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Oscillation in deviating differential equations using an iterative method

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Abstract. Sufficient oscillation conditions involving lim sup and lim inf for first-order differential equations with non-monotone deviating arguments and nonnegative coefficients are obtained. The results are based on the iterative application of the Grönwall inequality. Examples, numerically solved in MATLAB, are also given to illustrate the applicability and strength of the obtained conditions over known ones.

1 Introduction

Consider the first-order linear delay differential equation (*DDE*) of the form

$$x'(t) + p(t)x(\tau(t)) = 0 \tag{1}$$

and the (dual) first-order linear advanced differential equation (*ADE*) of the form

$$x'(t) - q(t)x(\sigma(t)) = 0 \tag{2}$$

with $t \geq t_0 > 0$, where p, q, τ, σ are continuous functions on $[t_0, \infty)$ such that p, q are nonnegative and do not vanish eventually, $\tau(t) < t$ for $t \geq t_0$, $\lim_{t\to\infty} \tau(t) = \infty$ and $\sigma(t) > t$ for $t \geq t_0$.

Definition 1. By a solution of (1) we mean a function which is continuous on $[\bar{t}_*, \infty)$ for some $t_* \geq t_0$, where $\bar{t}_* = \inf\{\tau(t) : t \geq t_*\}$ and satisfies (1) for all

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$t \geq t_*$. Similarly, by a solution of (2) we mean a function which is continuous on $[t_0, \infty)$ and satisfies (2) for all $t \geq t_0$.

Definition 2. A solution of (1) or (2) is said to be *oscillatory*, if it is neither eventually positive nor eventually negative. If there exists an eventually positive or an eventually negative solution, the equation is *nonoscillatory*. An equation is *oscillatory* if all its solutions oscillate.

Mathematical modeling involving *DDE*s is widely used for analysis and predictions in various areas of the life sciences, e.g., population dynamics, epidemiology, immunology, physiology, neural networks, etc. Here, the delay is proposed to play an essential role in order to represent basically a time taken to complete some hidden processes, which are known to cause a time lag, like the stages of the life cycle, the time between infection of a cell and the production of new viruses, the duration of the infectious period, the immune period and so on. Time delays also allow to model the memory effects of the studied phenomenon, taking into account the dependence of the model's present state on its past history.

Analogously, *ADE*s are used in many applied problems where the evolution rate depends not only on the present, but also on the future. While delays in *DDE*s represent the retrospective memory of the past, advances in *ADE*s represent the prospective memory of the future, accounting for the influence on the system of potential future actions, which are available at the present time. For instance, population dynamics, economics problems or mechanical control engineering are typical fields where such phenomena are thought to occur.

The problem of establishing sufficient conditions for the oscillation of all solutions of equations (1) and (2) has been the subject of many investigations. The reader is referred to [1], [2], [3], [4], [5], [6], [7], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [20], [21], [22], [23], [24], [25] and the references cited therein. Most of these papers, however, concern the special case where the arguments are nondecreasing. Some of these papers study the general case where the arguments are not necessarily monotone. See, for example, [1], [2], [3], [4], [5], [6], [7], [15], [21] and the references cited therein.

Apart from the pure mathematical interest, the importance of considering nonmonotone arguments is justified by the fact that they approximate the natural phenomena described by equations of the type (1) or (2). That is because there are always natural disturbances (e.g. noise in communication systems) that affect all the parameters of the equation and therefore the fair (from a mathematical point of view) monotone arguments become non-monotone almost always. In view of this, an interesting question arising in the case where the arguments $\tau(t)$ and $\sigma(t)$ are non-monotone, is whether we can state further oscillation criteria which essentially improve all the known results in the literature.

In the present paper, we give a positive answer to the above question by establishing new iterative sufficient conditions for the oscillation of all solutions of (1) and (2), when the arguments are not necessarily monotone.

Throughout, we are going to use the following notation:

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s\,, \quad \beta := \liminf_{t\to\infty} \int_{t}^{\sigma(t)} q(s)\,\mathrm{d}s \tag{3}$$

and

$$D(\omega) := \begin{cases} 0 & \text{if } \omega > 1/\mathrm{e}\,, \\ \frac{1-\omega-\sqrt{1-2\omega-\omega^2}}{2} & \text{if } \omega \in [0, 1/\mathrm{e}]\,. \end{cases} \tag{4}$$

2 Chronological review of known results

2.1 DDEs

The first systematic study on the oscillation of all solutions to equation (1) was made by Myškis in 1950 [22] when he proved that every solution of (1) oscillates if

$$\limsup_{t\to\infty}[t-\tau(t)] < \infty \quad \text{and} \quad \liminf_{t\to\infty}[t-\tau(t)] \liminf_{t\to\infty} p(t) > \frac{1}{\mathrm{e}}\,. \tag{5}$$

In 1972, Ladas, Lakshmikantham and Papadakis [17] proved that if $\tau(t)$ is nondecreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s > 1\,, \tag{6}$$

then all solutions of (1) oscillate.

