \Big\{ \sum_{i=1}^{n} \lambda_i b_i \;\Big|\; n \in \mathbb{N}^*,\, \lambda_i \in \mathbb{R}(q),\, b_i \in \mathcal{B} \Big\}.$$

Denote by $\mathbb{U}_m$ the group of all m^{th} roots of unity, and $\mathfrak{S}_n$ the permutation group on $[n]$. We represent an element π of the colored permutation group of m colors $\mathbb{U}_m \wr \mathfrak{S}_n$ by

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ (\sigma(1), k_1) & (\sigma(2), k_2) & \dots & (\sigma(n), k_n) \end{pmatrix},$$

where $k_1, \ldots, k_n \in [m]$, and σ is a permutation of $[n]$. But we also adopt the notation $\pi = (\sigma, \alpha)$ meaning that $\sigma \in \mathfrak{S}_n$ and $\alpha\colon [n] \to [m]$ such that

$$\forall i \in [n], \pi(i) = \big(\sigma(i), \alpha(i)\big).$$

More generally, let I be a multiset of n elements in $\mathbb{N}^*$, and $\mathfrak{S}_I$ its permutation set. An element θ of the colored permutation set $\mathbb{U}_m \wr \mathfrak{S}_I$ is defined by $\theta := (\varphi, \epsilon)$ meaning that $\varphi \in \mathfrak{S}_I$ and $\epsilon\colon [n] \to [m]$ such that

$$\forall i \in [n], \theta(i) = \big(\varphi(i), \epsilon(i)\big).$$

Denote the infinite matrix associated to the bilinear form in Theorem 1 by

$$\mathbf{M} := \big((f, g)\big)_{f,g \in \mathcal{B}}.$$

Let $\left[{\mathbb{N}^* \atop n} \right]$ be the set of multisets of n elements in $\mathbb{N}^*$. We also prove in Section 3 that

$$\mathbf{M} = \bigoplus_{n \in \mathbb{N}} \bigoplus_{I \in \left[{\mathbb{N}^* \atop n} \right]} \mathbf{M}_I \quad \text{with} \quad \mathbf{M}_I = \Big(\langle 0| \, a_{\vartheta(n)} \ldots a_{\vartheta(1)} \, a^\dagger_{\theta(1)} \ldots a^\dagger_{\theta(n)} \, |0\rangle \Big)_{\vartheta, \theta \in \mathbb{U}_m \wr \mathfrak{S}_I}.$$

For $m = 3$ for example, we have

$$\mathbf{M}_{[2]} = \begin{pmatrix}
1 & q & q & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 \\
q & 1 & q & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 \\
q & q & 1 & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 \\
q & q^2 & q^2 & 1 & q & q & q & q^2 & q^2 & q^2 & q^3 & q^3 & q & q^2 & q^2 & q^2 & q^3 & q^3 \\
q^2 & q & q^2 & q & 1 & q & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q^3 \\
q^2 & q^2 & q & q & q & 1 & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q^2 \\
q & q^2 & q^2 & q & q^2 & q^2 & 1 & q & q & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & q & q^2 & q^2 \\
q^2 & q & q^2 & q^2 & q & q^2 & q & 1 & q & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q^2 & q & q^2 \\
q^2 & q^2 & q & q^2 & q^2 & q & q & q & 1 & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q^2 & q^2 & q \\
q & q^2 & q^2 & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & 1 & q & q & q & q^2 & q^2 & q & q^2 & q^2 \\
q^2 & q & q^2 & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q & 1 & q & q^2 & q & q^2 & q^2 & q & q^2 \\
q^2 & q^2 & q & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q & q & 1 & q^2 & q^2 & q & q^2 & q^2 & q \\
q^2 & q^3 & q^3 & q & q^2 & q^2 & q^2 & q^3 & q^3 & q & q^2 & q^2 & 1 & q & q & q & q^2 & q^2 \\
q^3 & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q & 1 & q & q^2 & q & q^2 \\
q^3 & q^3 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^2 & q^2 & q & q & q & 1 & q^2 & q^2 & q \\
q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & 1 & q & q \\
q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q & 1 & q \\
q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q & q & 1
\end{pmatrix}.$$

