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New class of boundary value problem for nonlinear fractional differential equations involving Erdélyi-Kober derivative

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Abstract. In this paper, we introduce a new class of boundary value problem for nonlinear fractional differential equations involving the Erdélyi--Kober differential operator on an infinite interval. Existence and uniqueness results for a positive solution of the given problem are obtained by using the Banach contraction principle, the Leray-Schauder nonlinear alternative, and Guo-Krasnosel'skii fixed point theorem in a special Banach space. To that end, some examples are presented to illustrate the usefulness of our main results.

1 Introduction

Fractional-order differential equations have been used in the study of models of many phenomena in various fields of science and engineering, such as viscoelasticity, fluid mechanics, electrochemistry, control, porous media, mathematical biology, and electromagnetic bioengineering. More details are available, for instance, in the books Samko et al. 1993 [22], Podlubny 1999 [20], Kilbas et al. 2006 [9], Sabatier et al. 2007 [21], Das 2008 [7], Diethelm 2010 [8], and Mathai and Haubold 2018 [17].

The classical fractional calculus is based on several definitions for the operators of integration and differentiation of arbitrary order [10]. Among the various definitions of fractional differentiation, the Riemann-Liouville and Caputo fractional derivatives are widely used in the literature. The most useful classical fractional

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integrals, however, seem to be the Erdélyi-Kober operators. These were introduced by Sneddon (see, for example, [24], [25], [26]), who studied their basic properties and emphasized their useful applications to generalized axially symmetric potential theory and other physical problems, such as in electrostatics and elasticity.

The theory of boundary value problems on infinite intervals arises quite naturally and has many applications [4]; it is important and several authors have done much work on this topic [2], [3], [12], [13], [14], [19], [23], [27], [28], [29], [30], [31]. For instance, the author in [18] considered the following nonlinear fractional differential problem on the half-line $\mathbb{R}_+ = (0, +\infty)$,

$$\begin{cases} \mathcal{D}^{\alpha} u(t) + f\big(t, u(t)\big) = 0, \quad u > 0, \\ \lim_{t\to 0} u(t) = 0, \end{cases} \tag{1}$$

where $1 < \alpha \le 2$ and f is a measurable function in $\mathbb{R}_+ \times \mathbb{R}_+$ that satisfies an appropriate condition. Then, he established the existence of infinitely many solutions of (1). More recently, Dhifli and Maagli [16] explored the following boundary value problem:

$$\begin{cases} \mathcal{D}^{\alpha} u(t) + f\big(t, u(t), \mathcal{D}^{\alpha-1} u(t)\big) = 0, \quad t > 0, \\ \lim_{t\to 0} u(t) = 0, \end{cases}$$

where $1 < \alpha \le 2$ and f is a Borel measurable function in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$.

In [32], Zhao and Ge were the first to prove the existence of unbounded solutions for the following nonlinear fractional boundary value problem on an infinite interval:

$$\begin{cases} \mathcal{D}^{\alpha} u(t) + f\big(t, u(t)\big) = 0, \qquad t > 0, \\ u(0) = 0, \\ \lim_{t\to +\infty} \mathcal{D}^{\alpha-1} u(t) = \alpha u(\xi), \end{cases}$$

by using the Leray-Schauder nonlinear alternative. Here, $1 < \alpha \le 2$ and $\mathcal{D}^{\alpha}$ denotes the Riemann-Liouville fractional derivative.

In [23], the authors studied explicit solutions of fractional integral and differential equations involving Erdélyi-Kober operators:

$$t^{-\beta\delta} \mathcal{D}_{\beta}^{\alpha,\delta} u(t) - \lambda u(t) = f(t) = 0, \qquad t > 0,$$
$$u(t) - \lambda t^{\beta\delta} \mathcal{I}_{\beta}^{\alpha,\delta} u(t) = f(t) = 0, \qquad t > 0,$$

by using the transmutation method, where $\alpha, \lambda \in \mathbb{R}$, $\beta, \delta > 0$, f is a given function, and $\mathcal{I}_{\beta}^{\alpha,\delta}, \mathcal{D}_{\beta}^{\alpha,\delta}$ denote the Erdélyi-Kober fractional integral and derivative, respectively.

The aim of this study is to investigate the existence and uniqueness of a positive solution to boundary value problem of a nonlinear fractional differential equation involving Erdélyi-Kober differential operators on an infinite interval:

$$\mathcal{D}_{\beta}^{\gamma,\delta} u(t) + f\big(t, u(t)\big) = 0, \qquad t > 0, \tag{2}$$

with the boundary conditions

$$\lim_{t\to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta+\gamma, 2-\delta} u(t) = 0, \qquad \lim_{t\to\infty} t^{\beta(1+\gamma)} \mathcal{I}^{\delta+\gamma, 2-\delta} u(t) = 0, \tag{3}$$

where $\mathcal{D}_{\beta}^{\gamma,\delta}$ denotes the Erdélyi-Kober fractional derivative operator of order δ and $\mathcal{I}^{\delta+\gamma,2-\delta}$ is the Erdélyi-Kober fractional integral of order $2-\delta$, with $1<\delta\leq 2$, $-2<\gamma<-1$, $\beta>0$, and f is a given function satisfying certain conditions.

We obtain several existence and uniqueness results for the nonlinear fractional boundary value problem (2)–(3). The methods used in this work are the Leray--Schauder nonlinear alternative, Guo-Krasnosel'skii fixed point theorem, and Banach contraction principle, in a special Banach space.

Throughout this paper, we will refer to the following hypotheses:

(H1) $f\colon (0,\infty)\times\mathbb{R}\to(0,\infty)$ is continuous.

(H2) For all $(t,u)\in(0,\infty)\times\mathbb{R}$,

$$F(t,u)=t^{\beta(1+\gamma)-1}f(t,(1+t^{-\beta(1+\gamma)})u)\,,$$

such that

$$|F(t,u)|\leq\psi(t)\omega(|u|),$$

with $\omega\in C\big((0,\infty),(0,\infty)\big)$ nondecreasing and $\psi\in L^1(0,\infty)$.

(H3) $f\colon(0,\infty)\times(0,\infty)\to(0,\infty)$ is continuous, such that

$$t^{\beta(1+\gamma)-1}f(t,u)=a(t)g(t,u)\,,$$

where $a\in L^1(\mathbb{R}_+,\mathbb{R}_+)$, $g\in C(\mathbb{R}_+^2,\mathbb{R}_+)$ and $0<\int_{\frac{\tau}{\lambda}}^{\tau}a(t)\,\mathrm{d}t<\infty$, with $\tau>0$, $\lambda>1$.

(H4) There exists a positive function $q(t)$ with

$$q^*=\int_0^{+\infty}(1+t^{-\beta(1+\gamma)})q(t)\,\mathrm{d}t<\infty,$$

such that

$$t^{\beta(\gamma+1)-1}\big|f(t,u)-f(t,v)\big|\leq q(t)\,|u-v|\,,\qquad t\in(0,\infty),\ u,v\in\mathbb{R}.$$

The remainder of this paper is organized as follows. In Section 2, we recall some necessary preliminary facts. In Section 3, we prove our main results, after we have established sufficient conditions for the existence and uniqueness results for the solution to the problem (2)–(3). In Section 4, an example is given to demonstrate the application of our main results. Finally, we present our conclusions and discuss future research in Section 5.

2 Preliminaries

In this section, we present the necessary definitions and lemmas from fractional calculus theory that will be used to derive our main results.

Definition 1 ([15]). The space of functions C_{α}^{n}, $\alpha\in\mathbb{R}$, $n\in\mathbb{N}$, consists of all functions $f(t)$, $t>0$, that can be represented in the form $f(t)=t^pf_1(t)$ with $p>\alpha$ and $f_1\in C^n([0,\infty))$.

Definition 2 (Erdélyi-Kober fractional integrals [15]). The right and left-hand Erdélyi-Kober fractional integrals of the orders δ and α, respectively, of the function $u \in C_\alpha$ are defined by

$$(\mathcal{I}_\beta^{\gamma,\delta} u)(t) = \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+\delta)} \int_0^t (t^\beta - s^\beta)^{\delta-1} s^{\beta(\gamma+1)-1} u(s)\, \mathrm{d}s, \qquad \delta, \beta > 0,\ \gamma \in \mathbb{R}, \tag{4}$$

and

$$(\mathcal{J}_\beta^{\tau,\alpha} u)(t) = \frac{\beta}{\Gamma(\alpha)} t^{\beta\tau} \int_t^\infty (s^\beta - t^\beta)^{\alpha-1} s^{-\beta(\tau+\alpha-1)-1} u(s)\, \mathrm{d}s, \qquad \alpha, \beta > 0,\ \tau \in \mathbb{R}, \tag{5}$$

where Γ is the Euler gamma function.

Similarly, we define generalized fractional derivatives that correspond to the generalized fractional integrals (4) and (5).

Definition 3 (Erdélyi-Kober fractional derivatives [15]). Let $n-1 < \delta \le n$, $n \in \mathbb{N}$ and $m-1 < \alpha \le m$, $m \in \mathbb{N}$. The right-hand Erdélyi-Kober fractional derivative of the order δ of the function $u \in C_\alpha^n$ is defined by

$$(\mathcal{D}_\beta^{\gamma,\delta} u)(t) = \prod_{j=1}^{n} \Big(\gamma + j + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big) \big(\mathcal{I}_\beta^{\gamma+\delta, n-\delta} u\big)(t)\,. \tag{6}$$

The left-hand Erdélyi-Kober fractional derivative of the order α of the function $u \in C_\alpha^m$ is defined by

$$(\mathcal{P}_\beta^{\tau,\alpha} u)(t) = \prod_{j=0}^{m-1} \Big(\tau + j - \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big) \big(\mathcal{J}_\beta^{\tau+\alpha, m-\alpha} u\big)(t), \tag{7}$$

where

$$\prod_{j=1}^{n} \Big(\gamma + j + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big) (\mathcal{I}_\beta^{\gamma+\delta, n-\delta} u) = \Big(\gamma + 1 + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big) \dots \Big(\gamma + n + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big) (\mathcal{I}_\beta^{\gamma+\delta, n-\delta} u).$$

Lemma 1 ([15]). *Let $\delta, \beta > 0$, $\gamma \in \mathbb{R}$ and $u \in C_\alpha$. The Erdélyi-Kober fractional integrals defined by* (4) *have the following properties:*

$$(\mathcal{I}_\beta^{\gamma,\delta} x^{\lambda\beta} u)(t) = x^{\lambda\beta} (\mathcal{I}_\beta^{\gamma+\lambda,\delta} u)(t),$$

$$(\mathcal{I}_\beta^{\gamma,\delta} \mathcal{I}_\beta^{\gamma+\delta,\alpha} u)(t) = (\mathcal{I}_\beta^{\gamma,\delta+\alpha} u)(t), \tag{8}$$

$$(\mathcal{I}_\beta^{\gamma,\delta} \mathcal{I}_\beta^{\alpha,\eta} u)(t) = (\mathcal{I}_\beta^{\alpha,\eta} \mathcal{I}_\beta^{\gamma,\delta} u)(t). \tag{9}$$