In 1982, Koplatadze and Chanturija [14] improved (5) to

$$\alpha > \frac{1}{\mathrm{e}}\,. \tag{7}$$

Concerning the constant $1/\mathrm{e}$ in (7), it is to be pointed out that if the inequality

$$\int_{\tau(t)}^{t} p(s)\,\mathrm{d}s \le \frac{1}{\mathrm{e}}$$

holds eventually, then, according to a result in [14], (1) has a nonoscillatory solution. Obviously, when the limit $\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s$ does not exist, a gap appears between the conditions (6) and (7). How to fill this gap is an interesting problem which has attracted the attention of several authors. For example, in 2000, Jaroš and Stavroulakis [11] proved that if $\tau(t)$ is nondecreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s > \frac{1+\ln\lambda_0}{\lambda_0} - D(\alpha)\,, \tag{8}$$

where λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\alpha\lambda}$, then all solutions of (1) oscillate.

Now we come to the case that the argument $\tau(t)$ is not necessarily monotone. Set

$$h(t) := \sup_{s\le t} \tau(s)\,, \quad t \ge t_0\,. \tag{9}$$

Clearly, the function $h(t)$ is nondecreasing and $\tau(t) \le h(t) < t$ for all $t \ge t_0$.

In 1994, Koplatadze and Kvinikadze [15] proved that if

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{h(s)}^{h(t)} p(u)\psi_j(u)\,\mathrm{d}u\right) \mathrm{d}s > 1 - D(\alpha)\,, \tag{10}$$

where

$$\psi_1(t) = 0\,, \quad \psi_j(t) = \exp\left(\int_{\tau(t)}^{t} p(u)\psi_{j-1}(u)\,\mathrm{d}u\right)\,, \quad j \geq 2\,,$$

then all solutions of (1) oscillate.

In 2011, Braverman and Karpuz [2] proved that if

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)\,\mathrm{d}u\right)\mathrm{d}s > 1\,, \tag{11}$$

then all solutions of (1) oscillate.

In 2016, El-Morshedy and Attia [21] proved that if

$$\limsup_{t\to\infty}\left[\int_{g(t)}^{t} p_n(s)\,\mathrm{d}s + D(\alpha)\exp\left(\int_{g(t)}^{t}\sum_{j=0}^{n-1} p_j(s)\,\mathrm{d}s\right)\right] > 1\,, \tag{12}$$

where $p_0(t) = p(t)$ and

$$p_n(t) = p_{n-1}(t)\int_{g(t)}^{t} p_{n-1}(s)\exp\left(\int_{g(s)}^{t} p_{n-1}(u)\,\mathrm{d}u\right)\mathrm{d}s\,, \quad n \geq 1\,,$$

then all solutions of (1) are oscillatory. Here, $g(t)$ is a nondecreasing continuous function such that $\tau(t) \leq g(t) \leq t$, $t \geq t_1$ for some $t_1 \geq t_0$. Clearly, $g(t)$ is more general than $h(t)$ defined by (9).

In 2016, Chatzarakis [3] proved that if for some $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_j(u)\,\mathrm{d}u\right)\mathrm{d}s > 1\,, \tag{13}$$

where

$$p_j(t) = p(t)\left[1 + \int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_{j-1}(u)\,\mathrm{d}u\right)\mathrm{d}s\right], \tag{14}$$

with $p_0(t) = p(t)$, then all solutions of (1) oscillate.

Remark 1. Since

$$p_1(t) = p(t)\left[1 + \int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)\,\mathrm{d}u\right)\mathrm{d}s\right] \geq p(t)\,,$$

clearly

$$\exp\left(\int_{\tau(s)}^{h(t)} p_1(u)\,\mathrm{d}u\right) \geq \exp\left(\int_{\tau(s)}^{h(t)} p(u)\,\mathrm{d}u\right).$$

Thus

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_1(u)\,\mathrm{d}u\right)\,\mathrm{d}s$$
$$\geq \limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)\,\mathrm{d}u\right)\,\mathrm{d}s\,.$$

This means that the condition (13) (for $j = 1$) is weaker than the condition (11) (for $m = 1$ and $r = 1$) of Theorem 4 [1]. We can easily prove that this is valid for every $j = r > 1$.