We need to introduce the statistic $\operatorname{cinv}\colon \mathbb{U}_m \wr \mathfrak{S}_n \to \mathbb{N}$ defined by

$$\operatorname{cinv}(\sigma, \alpha) := \#\big\{(i,j) \in [n]^2 \;\big|\; i < j, \sigma(i) > \sigma(j)\big\} + \#\big\{i \in [n] \big| \alpha(i) \neq m\big\}.$$

Still in Section 3, we prove that $\mathbf{M}_I$ is the representation of

$$\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\operatorname{cinv} \pi} \pi \tag{1}$$

on the $\mathbb{U}_m \wr \mathfrak{S}_n$-module $\mathbb{R}[\mathbb{U}_m \wr \mathfrak{S}_I]$. Hence if the regular representation of (1), which is $\mathbf{M}_{[n]}$, is positive definite, then $\mathbf{M}_I$ is positive definite.

We prove in Section 4 that

$$\det \mathbf{M}_{[n]} = \Big(\big(1+(m-1)q\big)(1-q)^{m-1} \prod_{i=1}^{n-1} (1-q^{i^2+i})^{\frac{(n-i)}{(i^2+i)}}\Big)^{m^n n!}.$$

We particularly can infer that $\mathbf{M}_{[n]}$ is nonsingular for

$$-1 < q < 1 \text{ if } m = 1 \quad \text{and} \quad \frac{1}{1-m} < q < 1 \text{ if } m > 1.$$

Since $\mathbf{M}_{[n]}$ is the identity matrix of order $m^n n!$ if $q = 0$, we deduce by continuity that $\mathbf{M}_{[n]}$ is positive definite for the values of q mentioned above. For these suitable values of q, $\mathbf{M}$ is then a symmetric positive definite matrix or, in other terms, the bilinear form of Theorem 1 is an inner product on $\mathbf{H}$.

But before investigating the deformed quon algebra, it is necessary to recall some notions in representation theory and do some computations in Section 2.

The author would like to thank Patrick Rabarison for the discussions on quantum statistics.

2 Representation Theory

We recall the useful notions on representation theory of group and do some calculations for the cyclic groups.

Take a group G and a finite-dimensional vector space V over a field $\mathbb{K}$. Let $g, h \in G$, $a, b \in \mathbb{K}$, and $u, v \in V$. Then V is a G-module if there is a multiplication $\cdot$ of elements of V by elements of G such that

- $u \cdot g \in V$.
- $(au + bv) \cdot g = a(u \cdot g) + b(v \cdot g)$,
- $u \cdot (gh) = (u \cdot g) \cdot h$,
- $u \cdot 1 = u$ where 1 is the neutral element of G.

Take an element x in the group algebra $\mathbb{K}[G]$. Suppose that $\{v_1, \ldots, v_n\}$ is a basis of V, and that $v_j \cdot x = \sum_{i \in [n]} \mu_{i,j} v_i$. Then the representation of x on the G-module V is the matrix

$$R_V(x) := (\mu_{i,j})_{i,j \in [n]}.$$

In particular if $x = \sum_{g \in G} \lambda_g g \in \mathbb{K}[G]$ with $\lambda_g \in \mathbb{R}$, then the regular representation of x is

$$R_{\mathbb{K}[G]}(x) := \big(\lambda_{h^{-1}g}\big)_{g,h \in G}.$$

Lemma 1. *Let G be a finite group, $H \leq G$, and $x \in \mathbb{K}[H]$. Then,*

$$\det R_{\mathbb{K}[G]}(x) = \big(\det R_{\mathbb{K}[H]}(x)\big)^{|G:H|}.$$

Proof. Let $H = \{h_1, \ldots, h_r\}$, and $\{g_1, \ldots, g_k\}$ be a left coset representative set of H. On the ordered basis $(g_1h_1, \ldots, g_1h_r, g_2h_1, \ldots, g_2h_r, \ldots, g_kh_1, \ldots, g_kh_r)$ of $\mathbb{K}[G]$, we have