Remark 1. Let $\delta, \beta > 0$ and $\gamma \in \mathbb{R}$. Then we have

$$\mathcal{D}_\beta^{\gamma,\delta} t^p = \frac{\Gamma(\gamma + \delta + \frac{p}{\beta} + 1)}{\Gamma(\gamma + \frac{p}{\beta} + 1)} t^p, \qquad p + \beta(\gamma+1) > 0.$$

In particular,

$$\mathcal{D}_\beta^{\gamma,\delta} t^{-\beta(\gamma-i)} = 0, \quad \text{for each } i = 1, 2, \ldots, n\,.$$

In fact, from Definition 3, for $\delta, \beta > 0$, $\gamma \in \mathbb{R}$, and $p + \beta(\gamma+1) > 0$, we obtain

$$\mathcal{D}_\beta^{\gamma,\delta} t^p = \prod_{j=1}^{n} \Big(\gamma + j + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big) \mathcal{I}_\beta^{\gamma+\delta, n-\delta} t^p;$$

also from Definition 2, we find

$$\mathcal{I}_\beta^{\gamma+\delta, n-\delta} t^p = \frac{\beta}{\Gamma(n-\delta)} t^{-\beta(\gamma+n)} \int_0^t (t^\beta - s^\beta)^{n-\delta-1} s^{p+\beta(\gamma+\delta+1)-1}\, \mathrm{d}s.$$

If we put $x = \dfrac{s^\beta}{t^\beta}$, then we get

$$\mathcal{I}_\beta^{\gamma+\delta, n-\delta} t^p = \frac{1}{\Gamma(n-\delta)} t^p \int_0^1 (1-x)^{n-\delta-1} x^{\gamma+\delta+\frac{p}{\beta}}\, \mathrm{d}x,$$

By the definition of the beta function, we obtain

$$\mathcal{I}_\beta^{\gamma+\delta, n-\delta} t^p = \frac{\Gamma(\gamma+\delta+\frac{p}{\beta}+1)}{\Gamma(n+\gamma+\frac{p}{\beta}+1)} t^p, \qquad p + \beta(\gamma+\delta+1) > 0. \tag{10}$$

Now, if we choose $h = \dfrac{\Gamma(\gamma+\delta+\frac{p}{\beta}+1)}{\Gamma(n+\gamma+\frac{p}{\beta}+1)}$, then it holds that

$$\begin{aligned}
\mathcal{D}_\beta^{\gamma,\delta} t^p &= \prod_{j=1}^{n} (\gamma + j + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}) h t^p \\
&= \Big(\gamma + 1 + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big)\Big(\gamma + 2 + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big) \ldots \Big(\gamma + n + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big) h t^p \\
&= \Big(\gamma + 1 + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big) \ldots \Big(\gamma + n - 1 + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big)\Big(\gamma + n + \frac{p}{\beta}\Big) h t^p \\
&= \Big(\gamma + 1 + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big) \ldots \Big(\gamma + n - 1 + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big) h_1 t^p, \qquad h_1 = \Big(\gamma + n + \frac{p}{\beta}\Big) h \\
&\ \ \vdots \\
&= \Big(\gamma + 1 + \frac{p}{\beta}\Big)\Big(\gamma + 2 + \frac{p}{\beta}\Big) \ldots \Big(\gamma + n - 1 + \frac{p}{\beta}\Big)\Big(\gamma + n + \frac{p}{\beta}\Big) h t^p.
\end{aligned}$$

Thus,

$$\mathcal{D}_\beta^{\gamma,\delta} t^p = \frac{\big(\gamma+1+\frac{p}{\beta}\big)\big(\gamma+2+\frac{p}{\beta}\big)\ldots\big(\gamma+n+\frac{p}{\beta}\big)\Gamma\big(\gamma+\delta+\frac{p}{\beta}+1\big)}{\Gamma\big(n+\gamma+\frac{p}{\beta}+1\big)} t^p, \tag{11}$$

furthermore, $\Gamma\big(n+\gamma+\frac{p}{\beta}+1\big) = \big(n+\gamma+\frac{p}{\beta}\big)\ldots\big(1+\gamma+\frac{p}{\beta}\big)\Gamma\big(1+\gamma+\frac{p}{\beta}\big)$, it follows that

$$\mathcal{D}_\beta^{\gamma,\delta} t^p = \frac{\Gamma\big(\gamma+\delta+\frac{p}{\beta}+1\big)}{\Gamma\big(1+\gamma+\frac{p}{\beta}\big)} t^p, \quad p + \beta(1+\gamma) > 0.$$

In particular, if we put $i = -(\gamma + \frac{p}{\beta})$, then from (11), we obtain that

$$\mathcal{D}_{\beta}^{\gamma,\delta} t^{-\beta(\gamma+i)} = (1-i)(2-i)\cdots(n-1-i)(n-i)\frac{\Gamma(\delta-i+1)}{\Gamma(n-i+1)}t^{p}.$$

Therefore, for $i = 1, 2, \ldots, n$, we can conclude that

$$\mathcal{D}_{\beta}^{\gamma,\delta} t^{-\beta(\gamma+i)} = 0, \qquad \forall \delta, \beta > 0,\ \gamma \in \mathbb{R}.$$

Lemma 2. *Given $u \in C_{\alpha}^{n}$, $n \in \mathbb{N}$, $\delta, \beta > 0$ and $\gamma \in \mathbb{R}$, such that $\alpha \geq -\beta(\gamma+1)$, the fractional deferential equation $\mathcal{D}_{\beta}^{\gamma,\delta} u(t) = 0$ has the following solutions:*

$$u(t) = C_1 t^{-\beta(\gamma+1)} + C_2 t^{-\beta(\gamma+2)} + \cdots + C_n t^{-\beta(\gamma+n)}\,, \tag{12}$$

where $C_i \in \mathbb{R}$ and $i = 1, 2, \ldots, n$.

In fact, from Remark 1, we have $\mathcal{D}_{\beta}^{\gamma,\delta} t^{-\beta(\gamma-i)} = 0$, for each $i = 1, 2, \ldots, n$. Then, the fractional differential equation $\mathcal{D}_{\beta}^{\gamma,\delta} u(t) = 0$ admits a solution as follows:

$$u(t) = C_1 t^{-\beta(\gamma+1)} + C_2 t^{-\beta(\gamma+2)} + \cdots + C_n t^{-\beta(\gamma+n)}\,,$$

where $C_i \in \mathbb{R}$ and $i = 1, 2, \ldots, n$.

Lemma 3 ([15]). *Let $n-1 < \delta < n$, $n \in \mathbb{N}$, $\alpha \geq -\beta(\gamma+1)$, and $u \in C_{\alpha}^{n}$. Then, the following relationship between the Erdélyi-Kober fractional derivative and the Erdélyi-Kober fractional integral of order δ is given by*

$$(\mathcal{I}_{\beta}^{\gamma,\delta} \mathcal{D}_{\beta}^{\gamma,\delta} u)(t) = u(t) - \sum_{k=0}^{n-1} c_k t^{-\beta(1+\gamma+k)}\,,$$

where

$$c_k = \frac{\Gamma(n-k)}{\Gamma(\delta-k)} \lim_{t\to 0} t^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1} \Big(1 + \gamma + i + \frac{1}{\beta} t \frac{\mathrm{d}}{\mathrm{d}t}\Big)(\mathcal{I}_{\beta}^{\gamma+\delta, n-\delta} u)(t).$$

Definition 4. Let E be a real Banach space; a nonempty closed convex set $P \subset E$ is called a cone of E if it satisfies the following conditions:

(i) $u \in P$, $\lambda \geq 0$ implies $\lambda u \in P$,

(ii) $u \in P$, $-u \in P$ implies $u = 0$.

Definition 5 (Equicontinuous). Let E be a Banach space; a subset P in $C(E)$ is called equicontinuous if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall u, v \in E,\ \forall \mathcal{A} \in P: \ \|u - v\| < \delta \Rightarrow |\mathcal{A}(u) - \mathcal{A}(v)| < \varepsilon.$$

Theorem 1 (Ascoli-Arzelà). *Let E be a compact space. If $\mathcal{P}$ is an equicontinuous bounded subset of $C(E)$ then $\mathcal{P}$ is relatively compact.*

Definition 6 (Completely continuous). Let E be a Banach space. We say that $\mathcal{A}\colon E \to E$ is completely continuous if for any bounded subset P of E, the set $\mathcal{A}(P)$ is relatively compact.

The following fixed-point theorems are fundamental in the proofs of our main results.

Theorem 2 (Leray-Schauder Nonlinear Alternative theorem [11]). *Let E be a Banach space, and Ω a bounded open subset of E with $0 \in \Omega$. Then every completely continuous map $\mathcal{A}\colon \overline{\Omega} \to E$ has at least one of the following two properties:*

(i) $\mathcal{A}$ has a fixed point in $\overline{\Omega}$.

(ii) There is an $x \in \partial\Omega$ and $\lambda \in (0,1)$ with $x = \lambda\mathcal{A}x$.

Theorem 3 (Guo-Krasnosel'skii fixed point theorem [1]). *Let E be a Banach space and let $P \subseteq E$ be a cone. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega$ and let $\mathcal{A}\colon P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that:*

(i) $\|\mathcal{A}x\| \le \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|\mathcal{A}x\| \ge \|x\|$, $x \in P \cap \partial\Omega_2$, or

(ii) $\|\mathcal{A}x\| \ge \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|\mathcal{A}x\| \le \|x\|$, $x \in P \cap \partial\Omega_2$

holds. Then $\mathcal{A}$ has at least one positive solution in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 4 (Banach's fixed point theorem [1]). *Let E be a Banach space, D be closed subset of E and $\mathcal{A}\colon D \to D$ be a strict contraction, i.e.,*

$$\|\mathcal{A}u - \mathcal{A}v\| \le k\|u - v\| \text{ for some } k \in (0,1) \text{ and all } u, v \in D.$$

Then $\mathcal{A}$ has a unique fixed point.

3 Main results

In this section, we prove a preparatory lemma for the boundary value problem of nonlinear fractional differential equations with an Erdélyi-Kober derivative.