Several improvements were made to the above condition, see [4], [5], [6] to arrive at the recent form [5]

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)\exp\left(\int_{\tau(u)}^{u} R_\ell(\xi)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s > 1\,, \tag{15}$$

where

$$R_\ell(t) = p(t)\left[1+\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} R_{\ell-1}(\xi)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s\right],$$

with

$$R_0(t) = p(t)\left[1+\lambda_0\int_{\tau(t)}^{t} p(s)\mathrm{d}s\right]$$

and λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\alpha\lambda}$.

2.2 ADEs

By Theorem 2.4.3 in [19], if $\sigma(t)$ is nondecreasing and

$$\limsup_{t\to\infty}\int_{t}^{\sigma(t)} q(s)\,\mathrm{d}s > 1\,, \tag{16}$$

then all solutions of (2) oscillate.

In 1983, Fukagai and Kusano [10] proved that if

$$\beta > \frac{1}{\mathrm{e}}\,, \tag{17}$$

then all solutions of (2) oscillate, while if

$$\int_{t}^{\sigma(t)} q(s)\,\mathrm{d}s \leq \frac{1}{\mathrm{e}} \quad \text{for all sufficiently large } t,$$

then Eq. (2) has a nonoscillatory solution.

Assume that the argument $\sigma(t)$ is not necessarily monotone. Set

$$\rho(t) = \inf_{s \geq t} \sigma(s)\,, \quad t \geq t_0\,. \tag{18}$$

Clearly, the function $\rho(t)$ is nondecreasing and $\sigma(t) \geq \rho(t) > t$ for all $t \geq t_0$.

In 2015, Chatzarakis and Ocalan [7] proved that if

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q(u)\,\mathrm{d}u \right) \mathrm{d}s > 1\,, \tag{19}$$

then all solutions of (2) oscillate.

In 2016, Chatzarakis [3] proved that if for some $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q_j(u)\,\mathrm{d}u \right) \mathrm{d}s > 1\,, \tag{20}$$

where

$$q_j(t) = q(t) \left[1 + \int_t^{\sigma(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q_{j-1}(u)\,\mathrm{d}u \right) \mathrm{d}s \right], \quad j \geq 1$$

with $q_0(t) = q(t)$, then all solutions of (2) oscillate.

Several improvements were made to above conditions, see [4], [5], [6] to arrive at the recent form [5]

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q(u) \exp\left(\int_u^{\sigma(u)} G_\ell(\xi)\,\mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s > 1, \tag{21}$$

where

$$G_\ell(t) = q(t) \left[1 + \int_t^{\sigma(t)} q(s) \exp\left(\int_t^{\sigma(s)} q(u) \exp\left(\int_u^{\sigma(u)} G_{\ell-1}(\xi)\,\mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s \right],$$

with

$$G_0(t) = q(t) \left[1 + \lambda_0 \int_t^{\sigma(t)} q(s)\mathrm{d}s \right]$$

and λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\beta\lambda}$.

3 Main results

3.1 DDEs

We further study (1) and derive new sufficient oscillation conditions, involving lim sup and lim inf, which improve all the previous results. The method we apply is based on the iterative construction of solution estimates and repetitive application of the Grönwall inequality.

The proofs of our main results are essentially based on the following lemmas.

Lemma 1 (See [8, Lemma 2.1.1]). *Assume that $h(t)$ is defined by* (9)*. Then*

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s = \liminf_{t\to\infty} \int_{h(t)}^{t} p(s)\,\mathrm{d}s\,.$$

Lemma 2 (See [8, Lemma 2.1.3]). *Assume that $h(t)$ is defined by* (9)*, $\alpha \in (0, 1/\mathrm{e}]$ and $x(t)$ is an eventually positive solution of* (1)*. Then*

$$\liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \geq D(\alpha)\,, \tag{22}$$

where $D(\alpha)$ is defined by (4)*.*

Lemma 3 (See [25]). *Assume that $h(t)$ is defined by* (9)*, $\alpha \in (0, 1/\mathrm{e}]$ and $x(t)$ is an eventually positive solution of* (1)*. Then*

$$\liminf_{t\to\infty} \frac{x(h(t))}{x(t)} \geq \lambda_0\,, \tag{23}$$

where λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\alpha\lambda}$.