$$R_{\mathbb{K}[G]}(x) = R_{\mathbb{K}[H]}(x) \otimes I_{|G:H|},$$

where $I_{|G:H|}$ is the unit matrix of size $|G : H|$. □

Now consider the cyclic group Z_m of order m generated by γ, and take a variable z. We need the following equalities on the group algebra $\mathbb{R}(z)[Z_m]$.

Lemma 2. *We have*

$$\det R_{\mathbb{R}(z)[Z_m]}\big(1 + z \sum_{k\in[m-1]} \gamma^k\big) = \big(1 + (m-1)z\big)(1-z)^{m-1}.$$

Proof. The regular representation of $1 + z \sum_{k\in[m-1]} \gamma^k$ is the $m \times m$ circulant matrix with associated polynomial $f(x) = 1 + z \sum_{j\in[m-1]} x^j$. The determinant of this circulant matrix is $\prod_{i\in[m]} f(\zeta^i)$. If $i \in [m-1]$, then

$$\sum_{j\in[m-1]} \zeta^{ij} = \frac{1-\zeta^i}{1-\zeta^i} \sum_{j\in[m-1]} \zeta^{ij} = \frac{\zeta^i - 1}{1-\zeta^i} = -1.$$

Thus $f(1) = 1 + (m-1)z$, and $f(\zeta^i) = 1 - z$ for $i \in [m-1]$. □

Lemma 3. *We have*

$$\Big(1 + z \sum_{k\in[m-1]} \gamma^k\Big)^{-1} = \frac{1}{\big(1+(m-1)z\big)(1-z)} \Big(1 + (m-2)z - z \sum_{k\in[m-1]} \gamma^k\Big).$$

Proof. The form of $1 + z \sum_{k\in[m-1]} \gamma^k$ gives us the intuition that its inverse has the form $x + y \sum_{k\in[m-1]} \gamma^k$. The calculation

$$\Big(1 + z \sum_{k\in[m-1]} \gamma^k\Big) \cdot \Big(x + y \sum_{k\in[m-1]} \gamma^k\Big)$$
$$= x + (m-1)zy + \Big(zx + \big(1 + (m-2)z\big)y\Big) \sum_{k\in[m-1]} \gamma^k$$

confirms the intuition since it leads us to solve the equation system

$$\begin{cases} x + (m-1)zy = 1 \\ zx + \big(1 + (m-2)z\big)y = 0 \end{cases}$$

to get the inverse of $1 + z \sum_{k\in[m-1]} \gamma^k$. We obtain

$$x = \frac{1 + (m-2)z}{\big(1+(m-1)z\big)(1-z)} \quad \text{and} \quad y = -\frac{z}{\big(1+(m-1)z\big)(1-z)}.$$ □

Lemma 4. *We have*

$$(1 - z\gamma)^{-1} = \frac{1}{1 - z^m} \sum_{i=0}^{m-1} z^i \gamma^i.$$

Proof. It comes from $(1 - z\gamma)(1 + z\gamma + \cdots + z^{m-1}\gamma^{m-1}) = 1 - z^m$. □

3 The Bilinear Form $(\cdot, \cdot)$

We first show that $\mathbf{H}$ is linearly generated by the particle states obtained by applying combinations of $a^\dagger_{i,k}$'s to $|0\rangle$. Then we prove that

$$\mathbf{M} = \bigoplus_{n \in \mathbb{N}} \bigoplus_{I \in \left[{\mathbb{N}^* \atop n} \right]} \mathbf{M}_I \,,$$

where $\mathbf{M}_I$ is a representation of $\displaystyle\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\operatorname{cinv} \pi} \pi.$