Lemma 4. *Let $y \in C^2_\alpha$ with $\int_0^\infty s^{\beta(\gamma+m)-1} y(s)\,\mathrm{d}s < \infty$, $m = 1, 2$. Then the fractional differential equation*

$$\mathcal{D}_\beta^{\gamma,\delta} u(t) + y(t) = 0, \qquad t > 0,\ 1 < \delta \le 2,\ -2 < \gamma < -1,\ \beta > 0, \tag{13}$$

with the conditions

$$\lim_{t\to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} u(t) = 0, \tag{14}$$

$$\lim_{t\to\infty} t^{\beta(1+\gamma)} \mathcal{I}^{\delta+\gamma,2-\delta} u(t) = 0, \tag{15}$$

has a unique solution given by

$$u(t) = \int_0^\infty G(t,s) s^{\beta(\gamma+1)-1} y(s)\,\mathrm{d}s\,, \tag{16}$$

where

$$G(t,s)=\begin{cases}\frac{\beta}{\Gamma(\delta)}\left[t^{-\beta(\gamma+1)}-t^{-\beta(\delta+\gamma)}(t^{\beta}-s^{\beta})^{\delta-1}\right], & 0<s\le t<\infty\,,\\ \frac{\beta}{\Gamma(\delta)}t^{-\beta(\gamma+1)}, & 0<t\le s<\infty\,,\end{cases} \tag{17}$$

is called the Green function of the boundary value problem (13)-(15)*.*

Proof. Let $1<\delta\le 2$ with $-2<\gamma<-1$ and $\beta>0$. It is easy to prove that the operator $\mathcal{I}_{\beta}^{\gamma,\delta}$ has the linearity property for all $\delta>0$. Now, by applying $\mathcal{I}_{\beta}^{\gamma,\delta}$ to equation (13) we obtain

$$\mathcal{I}_{\beta}^{\gamma,\delta}\mathcal{D}_{\beta}^{\gamma,\delta}u(t)+\mathcal{I}_{\beta}^{\gamma,\delta}y(t)=0. \tag{18}$$

By using Lemma 3, for $1<\delta\le 2$, we can easily find that

$$\mathcal{I}_{\beta}^{\gamma,\delta}\mathcal{D}_{\beta}^{\gamma,\delta}u(t)=u(t)-c_0t^{-\beta(1+\gamma)}+c_1t^{-\beta(2+\gamma)}\,,$$

for some constants $c_0,c_1\in\mathbb{R}$. Thus, (18) gives

$$u(t)-c_0t^{-\beta(1+\gamma)}-c_1t^{-\beta(2+\gamma)}+\mathcal{I}_{\beta}^{\gamma,\delta}y(t)=0\,,$$

which means that

$$u(t)=c_0t^{-\beta(1+\gamma)}+c_1t^{-\beta(2+\gamma)}-\mathcal{I}_{\beta}^{\gamma,\delta}y(t). \tag{19}$$

The boundary condition (14) implies that

$$c_0\lim_{t\to 0}t^{\beta(2+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}t^{-\beta(1+\gamma)}+c_1\lim_{t\to 0}t^{\beta(2+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}t^{-\beta(2+\gamma)} - \lim_{t\to 0}t^{\beta(2+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}\mathcal{I}_{\beta}^{\gamma,\delta}y(t)=0\,,$$

consequently, from (8), (9) and (10) we obtain

$$\begin{aligned}\lim_{t\to 0}t^{\beta(2+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}t^{-\beta(1+\gamma)}&=\lim_{t\to 0}t^{\beta(2+\gamma)}\frac{\Gamma(\delta)}{\Gamma(2)}t^{-\beta(1+\gamma)}=0,\\ \lim_{t\to 0}t^{\beta(2+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}t^{-\beta(2+\gamma)}&=\lim_{t\to 0}t^{\beta(2+\gamma)}\frac{\Gamma(\delta-1)}{\Gamma(1)}t^{-\beta(2+\gamma)}=\Gamma(\delta-1),\\ \lim_{t\to 0}t^{\beta(2+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}\mathcal{I}_{\beta}^{\gamma,\delta}y(t)&=\lim_{t\to 0}\frac{\beta}{\Gamma(2)}\int_0^t(t^{\beta}-s^{\beta})s^{\beta(\gamma+1)-1}y(s)\,\mathrm{d}s\\ &=\lim_{t\to 0}\frac{\beta}{\Gamma(2)}t^{\beta}\int_0^t s^{\beta(\gamma+1)-1}y(s)\,\mathrm{d}s\\ &\quad-\lim_{t\to 0}\frac{\beta}{\Gamma(2)}\int_0^t s^{\beta(\gamma+2)-1}y(s)\,\mathrm{d}s=0\,,\end{aligned}$$

and therefore $c_1=0$.

In view of the boundary condition (15), we conclude that

$$c_0\lim_{t\to\infty}t^{\beta(1+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}t^{-\beta(1+\gamma)}-\lim_{t\to\infty}t^{\beta(1+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}\mathcal{I}_{\beta}^{\gamma,\delta}y(t)=0,$$

consequently, from (8), (9), (10), we find that

$$\lim_{t\to\infty} t^{\beta(1+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}t^{-\beta(1+\gamma)} = \lim_{t\to\infty} t^{\beta(1+\gamma)}\frac{\Gamma(\delta)}{\Gamma(2)}t^{-\beta(1+\gamma)} = \frac{\Gamma(\delta)}{\Gamma(2)},$$

$$\begin{aligned}\lim_{t\to\infty} t^{\beta(1+\gamma)}\mathcal{I}^{\delta+\gamma,2-\delta}\mathcal{I}_{\beta}^{\gamma,\delta}y(t) &= \lim_{t\to\infty}\frac{\beta}{\Gamma(2)}t^{-\beta}\int_0^t (t^{\beta}-s^{\beta})s^{\beta(\gamma+1)-1}y(s)\,\mathrm{d}s\\ &= \lim_{t\to\infty}\frac{\beta}{\Gamma(2)}\int_0^t s^{\beta(\gamma+1)-1}y(s)\,\mathrm{d}s\\ &\quad-\lim_{t\to\infty}\frac{\beta}{\Gamma(2)}t^{-\beta}\int_0^t s^{\beta(\gamma+2)-1}y(s)\,\mathrm{d}s\\ &= \frac{\beta}{\Gamma(2)}\int_0^{\infty} s^{\beta(\gamma+1)-1}y(s)\,\mathrm{d}s,\end{aligned}$$

and therefore, $c_0 = \frac{\beta}{\Gamma(\delta)}\int_0^{\infty} s^{\beta(\gamma+1)-1}y(s)\,\mathrm{d}s$.

Hence, the unique solution of the problem (13)–(15) is given by

$$\begin{aligned}u(t) &= \frac{\beta}{\Gamma(\delta)}t^{-\beta(1+\gamma)}\int_0^{\infty} s^{\beta(\gamma+1)-1}y(s)\,\mathrm{d}s\\ &\quad-\frac{\beta}{\Gamma(\delta)}t^{-\beta(\gamma+\delta)}\int_0^t (t^{\beta}-s^{\beta})^{\delta-1}s^{\beta(\gamma+1)-1}y(s)\,\mathrm{d}s\\ &= \frac{\beta}{\Gamma(\delta)}\int_0^t\left[t^{-\beta(1+\gamma)}-t^{-\beta(\gamma+\delta)}(t^{\beta}-s^{\beta})^{\delta-1}\right]s^{\beta(\gamma+1)-1}y(s)\,\mathrm{d}s\\ &\quad+\frac{\beta}{\Gamma(\delta)}\int_t^{\infty} t^{-\beta(1+\gamma)}s^{\beta(\gamma+1)-1}y(s)\,\mathrm{d}s\\ &= \int_0^{+\infty} G(t,s)s^{\beta(\gamma+1)-1}y(s)\,\mathrm{d}s\,.\end{aligned}$$

The proof is complete. □

Now, we present some properties of Green's function that form the basis of our main work.

Lemma 5. *For $1<\delta\le 2$, $-2<\gamma<-1$ and $\beta>0$, the function $G(t,s)$ in Lemma 4 satisfies the following conditions:*

1. $\dfrac{G(t,s)}{1+t^{-\beta(1+\gamma)}} > 0,\ \forall t,s\in(0,\infty),$
2. $\dfrac{G(t,s)}{1+t^{-\beta(1+\gamma)}} \le \dfrac{\beta}{\Gamma(\delta)},\ \forall t,s\in(0,\infty),$
3. *For all $0<\dfrac{\tau}{\lambda}\le t\le\tau$ and $\forall s>\dfrac{\tau}{\lambda^2}$, where $\lambda>1$, $\tau>0$, we have*

$$\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} \ge \frac{\beta(\delta-1)\tau^{-\beta(1+\gamma)}}{\Gamma(\delta)\lambda^{\beta(1-\gamma)}(1+\tau^{-\beta(1+\gamma)})} = \frac{\beta}{\Gamma(\delta)}p(\tau).$$

Proof. 1. For $t \leq s$. It is easy to check that $\dfrac{G(t,s)}{1+t^{-\beta(1+\gamma)}} > 0$.

For $s \leq t$, it holds that

$$\begin{aligned}
\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} &= \frac{\beta}{\Gamma(\delta)(1+t^{-\beta(1+\gamma)})}\left[t^{-\beta(\gamma+1)} - t^{-\beta(\delta+\gamma)}(t^{\beta}-s^{\beta})^{\delta-1}\right] \\
&= \frac{\beta t^{-\beta(\gamma+1)}}{\Gamma(\delta)(1+t^{-\beta(1+\gamma)})}\left[1-\left[1-\left(\frac{s}{t}\right)^{\beta}\right]^{\delta-1}\right] \\
&= \frac{\beta t^{-\beta(\gamma+1)}}{\Gamma(\delta)(1+t^{-\beta(1+\gamma)})}(\delta-1)\int_{1-\frac{s^{\beta}}{t^{\beta}}}^{1} u^{\delta-2}\mathrm{d}u \\
&\geq \frac{\beta t^{-\beta(\gamma+1)}}{\Gamma(\delta-1)(1+t^{-\beta(1+\gamma)})}(1)^{\delta-2}\left[1-\left(1-\frac{s^{\beta}}{t^{\beta}}\right)\right] \\
&\geq \frac{\beta t^{-\beta(\gamma+1)}}{\Gamma(\delta-1)(1+t^{-\beta(1+\gamma)})}\frac{s^{\beta}}{t^{\beta}} > 0\,. \qquad (20)
\end{aligned}$$

2. For each $t, s \in (0,\infty)$,

$$G(t,s) \leq \frac{\beta}{\Gamma(\delta)}t^{-\beta(\gamma+1)} \quad \text{implies that} \quad \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} \leq \frac{\beta}{\Gamma(\delta)}\,.$$

3. Let $0 < \dfrac{\tau}{\lambda} \leq t \leq \tau$, where $\lambda > 1$, $\tau > 0$. For $t \leq s$, we obtain

$$\begin{aligned}
\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} &= \frac{\beta t^{-\beta(1+\gamma)}}{\Gamma(\delta)(1+t^{-\beta(1+\gamma)})} \\
&\geq \frac{\beta \tau^{-\beta(1+\gamma)}}{\Gamma(\delta)\lambda^{-\beta(1+\gamma)}(1+\tau^{-\beta(1+\gamma)})} \\
&\geq \frac{(\delta-1)\beta \tau^{-\beta(1+\gamma)}}{\Gamma(\delta)\lambda^{\beta(1-\gamma)}(1+\tau^{-\beta(1+\gamma)})} \\
&= \frac{\beta}{\Gamma(\delta)}p(\tau).
\end{aligned}$$

For $s \leq t$, from (20), we obtain

$$\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} \geq \frac{\beta t^{-\beta(\gamma+1)}}{\Gamma(\delta-1)(1+t^{-\beta(1+\gamma)})}\frac{s^{\beta}}{t^{\beta}}.$$

We notice that there are two cases to consider.