Theorem 1. *Assume that $h(t)$ is defined by* (9) *and for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi)\,\mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s > 1\,, \tag{24}$$

where

$$g_\ell(t) = p(t) \left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} g_{\ell-1}(\xi)\,\mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s \right], \tag{25}$$

with

$$g_0(t) = p(t) \left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\lambda_0 \int_{\tau(s)}^{t} p(u)\mathrm{d}u \right) \mathrm{d}s \right]$$

and λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\alpha\lambda}$. Then all solutions of (1) *are oscillatory.*

Proof. Assume for the sake of contradiction that there exists a nonoscillatory solution $x(t)$ of (1). Since $-x(t)$ is also a solution of (1), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists a $t_1 > t_0$ such that $x(t)$ and $x(\tau(t)) > 0$ for all $t \geq t_1$. Thus, from (1) we have

$$x'(t) = -p(t)x(\tau(t)) \leq 0 \quad \text{for all } t \geq t_1,$$

which means that $x(t)$ is an eventually nonincreasing function of positive numbers.

Now we divide (1) by $x(t) > 0$ and integrate on $[s, t]$, so

$$\int_{s}^{t} \frac{x'(u)}{x(u)}\,\mathrm{d}u = -\int_{s}^{t} p(u) \frac{x(\tau(u))}{x(u)}\,\mathrm{d}u\,,$$

or

$$\ln \frac{x(s)}{x(t)} = \int_s^t p(u) \frac{x(\tau(u))}{x(u)} \, \mathrm{d}u \, .$$

Thus,

$$x(s) = x(t) \exp\left(\int_s^t p(u) \frac{x(\tau(u))}{x(u)} \, \mathrm{d}u \right) . \tag{26}$$

Since $\tau(s) < s \leq t$, (26) gives

$$x(\tau(s)) = x(t) \exp\left(\int_{\tau(s)}^t p(u) \frac{x(\tau(u))}{x(u)} \, \mathrm{d}u \right) . \tag{27}$$

Integrating (1) from $\tau(t)$ to t, we have

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^t p(s) x(\tau(s)) \, \mathrm{d}s = 0 \, . \tag{28}$$

Combining (27) and (28), we have

$$x(t) - x(\tau(t)) + x(t) \int_{\tau(t)}^t p(s) \exp\left(\int_{\tau(s)}^t p(u) \frac{x(\tau(u))}{x(u)} \, \mathrm{d}u \right) \mathrm{d}s = 0 \, .$$

Multiplying the last inequality by $p(t)$, we take

$$p(t)x(t) - p(t)x(\tau(t)) + p(t)x(t) \int_{\tau(t)}^t p(s) \exp\left(\int_{\tau(s)}^t p(u) \frac{x(\tau(u))}{x(u)} \, \mathrm{d}u \right) \mathrm{d}s = 0 \, ,$$

or

$$x'(t) + p(t)x(t) + p(t)x(t) \int_{\tau(t)}^t p(s) \exp\left(\int_{\tau(s)}^t p(u) \frac{x(\tau(u))}{x(u)} \, \mathrm{d}u \right) \mathrm{d}s = 0 \, .$$

Since $\tau(u) \leq h(u)$, it is clear that

$$x'(t) + p(t)x(t) + p(t)x(t) \int_{\tau(t)}^t p(s) \exp\left(\int_{\tau(s)}^t p(u) \frac{x(h(u))}{x(u)} \, \mathrm{d}u \right) \mathrm{d}s \leq 0 \, .$$

Taking into account the fact that (23) of Lemma 3 is satisfied, the last inequality becomes

$$x'(t) + p(t)x(t) + p(t)x(t) \int_{\tau(t)}^t p(s) \exp\left((\lambda_0 - \varepsilon) \int_{\tau(s)}^t p(u) \, \mathrm{d}u \right) \mathrm{d}s \leq 0 \, ,$$

where ε is small with t large enough. Thus,

$$x'(t) + p(t) \left[1 + \int_{\tau(t)}^t p(s) \exp\left((\lambda_0 - \varepsilon) \int_{\tau(s)}^t p(u) \, \mathrm{d}u \right) \mathrm{d}s \right] x(t) \leq 0 \, ,$$

or

$$x'(t) + g_0(t,\varepsilon)x(t) \le 0 \tag{29}$$

with

$$g_0(t,\varepsilon) = p(t)\left[1 + \int_{\tau(t)}^{t} p(s)\exp\left((\lambda_0 - \varepsilon)\int_{\tau(s)}^{t} p(u)\,\mathrm{d}u\right)\mathrm{d}s\right].$$