Lemma 5. *The vector space generated by our particle states is*

$$\mathbf{H} = \Big\{ \sum_{i=1}^{n} \lambda_i b_i \;\Big|\; n \in \mathbb{N}^*,\, \lambda_i \in \mathbb{R}(q),\, b_i \in \mathcal{B} \Big\}.$$

Proof. Let $(j, l) \in \mathbb{N}^* \times [m]$. We have,

$$\begin{aligned} a_{j,l}\, a^\dagger_{i_1,k_1} \ldots a^\dagger_{i_r,k_r} &= q^r a^\dagger_{i_1,k_1} \ldots a^\dagger_{i_r,k_r}\, a_{j,l} \\ &\quad + \sum_{\substack{u \in [r] \\ i_u = j}} q^{u-1} q^{\beta_{-k_u,l}}\, a^\dagger_{i_1,k_1} \ldots \widehat{a^\dagger_{i_u,k_u}} \ldots a^\dagger_{i_r,k_r}, \end{aligned}$$

where the hat over the u^{th} term of the product indicates that this term is omitted. So

$$a_{j,l}\, a^\dagger_{i_1,k_1} \ldots a^\dagger_{i_r,k_r} |0\rangle = \sum_{\substack{u \in [r] \\ i_u = j}} q^{u-1} q^{\beta_{-k_u,l}}\, a^\dagger_{i_1,k_1} \ldots \widehat{a^\dagger_{i_u,k_u}} \ldots a^\dagger_{i_r,k_r} |0\rangle.$$

Thus one can recursively remove every annihilation operator $a_{j,l}$ of an element $a|0\rangle$ of $\mathbf{H}$. □

Lemma 6. *Let* $\big((j_1, l_1), \ldots, (j_s, l_s)\big) \in (\mathbb{N}^* \times [m])^s$ *and* $\big((i_1, k_1), \ldots, (i_r, k_r)\big) \in (\mathbb{N}^* \times [m])^r$. *If, as multisets,* $\{j_1, \ldots, j_s\} \neq \{i_1, \ldots, i_s\}$, *then*

$$\langle 0 | a_{j_s,l_s} \ldots a_{j_1,l_1}\, a^\dagger_{i_1,k_1} \ldots a^\dagger_{i_r,k_r} |0\rangle = 0.$$

Proof. Suppose that v is the smallest integer in $[s]$ such that

$$j_v \notin \{i_1, \ldots, i_r\} \setminus \{j_1, \ldots, j_{v-1}\}.$$

Then

$$a_{j_s,l_s} \ldots a_{j_1,l_1} a^\dagger_{i_1,k_1} \ldots a^\dagger_{i_r,k_r} = P a_{j_v,l_v} \ldots a_{j_1,l_1} + Q a_{j_v,l_v} \quad \text{with} \quad P, Q \in \mathbf{A}.$$

We deduce that

$$a_{j_s,l_s}\dots a_{j_1,l_1}a^\dagger_{i_1,k_1}\dots a^\dagger_{i_r,k_r}|0\rangle = Pa_{j_v,l_v}\dots a_{j_1,l_1}|0\rangle + Qa_{j_v,l_v}|0\rangle = 0.$$

In the same way, suppose that u is the smallest integer in $[r]$ such that i_u does not belong to the multiset $\{j_1,\dots,j_s\}\setminus\{i_1,\dots,i_{u-1}\}$. Then

$$a_{j_s,l_s}\dots a_{j_1,l_1}\,a^\dagger_{i_1,k_1}\dots a^\dagger_{i_r,k_r} = a^\dagger_{i_1,k_1}\dots a^\dagger_{i_u,k_u}\,P' + a^\dagger_{i_u,k_u}\,Q' \text{ with } P',Q'\in\mathbf{A}.$$