First case: suppose that $0 < s \leq \frac{\tau}{\lambda}$, if we choose $s \in \left[\frac{\tau}{\lambda^2}, \frac{\tau}{\lambda}\right]$ then

$$\begin{aligned}
\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} &\geq \frac{\beta \tau^{-\beta(\gamma+1)}}{\lambda^{-\beta(1+\gamma)}\Gamma(\delta-1)(1+\tau^{-\beta(1+\gamma)})}\frac{\tau^{\beta}}{\lambda^{2\beta}\tau^{\beta}} \\
&\geq \frac{\beta \tau^{-\beta(\gamma+1)}}{\lambda^{\beta(1-\gamma)}\Gamma(\delta-1)(1+\tau^{-\beta(1+\gamma)})} \\
&= \frac{\beta}{\Gamma(\delta)}p(\tau).
\end{aligned}$$

Second case: suppose that $\frac{\tau}{\lambda} \le s \le t$ then it follows that

$$\begin{aligned}\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} &\ge \frac{\beta\tau^{-\beta(\gamma+1)}}{\lambda^{-\beta(1+\gamma)}\Gamma(\delta-1)(1+\tau^{-\beta(1+\gamma)})}\frac{\tau^{\beta}}{\lambda^{\beta}\tau^{\beta}}\\ &\ge \frac{\tau^{-\beta(\gamma+1)}}{\lambda^{-\beta\gamma}\Gamma(\delta-1)(1+\tau^{-\beta(1+\gamma)})}\\ &\ge \frac{\beta\tau^{-\beta(\gamma+1)}}{\lambda^{\beta(1-\gamma)}\Gamma(\delta-1)(1+\tau^{-\beta(1+\gamma)})}\\ &= \frac{\beta}{\Gamma(\delta)}p(\tau).\end{aligned}$$

□

We now turn to the question of existence for the boundary value problem (2)–(3). Let

$$C([0,\infty]) = \left\{ u \;\middle|\; \begin{array}{l} u \text{ is a continuous function on } (0,+\infty) \text{ such that} \\ \lim_{t\to 0} u(t) \text{ and } \lim_{t\to+\infty} u(t) \text{ exist} \end{array} \right\}.$$

It is easy to see that $C([0,\infty])$ is a Banach space with the norm

$$\|u\|_* = \sup_{t>0} |u(t)|$$

for instance see [6], [30].

In this work, we use the space C_∞ to study the problem (2)–(3), which is denoted by

$$C_\infty\big((0,\infty),\mathbb{R}\big) = \left\{ u \in C\big((0,\infty),\mathbb{R}\big) \;\middle|\; \begin{array}{l} \lim_{t\to 0} \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \text{ and} \\ \lim_{t\to+\infty} \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \text{ exist} \end{array} \right\};$$

C_∞ is a Banach space with the norm

$$\|u\|_\infty = \sup_{t>0} \left| \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \right|.$$

In fact, it is easy to see that $C_\infty\big((0,\infty),\mathbb{R}\big)$ is a normed linear space. Let $\{x_m\}$ be a Cauchy sequence in C_∞; then

$$\left\{ y_m \;\middle|\; y_m = \frac{x_m}{1+t^{-\beta(1+\gamma)}} \right\} \subset C([0,\infty])$$

is also a Cauchy sequence. Therefore there exists $y_0 \subset C([0,\infty])$ such that

$$\lim_{m\to+\infty} \|y_m - y_0\|_* = 0.$$

Let $x_0(t) = \big(1+t^{-\beta(1+\gamma)}\big)y_0(t)$. Then $x_0 \in C_\infty\big((0,\infty),\mathbb{R}\big)$ and

$$\begin{aligned}\lim_{m\to+\infty} \|x_m - x_0\|_\infty &= \lim_{m\to+\infty} \sup_{t>0} \left| \frac{x_m(t)-x_0(t)}{1+t^{-\beta(1+\gamma)}} \right|\\ &= \lim_{m\to+\infty} \|y_m - y_0\|_* = 0.\end{aligned}$$

Hence, C_∞ is Banach space.

Define an integral operator $\mathcal{A}\colon C_\infty \to C_\infty$ by

$$\mathcal{A}u(t) = \int_0^\infty G(t,s) s^{\beta(1+\gamma)-1} f\big(s, u(s)\big)\,\mathrm{d}s\,, \quad t \in (0,\infty)\,, \tag{21}$$

where $G(t,s)$ is defined by (17).

Clearly, from Lemma 4, the fixed points of the operator $\mathcal{A}$ coincide with the solutions of the problem (2)–(3).

To obtain the complete continuity of $\mathcal{A}$, the following Lemma is still needed.

Lemma 6 ([32]). *Let*

$$V = \left\{ u \in C_\infty \;\middle|\; \|u\|_\infty < l,\ l > 0 \right\}, \qquad V_1 = \left\{ \frac{u(t)}{1 + t^{-\beta(1+\gamma)}} \;\middle|\; u \in V \right\}.$$

If V_1 is equicontinous on any compact intervals of $(0,\infty)$ and equiconvergent at infinity, then V is relatively compact on C_∞.

Remark 2 ([32]). V_1 is called equiconvergent at infinity if and only if for all $\varepsilon > 0$, there exists $v(\varepsilon) > 0$ such that for all $u \in V_1$, $t_1, t_2 \geq v$, it holds that

$$\left| \frac{u(t_1)}{1 + t_1^{-\beta(1+\gamma)}} - \frac{u(t_2)}{1 + t_2^{-\beta(1+\gamma)}} \right| < \varepsilon.$$

Lemma 7. *If (H1)–(H2) hold, then $\mathcal{A}\colon C_\infty \to C_\infty$ is completely continuous.*

Proof. First, for all $u \in C_\infty$, we have

$$\begin{aligned}
\|\mathcal{A}u(t)\|_\infty &= \sup_{t>0} \frac{|\mathcal{A}u(t)|}{1 + t^{-\beta(1+\gamma)}} \\
&= \sup_{t>0} \left| \int_0^\infty \frac{G(t,s)}{1 + t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f\big(s, u(s)\big)\,\mathrm{d}s \right| \\
&\leq \frac{\beta}{\Gamma(\delta)} \int_0^\infty \left| s^{\beta(\gamma+1)-1} f\big(s, u(s)\big) \right| \mathrm{d}s
\end{aligned}$$

together with conditions (H1) and (H2), it then follows that

$$\begin{aligned}
\int_0^\infty \left| s^{\beta(\gamma+1)-1} f\big(s, u(s)\big) \right| \mathrm{d}s &= \int_0^\infty \left| s^{\beta(\gamma+1)-1} f\left(s, \frac{(1 + s^{-\beta(1+\gamma)})u(s)}{1 + s^{-\beta(1+\gamma)}} \right) \right| \mathrm{d}s \\
&= \int_0^\infty \left| F\left(s, \frac{u(s)}{1 + s^{-\beta(1+\gamma)}} \right) \right| \mathrm{d}s \\
&\leq \int_0^\infty \psi(s) \omega\Big(\frac{|u(s)|}{1 + s^{-\beta(1+\gamma)}} \Big)\,\mathrm{d}s \\
&\leq \omega(\|u\|_\infty) \int_0^\infty \psi(s)\,\mathrm{d}s < \infty\,.
\end{aligned}$$

Hence, $\mathcal{A}\colon C_\infty \to C_\infty$ is well-defined.

Let $\Omega = \{u \in C_\infty \mid \|u\|_\infty \le k,\ k > 0\}$ be a bounded subset of C_∞.

In the following, we divide the proof into several steps.

Step 1: $\mathcal{A}$ is continuous. Let $(u_n)_{n\in\mathbb{N}} \in C_\infty$ be a convergent sequence to u in C_∞ such that $\|u\|_\infty \le k$; from Lemma 5, we obtain that

$$\begin{aligned}\|\mathcal{A}u_n - \mathcal{A}u\|_\infty &= \sup_{t\in(0,\infty)} \left|\frac{\mathcal{A}u_n(t) - \mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}}\right| \\ &\le \frac{\beta}{\Gamma(\delta)} \sup_{t\in(0,\infty)} \left|\int_0^\infty s^{\beta(\gamma+1)-1} f\big(s, u_n(s)\big)\,\mathrm{d}s \right.\\ &\quad \left. - \int_0^\infty s^{\beta(\gamma+1)-1} f\big(s, u(s)\big)\,\mathrm{d}s\right|.\end{aligned}$$

By the condition (H2), we obtain

$$\begin{aligned}\left|s^{\beta(\gamma+1)-1} f\big(s, u(s)\big)\right| &= \left|s^{\beta(\gamma+1)-1} f\left(s, \frac{(1+s^{-\beta(1+\gamma)})u(s)}{1+s^{-\beta(1+\gamma)}}\right)\right| \\ &= \left|F\left(s, \frac{u(s)}{1+s^{-\beta(1+\gamma)}}\right)\right| \\ &\le \psi(s)\omega\left(\frac{|u(s)|}{1+s^{-\beta(1+\gamma)}}\right) \\ &\le \psi(s)\omega(\|u\|_\infty) \\ &\le \omega(k)\psi(s) \in L^1(0,\infty).\end{aligned}$$

Together with the continuity of the function $s^{\beta(\gamma+1)-1} f\big(s, u(s)\big)$, the Lebesgue dominated convergence theorem [5, Theorem 12.12, page 199] yields

$$u \to \int_0^\infty s^{\beta(\gamma+1)-1} f\big(s, u(s)\big)\,\mathrm{d}s$$

is continuous, and it follows that

$$\int_0^\infty s^{\beta(\gamma+1)-1} f\big(s, u_n(s)\big)\,\mathrm{d}s \to \int_0^\infty s^{\beta(\gamma+1)-1} f\big(s, u(s)\big)\,\mathrm{d}s \text{ as } n \to \infty.$$

Therefore,

$$\|\mathcal{A}u_n - \mathcal{A}u\|_\infty \to 0, \text{ as } n \to \infty.$$

Step 2: $\mathcal{A}(\Omega)$ is relatively compact. First, we show that $\mathcal{A}(\Omega)$ is uniformly bounded. Let $u \in \Omega$; by the condition (H2), we obtain

$$\begin{aligned}
\frac{|\mathcal{A}u(t)|}{1+t^{-\beta(1+\gamma)}} &= \left|\int_0^\infty \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\,\mathrm{d}s\right| \\
&\leq \frac{\beta}{\Gamma(\delta)} \int_0^\infty \left|s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\right| \mathrm{d}s \\
&\leq \frac{\beta\omega(k)}{\Gamma(\delta)} \int_0^\infty \psi(s)\,\mathrm{d}s < \infty.
\end{aligned}$$

Consequently,

$$\|\mathcal{A}u\|_\infty \leq \frac{\beta\omega(k)}{\Gamma(\delta)} \int_0^\infty \psi(s)\,\mathrm{d}s < \infty\,, \text{ for all } u \in \Omega, \tag{22}$$

and hence $\mathcal{A}(\Omega)$ is uniformly bounded.