And $\langle 0|\,a_{j_s,l_s}\dots a_{j_1,l_1}\,a^\dagger_{i_1,k_1}\dots a^\dagger_{i_r,k_r} = \langle 0|\,a^\dagger_{i_1,k_1}\dots a^\dagger_{i_u,k_u}\,P' + \langle 0|\,a^\dagger_{i_u,k_u}\,Q' = 0.$ □

We just then need to investigate the product $\langle 0|\,a_{j_n,l_n}\dots a_{j_1,l_1}\,a^\dagger_{i_1,k_1}\dots a^\dagger_{i_n,k_n}\,|0\rangle$, where $(j_1,\dots,j_n)$ is a permutation of $(i_1,\dots,i_n)$. Consider a multiset I of n elements in $\mathbb{N}^*$.

Lemma 7. *Let $\theta,\vartheta\in\mathbb{U}_m\wr\mathfrak{S}_I$. Then,*

$$\langle 0|\,a_{\vartheta(n)}\dots a_{\vartheta(1)}\,a^\dagger_{\theta(1)}\dots a^\dagger_{\theta(n)}\,|0\rangle = \sum_{\substack{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n\\ \vartheta=\theta\pi}} q^{\operatorname{cinv}\pi}.$$

Proof. Let $(j_1,\dots,j_n)$ be a permutation of $(i_1,\dots,i_n)$. Then,

$$\begin{aligned}
&a_{j_n,l_n}\dots a_{j_1,l_1}a^\dagger_{i_1,k_1}\dots a^\dagger_{i_n,k_n}|0\rangle\\
&\quad= \sum_{\substack{(u_1,\dots,u_n)\in[n]^n\\ i_{u_1}=j_1,\dots,i_{u_n}=j_n}} \prod_{s\in[n]} q^{u_s-1-\#\left\{r\in[s-1]\,\middle|\,u_r<u_s\right\}} q^{\beta_{k_{u_s},l_s}}|0\rangle\\
&\quad= \sum_{\substack{(u_1,\dots,u_n)\in[n]^n\\ i_{u_1}=j_1,\dots,i_{u_n}=j_n}} \prod_{s\in[n]} q^{\#\left\{r\in[s-1]\,\middle|\,u_r>u_s\right\}} q^{\beta_{k_{u_s},l_s}}|0\rangle\\
&\quad= \sum_{\substack{(u_1,\dots,u_n)\in[n]^n\\ i_{u_1}=j_1,\dots,i_{u_n}=j_n}} q^{\#\left\{(r,s)\in[n]^2\,\middle|\,r<s,u_r>u_s\right\}+\sum_{s\in[n]}\beta_{k_{u_s},l_s}}|0\rangle\\
&\quad= \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \forall s\in[n],j_s=i_{\sigma(s)}}} q^{\#\left\{(r,s)\in[n]^2\,\middle|\,r<s,\sigma(r)>\sigma(s)\right\}+\sum_{s\in[n]}\beta_{k_{\sigma(s)},l_s}}|0\rangle\\
&\quad= \sum_{\substack{\pi=(\sigma,\alpha)\in\mathbb{U}_m\wr\mathfrak{S}_n\forall s\in[n],\\ j_s=i_{\sigma(s)},l_s\equiv k_{\sigma(s)}+\alpha(s)\pmod m}} q^{\operatorname{cinv}\pi}|0\rangle.
\end{aligned}$$

We obtain the result by replacing $a_{j_n,l_n}\dots a_{j_1,l_1}$ and $a^\dagger_{i_1,k_1}\dots a^\dagger_{i_n,k_n}$ by $a_{\vartheta(n)}\dots a_{\vartheta(1)}$ and $a^\dagger_{\theta(1)}\dots a^\dagger_{\theta(n)}$ respectively. □

For example, take $m=4$,

$$\vartheta = \begin{pmatrix} 1 & 2 & 3\\ (2,4) & (5,1) & (2,4)\end{pmatrix} \quad\text{and}\quad \theta = \begin{pmatrix} 1 & 2 & 3\\ (5,2) & (2,3) & (2,1)\end{pmatrix}.$$

Then

$$\begin{aligned}\langle 0| a_{2,4} a_{5,1} a_{2,4} a_{5,2}^{\dagger} a_{2,3}^{\dagger} a_{2,1}^{\dagger} |0\rangle & \\ &= q^{\operatorname{cinv}\left(\begin{smallmatrix} 1 & 2 & 3 \\ (2,1) & (1,3) & (3,3)\end{smallmatrix}\right)} + q^{\operatorname{cinv}\left(\begin{smallmatrix} 1 & 2 & 3 \\ (3,3) & (1,3) & (2,1)\end{smallmatrix}\right)} \\ &= q^4 + q^5\end{aligned}$$

Define the multiplication of an element $\theta = (\varphi, \epsilon)$ of $\mathbb{U}_m \wr \mathfrak{S}_I$ by an element $\pi = (\sigma, \alpha)$ of $\mathbb{U}_m \wr \mathfrak{S}_n$ by

$$\theta \cdot \pi = (\psi, \eta) \in \mathbb{U}_m \wr \mathfrak{S}_I \quad \text{with} \quad \forall i \in [n],\ \psi(i) = \varphi\sigma(i),\ \eta(i) \equiv \epsilon\sigma(i) + \alpha(i) \mod m.$$

Consider the vector space of linear combinations of colored permutations

$$\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_I] := \Big\{ \sum_{\theta \in \mathbb{U}_m \wr \mathfrak{S}_I} z_\theta \theta \ \Big|\ z_\theta \in \mathbb{R}(q) \Big\}.$$

One can easily check that, relatively to the multiplication $\cdot$, $\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_I]$ is a $\mathbb{U}_m \wr \mathfrak{S}_n$-module.

Proposition 1. *We have*

$$\mathbf{M}_I = R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_I]}\Big(\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\operatorname{cinv} \pi} \Big).$$

Proof. Using Lemma 7, we obtain for $\theta \in \mathbb{U}_m \wr \mathfrak{S}_I$

$$\begin{aligned}\theta \cdot \sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\operatorname{cinv} \pi} &= \sum_{\vartheta \in \mathbb{U}_m \wr \mathfrak{S}_I} \Big(\sum_{\substack{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n \\ \vartheta = \theta\pi}} q^{\operatorname{cinv} \pi} \Big) \vartheta \\ &= \sum_{\vartheta \in \mathbb{U}_m \wr \mathfrak{S}_I} \langle 0|\, a_{\vartheta(n)} \dots a_{\vartheta(1)}\, a_{\theta(1)}^{\dagger} \dots a_{\theta(n)}^{\dagger}\, |0\rangle \vartheta\,.\end{aligned}$$ □

4 The Determinant of $\mathrm{M}_{[n]}$

We compute the determinant and the inverse of the regular representation of

$$\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\operatorname{cinv} \pi} \pi\,.$$

Consider the subgroup $\mathfrak{C}_n$ of $\mathbb{U}_m \wr \mathfrak{S}_n$ defined by

$$\mathfrak{C}_n := \big\{ \pi = (\sigma, \alpha) \in \mathbb{U}_m \wr \mathfrak{S}_n \ \big|\ \forall i \in [n],\ \sigma(i) = i \big\}\,.$$

For $i \in [n]$, let ξ_i be the colored permutation

$$\begin{pmatrix} 1 & 2 & \dots & i & \dots & n \\ (1,m) & (2,m) & \dots & (i,1) & \dots & (n,m) \end{pmatrix}$$

in $\mathfrak{C}_n$. We need the following lemma.