Next, letting $V = \left\{\dfrac{\mathcal{A}u}{1+t^{-\beta(1+\gamma)}} \;\middle|\; u \in \Omega\right\}$, we show that V is equicontinuous on any compact interval of $\mathbb{R}_+$.

For all $u \in \Omega$, $t_1, t_2 \in [a,b]$, $0 < a < b < \infty$ and $t_1 \leq t_2$, we can find

$$\begin{aligned}
&\left|\frac{\mathcal{A}u(t_2)}{1+t_2^{-\beta(1+\gamma)}} - \frac{\mathcal{A}u(t_1)}{1+t_1^{-\beta(1+\gamma)}}\right| \\
&\quad \leq \int_0^\infty \left|\frac{G(t_2,s)}{1+t_2^{-\beta(1+\gamma)}} - \frac{G(t_1,s)}{1+t_1^{-\beta(1+\gamma)}}\right| \left|s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\right| \mathrm{d}s \\
&\quad \leq \int_0^\infty \left|\frac{G(t_2,s)}{1+t_2^{-\beta(1+\gamma)}} - \frac{G(t_1,s)}{1+t_2^{-\beta(1+\gamma)}} + \frac{G(t_1,s)}{1+t_2^{-\beta(1+\gamma)}}\right. \\
&\qquad \left. - \frac{G(t_1,s)}{1+t_1^{-\beta(1+\gamma)}}\right| \left|s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\right| \mathrm{d}s \\
&\quad \leq \int_0^\infty \left|\frac{G(t_2,s)-G(t_1,s)}{1+t_2^{-\beta(1+\gamma)}} - \frac{G(t_1,s)\big(t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)}\big)}{\big(1+t_2^{-\beta(1+\gamma)}\big)\big(1+t_1^{-\beta(1+\gamma)}\big)}\right| \\
&\qquad \times \left|s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\right| \mathrm{d}s \\
&\quad \leq \int_0^\infty \frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}} \left|s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\right| \mathrm{d}s \\
&\qquad + \int_0^\infty \frac{G(t_1,s)(t_2^{-\beta(1+\gamma)} - t_1^{-\beta(1+\gamma)})}{(1+t_2^{-\beta(1+\gamma)})(1+t_1^{-\beta(1+\gamma)})} \left|s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\right| \mathrm{d}s.
\end{aligned}$$

It remains to show that the right-hand side of the above inequality tends to zero. It is easy to see that

$$\begin{aligned}
\int_0^\infty &\frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s \\
&\le \int_0^{t_1}\frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s \\
&\quad+\int_{t_1}^{t_2}\frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s \\
&\quad+\int_{t_2}^{\infty}\frac{|G(t_2,s)-G(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s \\
&\le \frac{\beta\omega(k)}{\Gamma(\delta)}\int_0^{t_1}\frac{1}{1+t_2^{-\beta(1+\gamma)}}\Big|t_2^{-\beta(1+\gamma)}-t_1^{-\beta(1+\gamma)} \\
&\qquad - t_2^{-\beta(\delta+\gamma)}(t_2^\beta-s^\beta)^{\delta-1}+t_1^{-\beta(\delta+\gamma)}(t_1^\beta-s^\beta)^{\delta-1}\Big|\,\psi(s)\,\mathrm{d}s \\
&\quad+\frac{\beta\omega(k)}{\Gamma(\delta)}\int_{t_1}^{t_2}\frac{1}{1+t_2^{-\beta(1+\gamma)}}\Big|t_2^{-\beta(1+\gamma)}-t_1^{-\beta(1+\gamma)} \\
&\qquad - t_2^{-\beta(\delta+\gamma)}(t_2^\beta-s^\beta)^{\delta-1}\Big|\,\psi(s)\,\mathrm{d}s \\
&\quad+\frac{\beta\omega(k)}{\Gamma(\delta)}\int_{t_2}^{\infty}\frac{t_2^{-\beta(1+\gamma)}-t_1^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}\psi(s)\,\mathrm{d}s, \\
&\to 0 \text{ uniformly as } t_1\to t_2 \text{ for all } u\in\Omega.
\end{aligned}$$

Analogously, we can obtain

$$\int_0^\infty \frac{G(t_1,s)\big(t_2^{-\beta(1+\gamma)}-t_1^{-\beta(1+\gamma)}\big)}{\big(1+t_2^{-\beta(1+\gamma)}\big)\big(1+t_1^{-\beta(1+\gamma)}\big)}s^{\beta(\gamma+1)-1}\left|f\big(s,u(s)\big)\right|\,\mathrm{d}s\to 0\,,$$

uniformly as $t_1\to t_2$ for all $u\in\Omega$. Hence, V is locally equicontinuous on $(0,\infty)$. Finally, we show that V is equiconvergent at ∞. We know that

$$\begin{aligned}
Au(t)&=\frac{\beta}{\Gamma(\delta)}t^{-\beta(1+\gamma)}\int_0^\infty s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\,\mathrm{d}s \\
&\quad-\frac{\beta}{\Gamma(\delta)}t^{-\beta(\gamma+\delta)}\int_0^t (t^\beta-s^\beta)^{\delta-1}s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\,\mathrm{d}s\,,
\end{aligned}\tag{23}$$

observing that for any $u\in\Omega$, the condition (H2) gives

$$\int_0^\infty \left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s\le\omega(\|u\|_\infty)\int_0^\infty\psi(s)\,\mathrm{d}s<\infty,\tag{24}$$

for a given $\varepsilon>0$, there exists a constant $L>0$, such that

$$\int_L^\infty \left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s<\varepsilon.\tag{25}$$

However, because $\lim\limits_{t\to+\infty}\frac{t^{-\beta(1+\gamma)}}{1+t^{-\beta(1+\gamma)}}=1$, there exists a constant $T_1>0$, such that for any $t_1,t_2\geq T_1$, and we obtain

$$\left|\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|\leq\left|1-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|+\left|1-\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}\right|<\varepsilon. \tag{26}$$

Similarly, $\lim\limits_{t\to+\infty}\frac{t^{-\beta(\delta+\gamma)}(t^\beta-s^\beta)^{\delta-1}}{1+t^{-\beta(1+\gamma)}}=1$ and thus there exists a constant $T_2>L>0$, such that for any $t_1,t_2\geq T_2$ and $0<s\leq L$, it holds that

$$\begin{aligned}&\left|\frac{t_2^{-\beta(\delta+\gamma)}(t_2^\beta-s^\beta)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(\delta+\gamma)}(t_1^\beta-s^\beta)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right|\\
&\leq\left|1-\frac{t_1^{-\beta(\delta+\gamma)}(t_1^\beta-s^\beta)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right|+\left|1-\frac{t_2^{-\beta(\delta+\gamma)}(t_2^\beta-s^\beta)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}\right|\\
&\leq\left|1-\frac{t_1^{-\beta(\delta+\gamma)}(t_1^\beta-L^\beta)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right|+\left|1-\frac{t_2^{-\beta(\delta+\gamma)}(t_2^\beta-L^\beta)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}\right|\\
&<\varepsilon.\end{aligned} \tag{27}$$

Now, we choose $T>\max\{T_1,T_2\}$ for all $t_1,t_2\geq T$. By (23), we can obtain

$$\begin{aligned}&\left|\frac{\mathcal{A}u(t_2)}{1+t_2^{-\beta(1+\gamma)}}-\frac{\mathcal{A}u(t_1)}{1+t_1^{-\beta(1+\gamma)}}\right|\\
&\leq\frac{\beta}{\Gamma(\delta)}\left|\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|\int_0^\infty\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\mathrm{d}s\\
&+\frac{\beta}{\Gamma(\delta)}\left|\int_0^{t_2}\frac{t_2^{-\beta(\gamma+\delta)}(t_2^\beta-s^\beta)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\,\mathrm{d}s\right.\\
&\left.-\int_0^{t_1}\frac{t_1^{-\beta(\gamma+\delta)}(t_1^\beta-s^\beta)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\,\mathrm{d}s\right|\\
&\leq\frac{\beta}{\Gamma(\delta)}\left|\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|\int_0^\infty\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\mathrm{d}s\\
&+\frac{\beta}{\Gamma(\delta)}\left|\int_0^{L}\frac{t_2^{-\beta(\gamma+\delta)}(t_2^\beta-s^\beta)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\,\mathrm{d}s\right.\\
&\left.-\int_0^{L}\frac{t_1^{-\beta(\gamma+\delta)}(t_1^\beta-s^\beta)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\,\mathrm{d}s\right|\\
&+\frac{\beta}{\Gamma(\delta)}\left|\int_L^{t_2}\frac{t_2^{-\beta(\gamma+\delta)}(t_2^\beta-s^\beta)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\,\mathrm{d}s\right.\\
&\left.-\int_L^{t_1}\frac{t_1^{-\beta(\gamma+\delta)}(t_1^\beta-s^\beta)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\,\mathrm{d}s\right|;\end{aligned}$$

a direct calculation yields

$$\begin{aligned}
&\left|\frac{\mathcal{A}u(t_2)}{1+t_2^{-\beta(1+\gamma)}}-\frac{\mathcal{A}u(t_1)}{1+t_1^{-\beta(1+\gamma)}}\right| \\
&\qquad\le \frac{\beta}{\Gamma(\delta)}\left|\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|\int_0^\infty\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s \\
&\qquad\quad+\frac{\beta}{\Gamma(\delta)}\int_0^L\left|\frac{t_2^{-\beta(\gamma+\delta)}(t_2^\beta-s^\beta)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}\right. \\
&\qquad\quad\left.-\frac{t_1^{-\beta(\gamma+\delta)}(t_1^\beta-s^\beta)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\right|\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s \\
&\qquad\quad+\frac{\beta}{\Gamma(\delta)}\int_L^{t_2}\frac{t_2^{-\beta(\gamma+\delta)}(t_2^\beta-s^\beta)^{\delta-1}}{1+t_2^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s \\
&\qquad\quad+\frac{\beta}{\Gamma(\delta)}\int_L^{t_1}\frac{t_1^{-\beta(\gamma+\delta)}(t_1^\beta-s^\beta)^{\delta-1}}{1+t_1^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s\,.
\end{aligned}$$

From (24), (25), (26), (27) and for $t_1, t_2 \to \infty$ we obtain

$$\begin{aligned}
&\left|\frac{\mathcal{A}u(t_2)}{1+t_2^{-\beta(1+\gamma)}}-\frac{\mathcal{A}u(t_1)}{1+t_1^{-\beta(1+\gamma)}}\right| \\
&\qquad< \frac{\beta}{\Gamma(\delta)}\varepsilon\int_0^\infty\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s \\
&\qquad\quad+\frac{\beta}{\Gamma(\delta)}\varepsilon\int_0^L\left|s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\right|\,\mathrm{d}s+\frac{2\beta}{\Gamma(\delta)}\varepsilon\,.
\end{aligned}$$

Hence, V is equiconvergent at ∞. Consequently, Lemma 6 yields that V is relatively compact.

Therefore, $\mathcal{A}\colon C_\infty \to C_\infty$ is completely continuous. □

3.1 Existence of at least one solution

Now, to prove the first following existence result, we use the Leray-Schauder nonlinear alternative fixed point theorem.

Theorem 5. *Assume that hypotheses (H1)–(H2) hold, and that there exists $k > 0$, such that*

$$\frac{\beta\omega(k)\int_0^\infty \psi(s)\,\mathrm{d}s}{k\Gamma(\delta)} < 1; \tag{28}$$

then, the fractional boundary value problem (2)–(3) *has at least one solution $u \in \Omega$.*

Proof. From the proof of Lemma 7, we know that $\mathcal{A}$ is a completely continuous operator. We apply the nonlinear alternative of Leray-Schauder to prove that $\mathcal{A}$

has at least one nontrivial solution in Ω. Let $u \in \partial\Omega$, such that $u = \lambda\mathcal{A}u$, $\lambda \in (0,1)$. From (22), we obtain

$$\|u\|_\infty = \lambda \|\mathcal{A}u\|_\infty \leq \|\mathcal{A}u\|_\infty \leq \frac{\beta\omega(k)}{\Gamma(\delta)} \int_0^\infty \psi(s)\,\mathrm{d}s,$$

and thus

$$k \leq \frac{\beta\omega(k)}{\Gamma(\delta)} \int_0^\infty \psi(s)\,\mathrm{d}s;$$

hence,

$$\frac{\beta\omega(k)\int_0^\infty \psi(s)\,\mathrm{d}s}{k\Gamma(\delta)} \geq 1,$$

which contradicts (28). By Theorem 2 and Lemma 7, the boundary value problem (2)–(3) has at least one solution $u \in \Omega$. □

3.2 Existence of at least one positive solution

In this subsection, we establish the existence of a positive solution for the boundary value problem (2)–(3). First, we introduce the following results.

Remark 3. Let $\delta, \beta, \gamma, \lambda, \varrho, l, \tau \in \mathbb{R}$, such that $1 < \delta \leq 2$, $\beta > 0$, $-2 < \gamma \leq -1$, $\lambda > 1$, and $l, \tau, \varrho > 0$. If the conditions (H2)–(H3) hold, then

$$\int_0^{+\infty} s^{\beta(1+\gamma)-1} f(s,u(s))\,\mathrm{d}s \leq \eta \int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f(s,u(s))\,\mathrm{d}s, \tag{29}$$

where $\eta = \frac{l}{\varrho(\lambda^2-1)} + 1 > 1$.

In fact, for all $t \in \left[\frac{\tau}{\lambda^2}, \tau\right]$, there exists a finite constant $\varrho > 0$, such that $t^{\beta(1+\gamma)-1} f(t,u) \geq \varrho$. Thus,

$$\begin{aligned}\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f\big(s,u(s)\big)\,\mathrm{d}s &\geq \int_{\frac{\tau}{\lambda^2}}^{\tau} s^{\beta(1+\gamma)-1} f\big(s,u(s)\big)\,\mathrm{d}s \\ &\geq \frac{\tau(\lambda^2-1)}{\lambda^2}\varrho,\end{aligned}$$

and hence,

$$\frac{\lambda^2}{\tau(\lambda^2-1)\varrho} \int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1} f\big(s,u(s)\big)\,\mathrm{d}s \geq 1. \tag{30}$$

If $\sup\limits_{t\in(0,\frac{\tau}{\lambda^2}]} t^{\beta(1+\gamma)-1} f\big(t,u(t)\big)$ is bounded for $u \in (0,\infty)$, then there exists some $l_0 > 0$, such that

$$\left|t^{\beta(1+\gamma)-1} f\big(t,u(t)\big)\right| \leq l_0\,,\ \forall t \in (0, \frac{\tau}{\lambda^2}]\,.$$

Similarly, if $\sup_{t\in(0,\frac{\tau}{\lambda^2}]} t^{\beta(1+\gamma)-1}f\big(t,u(t)\big)$ is unbounded for $u\in(0,\infty)$, then there exists $M_0>0$, such that

$$\sup_{0<u\le M_0}\ \sup_{t\in(0,\frac{\tau}{\lambda^2}]} t^{\beta(1+\gamma)-1}f(t,u(t))\le l_1,\ \text{ for some } l_1>0\,.$$

In all cases, if we choose $l=\max\{l_0,l_1\}$, for all $t\in(0,\frac{\tau}{\lambda^2}]$, then we obtain

$$t^{\beta(1+\gamma)-1}f\big(t,u(t)\big)\le l\,,$$

and thus

$$\int_0^{\frac{\tau}{\lambda^2}} s^{\beta(1+\gamma)-1}f\big(s,u(s)\big)\,\mathrm{d}s\le\frac{l\tau}{\lambda^2}. \tag{31}$$

From (30) and (31), we can find that

$$\begin{aligned}\int_0^{+\infty} s^{\beta(1+\gamma)-1}f\big(s,u(s)\big)\,\mathrm{d}s &= \int_0^{\frac{\tau}{\lambda^2}} s^{\beta(1+\gamma)-1}f\big(s,u(s)\big)\,\mathrm{d}s\\ &\quad+\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1}f\big(s,u(s)\big)\,\mathrm{d}s\\ &\le\frac{l\tau}{\lambda^2}+\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1}f\big(s,u(s)\big)\,\mathrm{d}s\\ &\le\frac{l}{\varrho(\lambda^2-1)}\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1}f\big(s,u(s)\big)\,\mathrm{d}s\\ &\quad+\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1}f\big(s,u(s)\big)\,\mathrm{d}s\\ &\le\Big(\frac{l}{\varrho(\lambda^2-1)}+1\Big)\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1}f\big(s,u(s)\big)\,\mathrm{d}s\\ &\le\eta\int_{\frac{\tau}{\lambda^2}}^{+\infty} s^{\beta(1+\gamma)-1}f\big(s,u(s)\big)\,\mathrm{d}s.\end{aligned}$$

Let us define the cone K by

$$K=\left\{u\in C_\infty\ \middle|\ u(t)>0,\ \text{ and } \forall t>0;\ \min_{t\in[\frac{\tau}{\lambda},\tau]}\frac{u(t)}{1+t^{-\beta(1+\gamma)}}\ge\frac{p(\tau)}{\eta}\,\|u\|_\infty\right\}.$$

Lemma 8. *We have $\mathcal{A}(K)\subset K$.*

Proof. We know from Lemma 5 that

$$\begin{aligned}\|\mathcal{A}u(t)\|_\infty=\sup_{t>0}\frac{|\mathcal{A}u(t)|}{1+t^{-\beta(1+\gamma)}}&=\sup_{t>0}\left|\int_0^\infty\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\,\mathrm{d}s\right|\\ &\le\frac{\beta}{\Gamma(\delta)}\int_0^\infty s^{\beta(\gamma+1)-1}f\big(s,u(s)\big)\,\mathrm{d}s;\end{aligned} \tag{32}$$

also from (29) and Lemma 5, for all $t \in \left[\frac{\tau}{\lambda}, \tau\right]$, $\tau > 0$, and $\lambda > 1$, we obtain

$$\begin{aligned}
\frac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}} &= \int_0^{\infty} \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\,\mathrm{d}s \\
&= \int_0^{\frac{\tau}{\lambda^2}} \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\,\mathrm{d}s \\
&\quad + \int_{\frac{\tau}{\lambda^2}}^{\infty} \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\,\mathrm{d}s \\
&\geq \int_{\frac{\tau}{\lambda^2}}^{\infty} \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\,\mathrm{d}s \\
&\geq \frac{\beta}{\Gamma(\delta)} p(\tau) \int_{\frac{\tau}{\lambda^2}}^{\infty} s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\,\mathrm{d}s \\
&\geq \frac{p(\tau)}{\eta} \frac{\beta}{\Gamma(\delta)} \int_0^{+\infty} s^{\beta(1+\gamma)-1} f\big(s,u(s)\big)\,\mathrm{d}s\,. \qquad (33)
\end{aligned}$$

From (32), we obtain

$$\frac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}} \geq \frac{p(\tau)}{\eta} \|\mathcal{A}u(t)\|_{\infty}\,;$$

therefore, $\min\limits_{t\in\left[\frac{\tau}{\lambda},\tau\right]} \dfrac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}} \geq \frac{p(\tau)}{\eta} \|\mathcal{A}(u)\|_{\infty}$, which proves that $\mathcal{A}(K) \subset K$. □

For convenience, we denote some important constants:

$$F_0 = \lim_{u\to 0} \sup_{t>0} \frac{g\big(t,(1+t^{-\beta(1+\gamma)})u\big)}{u}, \qquad f_\infty = \lim_{u\to+\infty} \inf_{t>0} \frac{g\big(t,(1+t^{-\beta(1+\gamma)})u\big)}{u},$$
$$f_0 = \lim_{u\to 0} \inf_{t>0} \frac{g\big(t,(1+t^{-\beta(1+\gamma)})u\big)}{u}, \qquad F_\infty = \lim_{u\to+\infty} \sup_{t>0} \frac{g\big(t,(1+t^{-\beta(1+\gamma)})u\big)}{u}.$$

Theorem 6. *Assume that hypotheses (H2)–(H3) hold. If the following condition is satisfied:*

$$F_0 = 0, \qquad f_\infty = \infty,$$

then the boundary value problem (2)–(3) *has at least one positive solution.*

Proof. From Lemma 7, $\mathcal{A}$ is a completely continuous operator. Now, because $F_0 = 0$, we may choose $r_1 > 0$, such that

$$g(t,(1+t^{-\beta(1+\gamma)})u) \leq \varepsilon u, \text{ for } 0 < u \leq r_1,\ t > 0, \qquad (34)$$

where $\varepsilon > 0$ satisfies

$$\varepsilon \leq \frac{\Gamma(\delta)}{\beta \int_0^{+\infty} a(s)\,\mathrm{d}s}\,. \qquad (35)$$

Therefore, for all $u \in K$ and $\|u\|_\infty = r_1$, from Lemma 5, we have

$$\begin{aligned}\frac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}} &= \int_0^\infty \frac{G(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f\big(s,u(s)\big)\,\mathrm{d}s \\ &\le \frac{\beta}{\Gamma(\delta)} \int_0^\infty s^{\beta(1+\gamma)-1} f\big(s,u(s)\big)\,\mathrm{d}s.\end{aligned}$$

By the condition (H3) and from (34), we obtain

$$\begin{aligned}\frac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}} &\le \frac{\beta}{\Gamma(\delta)} \int_0^\infty a(s) g\big(s,u(s)\big)\,\mathrm{d}s \\ &\le \frac{\beta}{\Gamma(\delta)} \int_0^\infty a(s) g\Big(s, \big(1+s^{-\beta(1+\gamma)}\big)\frac{u(s)}{1+s^{-\beta(1+\gamma)}}\Big)\,\mathrm{d}s \\ &\le \frac{\beta}{\Gamma(\delta)} \int_0^\infty a(s)\varepsilon \frac{u(s)}{1+s^{-\beta(1+\gamma)}}\,\mathrm{d}s \\ &\le \frac{\beta}{\Gamma(\delta)}\varepsilon \|u\|_\infty \int_0^\infty a(s)\,\mathrm{d}s\,.\end{aligned}$$