Lemma 8. *We have*

$$\det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]}\Big(\sum_{\xi\in\mathfrak{C}_n} q^{\operatorname{cinv}\xi}\xi\Big) = \Big(\big(1+(m-1)q\big)\big(1-q\big)^{m-1}\Big)^{m^n n!}.$$

Proof. Remark that

$$\sum_{\xi\in\mathfrak{C}_n} q^{\operatorname{cinv}\xi}\xi = \prod_{i\in[n]}\big(1+q\sum_{k\in[m-1]}\xi_i^k\big).$$

Then, using Lemma 1 and Lemma 2, we obtain

$$\det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]}\big(1+q\sum_{k\in[m]}\xi_i^k\big) = \Big(\big(1+(m-1)q\big)\big(1-q\big)^{m-1}\Big)^{m^{n-1} n!}. \qquad \square$$

Now we can compute the determinant of $\sum\limits_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi}\pi$.

Theorem 2. *We have*

$$\det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]}\Big(\sum_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi}\pi\Big)$$
$$= \Big(\big(1+(m-1)q\big)(1-q)^{m-1}\prod_{i=1}^{n-1}(1-q^{i^2+i})^{\frac{(n-i)}{(i^2+i)}}\Big)^{m^n n!}.$$

Proof. Every $\pi\in\mathbb{U}_m\wr\mathfrak{S}_n$ has a decomposition $\pi=\sigma\xi$ such that

$$\sigma\in\mathfrak{S}_n,\ \xi\in\mathfrak{C}_n,\ \text{ and }\ \operatorname{cinv}\pi = \operatorname{cinv}\sigma + \operatorname{cinv}\xi.$$

Then,

$$\sum_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi}\pi = \Big(\sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{cinv}\sigma}\sigma\Big)\Big(\sum_{\xi\in\mathfrak{C}_n} q^{\operatorname{cinv}\xi}\xi\Big).$$

It is known that [3, Theorem 2]

$$\det R_{\mathbb{R}(q)[\mathfrak{S}_n]}\Big(\sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{cinv}\sigma}\sigma\Big) = \prod_{i=1}^{n-1}(1-q^{i^2+i})^{\frac{(n-i)n!}{(i^2+i)}}.$$

We finally obtain the result by using Lemma 1 and Lemma 8. $\square$

For $k\in[n]$, denote by $t_{k,n}$ the permutation $(n\ n-1\ \ldots\ k)$ in cycle notation. Let

$$\gamma_n = \overrightarrow{\prod_{k\in[n-1]}} 1-q^{n-k}t_{k,n} \quad\text{and}\quad \varepsilon_n = \overleftarrow{\prod_{k\in[n]}} \frac{\sum_{i=0}^{n-k} q^{(n-k+2)i}\, t_{k,n}^i}{1-q^{(n-k+1)(n-k+2)}}.$$

Furthermore, let

$$\rho_k = \frac{1+(m-2)q-q\sum_{i\in[m-1]}\xi_k^i}{\big(1+(m-1)q\big)(1-q)}.$$

We finish with the inverse of $\sum\limits_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi}\pi$.

Proposition 2. *We have*

$$\Big(\sum_{\pi\in\mathbb{U}_m\wr\mathfrak{S}_n} q^{\operatorname{cinv}\pi}\pi\Big)^{-1} = \prod_{i\in[n]}\rho_i\cdot\overleftarrow{\prod_{i\in[n-1]}}\gamma_{i+1}\varepsilon_i\,.$$

Proof. We obtain

$$\Big(\sum_{\xi\in\mathfrak{C}_n} q^{\operatorname{cinv}\xi}\xi\Big)^{-1} = \prod_{i\in[n]}\rho_i$$

by means of Lemma 3. Then [3, Proposition 2] and Lemma 4 permit us to write

$$\Big(\sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{cinv}\sigma}\sigma\Big)^{-1} = \overleftarrow{\prod_{i\in[n-1]}}\gamma_{i+1}\varepsilon_i\,.$$ □

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Received: 22 May 2018
Accepted for publication: 3 July 2019
Communicated by: Eric Swartz