Therefore, from (35), we obtain

$$\|\mathcal{A}u\|_\infty \le \|u\|_\infty\,,\ u \in K \text{ and } \|u\|_\infty = r_1,\ \forall t > 0. \tag{36}$$

If we choose

$$\Omega_1 = \{u \in C_\infty, \|u\|_\infty < r_1\}\,,$$

then (36) shows that

$$\|\mathcal{A}u\|_\infty \le \|u\|_\infty\,,\ \text{for } u \in K \cap \partial\Omega_1.$$

Furthermore, because $f_\infty = \infty$, there exists $r > 0$ such that

$$g\Big(t, \big(1+t^{-\beta(1+\gamma)}\big)u\Big) \ge mu\,,\ \text{for } u \ge r,\ t > 0, \tag{37}$$

where $m > 0$ is chosen so that

$$m \ge \frac{\eta^2\Gamma(\delta)}{\beta p^2(\tau)\int_{\frac{\tau}{\lambda}}^{\tau} a(s)\,\mathrm{d}s}. \tag{38}$$

Let $r_2 \ge \max\big\{r_1, \frac{\eta r}{p(\tau)}\big\}$ and $\Omega_2 = \{u \in C_\infty, \|u\|_\infty < r_2\}$ then $\Omega_1 \subset \Omega_2$.

Thus, for all $u \in K$ and $\|u\|_\infty = r_2$, we have that

$$\frac{u(t)}{1+t^{-\beta(1+\gamma)}} \geq \min_{\frac{\tau}{\lambda}\leq t\leq \tau} \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \geq \frac{p(\tau)}{\eta}\|u\|_\infty = \frac{p(\tau)}{\eta} r_2 \geq r\,.$$

From (33), (37), (38) and by the condition (H3), for all $\frac{\tau}{\lambda} \leq t \leq \tau$, we have

$$\begin{aligned}
\frac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}} &\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty s^{\beta(1+\gamma)-1} f\big(s, u(s)\big)\,\mathrm{d}s \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty a(s) g(s, u(s))\,\mathrm{d}s \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty a(s) g(s, (1+s^{-\beta(1+\gamma)})\frac{u(s)}{1+s^{-\beta(1+\gamma)}})\,\mathrm{d}s \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} m \int_0^\infty a(s) \frac{u(s)}{1+s^{-\beta(1+\gamma)}}\,\mathrm{d}s \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} m \min_{\frac{\tau}{\lambda}\leq t\leq\tau} \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \int_{\frac{\tau}{\lambda}}^{\tau} a(s)\,\mathrm{d}s \\
&\geq \frac{\beta}{\eta^2\Gamma(\delta)} p^2(\tau) m \|u\|_\infty \int_{\frac{\tau}{\lambda}}^{\tau} a(s)\,\mathrm{d}s \geq \|u\|_\infty\,.
\end{aligned}$$

Hence, $\|\mathcal{A}u\|_\infty \geq \|u\|_\infty$, for $u \in K \cap \partial\Omega_2$. Therefore, by the first part of Theorem 3, it follows that $\mathcal{A}$ has a fixed point in $K \cap \partial(\overline{\Omega}_2 \backslash \Omega_1)$. This completes the proof. □

Similarly to the previous theorem, we can prove the following theorem.

Theorem 7. *Assume that hypotheses (H2)–(H3) hold. If the following condition is satisfied:*

$$f_0 = \infty, \qquad F_\infty = 0,$$

then the boundary value problem (2)–(3) *has at least one positive solution.*

Proof. From Lemma 7, $\mathcal{A}$ is a completely continuous operator. Now, because $f_0 = \infty$, there exists $R_1 > 0$ such that

$$g\Big(t, \big(1+t^{-\beta(1+\gamma)}\big)u\Big) \geq Mu\,, \text{ for } 0 < u \leq R_1,\, t > 0, \tag{39}$$

where $M > 0$ is chosen so that

$$M \geq \frac{\eta^2\Gamma(\delta)}{\beta p^2(\tau)\int_0^\infty a(s)\,\mathrm{d}s}. \tag{40}$$

Let $\Omega_1 = \{u \in C_\infty, \|u\|_\infty < R_1\}$. Thus, if $u \in K$ and $\|u\|_\infty = R_1$ $(u \in \partial\Omega_1)$, then for all $t \in \left[\frac{\tau}{\lambda}, \tau\right]$, from (33), (39), (40) and by the condition (H3), we obtain

$$\begin{aligned}
\frac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}} &\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty s^{\beta(1+\gamma)-1} f\big(s, u(s)\big)\,\mathrm{d}s \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty a(s) g\big(s, u(s)\big)\,\mathrm{d}s \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} \int_0^\infty a(s) g\Big(s, \big(1+s^{-\beta(1+\gamma)}\big)\frac{u(s)}{1+s^{-\beta(1+\gamma)}}\Big)\,\mathrm{d}s \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} M \int_{\frac{\tau}{\lambda}}^{\tau} a(s) \frac{u(s)}{1+s^{-\beta(1+\gamma)}}\,\mathrm{d}s \\
&\geq \frac{\beta p(\tau)}{\eta\Gamma(\delta)} M \int_{\frac{\tau}{\lambda}}^{\tau} a(s) \frac{p(\tau)}{\eta} \|u\|_\infty \,\mathrm{d}s \\
&\geq \frac{\beta p^2(\tau)}{\eta^2\Gamma(\delta)} M \|u\|_\infty \int_{\frac{\tau}{\lambda}}^{\tau} a(s)\,\mathrm{d}s \geq \|u\|_\infty\,.
\end{aligned}$$

Hence

$$\|\mathcal{A}u\|_\infty \geq \|u\|_\infty\,, \text{ for } u \in K \cap \partial\Omega_1.$$

Furthermore, from $F_\infty = 0$, there exists $R > 0$, such that

$$g\Big(t, \big(1+t^{-\beta(1+\gamma)}\big)u\Big) \leq \varepsilon u, \text{ for } u > R, \text{ and } t > 0, \tag{41}$$

where $\varepsilon > 0$ satisfies

$$\varepsilon \leq \frac{\Gamma(\delta)}{\beta \int_0^\infty a(s)\,\mathrm{d}s}\,. \tag{42}$$

Let $\Omega_2 = \{u \in C_\infty, \|u\|_\infty < R_2\}$, where $R_2 > \max\{R_1, R\}$. Then, $\Omega_1 \subset \Omega_2$. Now, we define the function J as

$$\begin{gathered}
J \colon \mathbb{R}_+ \to \mathbb{R}_+, \\
J(a) = \sup_{0<u\leq a} \sup_{t>0} g\Big(t, \big(1+t^{-\beta(1+\gamma)}\big)u\Big).
\end{gathered}$$

Suppose $u \in K$ and $\|u\|_\infty = R_2$ ($u \in \partial\Omega_2$). Then from (41) we obtain

$$\begin{aligned}
\sup_{0<u\leq R_2} \sup_{t>0} g\Big(t, \big(1+t^{-\beta(1+\gamma)}\big)u\Big) &\leq \varepsilon \sup_{0<u\leq R_2} u = \varepsilon R_2 \\
&\implies J(R_2) \leq \varepsilon R_2 = \varepsilon \|u\|_\infty\,.
\end{aligned} \tag{43}$$

Given Lemma 5 and the condition (H3), (42) and (43) yield that

$$\begin{aligned}
\frac{\mathcal{A}u(t)}{1+t^{-\beta(1+\gamma)}} &\leq \frac{\beta}{\Gamma(\delta)}\int_0^{\infty} s^{\beta(1+\gamma)-1} f\big(s,u(s)\big)\,\mathrm{d}s \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^{\infty} a(s)g\big(s,u(s)\big)\,\mathrm{d}s \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^{\infty} a(s)g\Big(s,\big(1+s^{-\beta(1+\gamma)}\big)\frac{u(s)}{1+s^{-\beta(1+\gamma)}}\Big)\,\mathrm{d}s \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^{\infty} a(s)\sup_{0<u\leq R_2}\sup_{s>0} g\Big(s,\big(1+s^{-\beta(1+\gamma)}\big)\frac{u(s)}{1+s^{-\beta(1+\gamma)}}\Big)\,\mathrm{d}s \\
&= \frac{\beta}{\Gamma(\delta)}\int_0^{\infty} a(s)J(R_2)\,\mathrm{d}s \\
&\leq \frac{\beta}{\Gamma(\delta)}\varepsilon\,\|u\|_{\infty}\int_0^{\infty} a(s)\,\mathrm{d}s\,.
\end{aligned}$$

Hence, $\|\mathcal{A}u\|_{\infty} \leq \|u\|_{\infty}$, for $u \in K \cap \partial\Omega_2$.

Therefore, by the second part of Theorem 3, it follows that $\mathcal{A}$ has a fixed point in $K \cap \partial(\overline{\Omega}_2 \setminus \Omega_1)$. The proof is complete. □

3.3 Uniqueness of solution

The last result of the existence is based on the Banach contraction principle theorem.

Theorem 8. *Assume that hypotheses (H1), (H2), and (H4) hold. If*

$$\frac{q^*\beta}{\Gamma(\delta)} < 1, \tag{44}$$

then the boundary value problem (2)–(3) *has a unique solution* $u \in C_{\infty}$.

Proof. We shall show that the operator $\mathcal{A}$ defined by (21) is a contraction mapping.

Let $u, v \in C_{\infty}$. From Lemma 5 and by the condition (H4), we can obtain that

$$\begin{aligned}
\left|\frac{\mathcal{A}u(t)-\mathcal{A}v(t)}{1+t^{-\beta(1+\gamma)}}\right| &= \int_0^{\infty}\frac{G(t,s)}{1+t^{-\beta(1+\gamma)}}s^{\beta(\gamma+1)-1}\left|f\big(s,u(s)\big)-f\big(s,v(s)\big)\right|\,\mathrm{d}s \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^{\infty} s^{\beta(\gamma+1)-1}\left|f\big(s,u(s)\big)-f\big(s,v(s)\big)\right|\,\mathrm{d}s \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^{\infty} q(s)\,|u-v|\,\mathrm{d}s \\
&\leq \frac{\beta}{\Gamma(\delta)}\int_0^{\infty} q(s)\big(1+s^{-\beta(1+\gamma)}\big)\left|\frac{u-v}{1+s^{-\beta(1+\gamma)}}\right|\,\mathrm{d}s\,,
\end{aligned}$$

this implies that

$$\begin{aligned}
\|\mathcal{A}(u)-\mathcal{A}(v)\|_{\infty} &\leq \frac{\beta}{\Gamma(\delta)}\|u-v\|_{\infty}\int_0^{\infty} q(s)\big(1+s^{-\beta(1+\gamma)}\big)\,\mathrm{d}s \\
&\leq \frac{\beta q^*}{\Gamma(\delta)}\|u-v\|_{\infty}\,.
\end{aligned}$$

It follows from the assumption (44) and the preceding estimate that $\mathcal{A}$ is a contraction mapping. Applying Banach's fixed point Theorem 4, the operator $\mathcal{A}$ has a fixed point that corresponds to the unique solution of problem (2)–(3). □

4 Examples

In this section, we present some examples to illustrate the usefulness of our main results.

Example 1. Consider the following boundary value problem:

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{3}{2}} u(t) + t^{\frac{3}{2}}\sqrt{\left|\frac{u}{1+t^{\frac{1}{2}}}\right|}\mathrm{e}^{-t} = 0, \quad t > 0, \\ \lim\limits_{t\to 0} t^{\frac{1}{2}} I_1^{0,\frac{1}{2}} u(t) = 0, \\ \lim\limits_{t\to \infty} t^{-\frac{1}{2}} I_1^{0,\frac{1}{2}} u(t) = 0. \end{cases} \tag{45}$$

Here, $f(t,u) = t^{\frac{3}{2}}\sqrt{\left|\frac{u}{1+t^{\frac{1}{2}}}\right|}\mathrm{e}^{-t}$, $\delta = \frac{3}{2}$, $\gamma = -\frac{3}{2}$ and $\beta = 1$.

(H1) It is easy to show that the function f is continuous for any $(t,u) \in (0,\infty)\times\mathbb{R}$.
(H2) From the expression of the function f, it follows that

$$F(t,u) = t^{\beta(1+\gamma)-1} f\Big(t, \big(1 + t^{-\beta(1+\gamma)}\big)u\Big) = \sqrt{|u|}\mathrm{e}^{-t}.$$

If we choose $\omega(u) = \sqrt{u}$, $\psi(t) = \mathrm{e}^{-t}$, then we obtain

$$|F(t,u)| \le \psi(t)\omega(|u|), \text{ on } (0,\infty)\times\mathbb{R},$$

with $\omega \in C\big((0,\infty),(0,\infty)\big)$ nondecreasing and $\psi \in L^1(0,\infty)$. Then, the condition (H2) holds.

If we choose $k > \frac{4}{\pi}$, we show that

$$\frac{\beta\omega(k)\int_0^\infty \psi(s)\,\mathrm{d}s}{k\Gamma(\delta)} = \frac{2}{k\sqrt{\pi}} < 1;$$

therefore, (28) is satisfied. Hence, all the conditions of Theorem 5 hold, and problem (45) has at least one solution.

Example 2. Consider the following problem:

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{5}{3}} u(t) + \frac{t^{\frac{3}{2}}\exp(-t)u^2}{(1+\sqrt{t})} = 0, \quad t > 0, \\ \lim\limits_{t\to 0} t^{\frac{1}{2}} I_1^{\frac{1}{6},\frac{1}{3}} u(t) = 0, \\ \lim\limits_{t\to \infty} t^{-\frac{1}{2}} I_1^{\frac{1}{6},\frac{1}{3}} u(t) = 0. \end{cases} \tag{46}$$

Here, $f(t,u) = \frac{t^{\frac{3}{2}}\exp(-t)u^2}{(1+\sqrt{t})}$, $\beta = 1$, $\gamma = -\frac{3}{2}$ and $\delta = \frac{5}{3}$.

(H2) $F(t,u) = t^{\beta(1+\gamma)-1} f\Big(t, \big(1 + t^{-\beta(1+\gamma)}\big)u\Big) = \exp(-t)u^2$, verify

$$|F(t,u)| \le \psi(t)\omega(|u|), \text{ on } (0,+\infty)\times\mathbb{R},$$

where $\psi(t) = \exp(-t) \in L^1(0,+\infty)$ and $\omega(u) = u^2 \in C\big((0,+\infty),(0,+\infty)\big)$. Then, the condition (H2) holds.
(H3) It is clear that the function f is continuous on $(0,+\infty)\times(0,+\infty)$, and

$$t^{-\frac{3}{2}} f(t,u) = \frac{\exp(-t)u^2}{(1+\sqrt{t})} = a(t)g(t,u),$$

where $a(t) = \exp(-t)$ and $g(t,u) = \frac{u^2}{1+\sqrt{t}} \in C\big((0,+\infty)\times(0,+\infty)\big)$. Hence, the condition (H3) is satisfied.

We have $g(t,(1+\sqrt{t})u) = u^2$, which implies that

$$F_0 = \lim_{u\to 0}\sup_{t>0} \frac{g\big(t,(1+\sqrt{t})u\big)}{u} = 0,$$

$$f_\infty = \lim_{u\to\infty}\inf_{t>0} \frac{g\big(t,(1+\sqrt{t})u\big)}{u} = \infty.$$

It follows from Theorem 6 that problem (46) has at least one positive solution.

Example 3. We take $\beta = 1$, $\gamma = -\frac{3}{2}$ and $\delta = \frac{7}{6}$. Consider the following problem:

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{7}{6}} u(t) + f(t,u) = 0, & t>0, \\ \lim_{t\to 0} t^{\frac{1}{2}} I_1^{\frac{16}{6},\frac{5}{6}} u(t) = 0, \\ \lim_{t\to\infty} t^{-\frac{1}{2}} I_1^{\frac{16}{6},\frac{5}{6}} u(t) = 0, \end{cases} \tag{47}$$

where $f(t,u) = t^{\frac{3}{2}}\Big[\arctan\big(\frac{|u|}{1+\sqrt{t}}\big) + 1\Big]\exp(-t)$.
(H2) It is clear that $F(t,u) = [\arctan(|u|)+1]\exp(-t)$, verify

$$|F(t,u)| \le \psi(t)\omega(|u|),$$

where $\psi(t) = \exp(-t) \in L^1(0,+\infty)$, $\omega(u) = \arctan(|u|)+1 \in C((0,+\infty),(0,+\infty))$. Then, the condition (H2) holds.
(H3) $f(t,u)$ is continuous on $\big((0,+\infty)\times(0,+\infty),(0,+\infty)\big)$, and

$$t^{-\frac{3}{2}} f(t,u) = \Big[\arctan\Big(\frac{|u|}{1+\sqrt{t}}\Big) + 1\Big]\exp(-t) = a(t)g(t,u),$$

where $a(t) = \exp(-t)$, $g(t,u) = \Big[\arctan\big(\frac{|u|}{1+\sqrt{t}}\big) + 1\Big] \in C((0,+\infty)\times(0,+\infty))$. Hence, the condition (H3) is satisfied.

We have $g(t,(1+\sqrt{t})u) = \arctan(|u|)+1$, which implies that

$$F_0 = \lim_{u\to 0}\sup_{t>0} \frac{g(t,(1+\sqrt{t})u)}{u} = \lim_{u\to 0} \frac{\arctan(|u|)+1}{u} = \infty\,,$$

$$f_\infty = \lim_{u\to\infty}\inf_{t>0} \frac{g(t,(1+\sqrt{t})u)}{u} = \lim_{u\to\infty} \frac{\arctan(|u|)+1}{u} = 0\,.$$

It follows from Theorem 7 that problem (47) has at least one positive solution.

Example 4. We take $\beta = 1$, $\gamma = -\frac{3}{2}$ and $\delta = \frac{3}{2}$. Consider the following problem:

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{3}{2}} u(t) + f(t,u) = 0, \qquad t > 0, \\ \lim_{t\to 0} t^{\frac{1}{2}} I_1^{0,\frac{1}{2}} u(t) = 0\,, \\ \lim_{t\to\infty} t^{-\frac{1}{2}} I_1^{0,\frac{1}{2}} u(t) = 0\,, \end{cases} \tag{48}$$

where $f(t,u) = \frac{t^{\frac{3}{2}} \exp(-t)}{2\sqrt{\pi}(1+t^{\frac{1}{2}})} \arctan(|u|)$.

(H1) $f(t,u)$ is continuous on $\big((0,+\infty)\times\mathbb{R},(0,+\infty)\big)$.

(H2) It is clear that $F(t,u) = \frac{\exp(-t)}{2\sqrt{\pi}} \arctan(|u|)$, verify

$$|F(t,u)| \le \psi(t)\omega(|u|),$$

where $\psi(t) = \exp(-t) \in L^1(0,+\infty)$, $\omega(u) = \frac{\arctan(|u|)}{2\sqrt{\pi}} \in C\big((0,+\infty),(0,+\infty)\big)$. Then (H2) holds.

(H4) We have

$$\begin{aligned} \left| t^{-\frac{3}{2}} f(t,u) - t^{-\frac{3}{2}} f(t,v) \right| &= \left| \frac{\exp(-t)}{2\sqrt{\pi}(1+t^{\frac{1}{2}})} \arctan(|u|) - \frac{\exp(-t)}{2\sqrt{\pi}(1+t^{\frac{1}{2}})} \arctan(|v|) \right| \\ &= \frac{\exp(-t)}{2\sqrt{\pi}(1+t^{\frac{1}{2}})} \Big| \arctan(|u|) - \arctan(|v|) \Big| \\ &\le \frac{\exp(-t)}{2\sqrt{\pi}(1+t^{\frac{1}{2}})} \Big| |u| - |v| \Big| \\ &\le \frac{\exp(-t)}{2\sqrt{\pi}(1+t^{\frac{1}{2}})} |u-v|\,. \end{aligned}$$

If we put $q(t) = \frac{\exp(-t)}{2\sqrt{\pi}(1+t^{\frac{1}{2}})}$, then we obtain

$$q^* = \int_0^{+\infty} (1+t^{\frac{1}{2}}) \frac{\exp(-t)}{2\sqrt{\pi}(1+t^{\frac{1}{2}})}\, \mathrm{d}t = \frac{1}{2\sqrt{\pi}} < \infty.$$

Hence, the condition (H4) is satisfied.

Moreover, we have

$$\frac{\beta q^*}{\Gamma(\delta)} = \frac{1}{2\sqrt{\pi}\Gamma(\frac{3}{2})} = \frac{1}{\pi} < 1\,,$$

and the condition (44) is satisfied. It follows from Theorem 8 that the boundary value problem (48) has a unique solution $u \in C_\infty$.

5 Conclusion

In this work, the existence and uniqueness of a positive solution for the nonlinear fractional differential equations with initial conditions comprising the Erdélyi--Kober fractional derivatives have been discussed in a special Banach space $C_\infty(0,\infty)$. For our discussion, we have used the Leray-Schauder nonlinear alternative and Guo--Krasnosel'skii fixed point theorems, as well as the Banach contraction principle.

The differential operator used has two additional parameters (δ and γ), which may give higher degrees of freedom than the fractional differential equation reported in literature. Future work will be directed toward the Caputo version of the Erdélyi--Kober fractional differential equation and fractional coupled systems of differential equations involving Erdélyi-Kober derivatives.

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