Applying the Grönwall inequality in (29), we obtain

$$x(s) \ge x(t)\exp\left(\int_s^t g_0(\xi,\varepsilon)\mathrm{d}\xi\right), \quad t \ge s.$$

Thus,

$$x(\tau(u)) \ge x(u)\exp\left(\int_{\tau(u)}^{u} g_0(\xi,\varepsilon)\,\mathrm{d}\xi\right). \tag{30}$$

Now we divide (1) by $x\,(t) > 0$ and integrate on $[s,t]$, so

$$-\int_s^t \frac{x'(u)}{x(u)}\,\mathrm{d}u = \int_s^t p(u)\frac{x\,(\tau(u))}{x(u)}\,\mathrm{d}u$$

or

$$\ln\frac{x(s)}{x(t)} = \int_s^t p(u)\frac{x\,(\tau(u))}{x(u)}\,\mathrm{d}u\,. \tag{31}$$

Combining (30) and (31), we have

$$\ln\frac{x(s)}{x(t)} = \int_s^t p(u)\frac{x\,(\tau(u))}{x(u)}\,\mathrm{d}u \ge \int_s^t p(u)\exp\left(\int_{\tau(u)}^{u} g_0(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,du$$

or

$$x(s) \ge x(t)\exp\left(\int_s^t p(u)\exp\left(\int_{\tau(u)}^{u} g_0(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right). \tag{32}$$

Setting $s = \tau\,(s)$ in (32), we take

$$x(\tau\,(s)) \ge x(t)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} g_0(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right). \tag{33}$$

Combining (28) and (33) we obtain

$$x(t) - x(\tau(t)) + x(t)\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} g_0(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s \le 0\,.$$

Multiplying the last inequality by $p(t)$, we find

$$\begin{aligned} &p(t)x(t) - p(t)x(\tau(t)) \\ &\quad + p(t)x(t)\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} g_0(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s \le 0\,, \end{aligned}$$

which, in view of (1), becomes

$$x'(t) + p(t)x(t) + p(t)x(t) \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} g_0(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s \le 0 .$$

Hence, for all sufficiently large t,

$$x'(t) + p(t)\left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} g_0(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s\right] x(t) \le 0$$

or

$$x'(t) + g_1(t,\varepsilon)x(t) \le 0 , \tag{34}$$

where

$$g_1(t,\varepsilon) = p(t)\left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} g_0(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s\right] .$$

Now, it becomes apparent that by repeating the above steps, we can build inequalities on $x'(t)$ with progressively higher indices $g_\ell(t)$, $\ell \in \mathbb{N}$. In general, for sufficiently large t, the positive solution $x(t)$ satisfies the inequality

$$x'(t) + g_\ell(t)x(t,\varepsilon) \le 0, \quad (\ell \in \mathbb{N}) ,$$

where

$$g_\ell(t,\varepsilon) = p(t)\left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} g_{\ell-1}(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s\right]$$

and

$$x(\tau(s)) \ge x(h(t)) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right) . \tag{35}$$

Integrating (1) from $h(t)$ to t, and using (35) we have

$$x(t) - x(h(t)) + x(h(t)) \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s \le 0 . \tag{36}$$

The inequality is valid if we omit $x(t) > 0$ in the left-hand side. Therefore,

$$x(h(t))\left[\int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s - 1\right] < 0 ,$$

which means that

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s \le 1 .$$

Since ε may be taken arbitrarily small, this inequality contradicts (24).

The proof of the theorem is complete. □

Theorem 2. *Assume that $h(t)$ is defined by* (9) *and $\alpha \in (0, 1/\mathrm{e}]$. If for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s > 1 - D(\alpha)\,, \quad (37)$$

where g_ℓ is defined by (25) *and $D(\alpha)$ by* (4), *then all solutions of* (1) *are oscillatory.*

Proof. Let x be an eventually positive solution of (1). Then, as in the proof of Theorem 1, we obtain (36), i.e., for sufficiently large t we have

$$\begin{aligned} &x(t) - x(h(t)) \\ &\quad + x(h(t)) \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s \le 0\,. \end{aligned}$$

Thus,

$$\int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s \le 1 - \frac{x(t)}{x(h(t))}\,,$$

which gives

$$\begin{aligned} \limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s& \\ \le 1 - \liminf_{t\to\infty} \frac{x(t)}{x(h(t))}\,.& \quad (38) \end{aligned}$$

By Lemma 2, it is obvious that inequality (22) is fulfilled. So, (38) leads to

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s \le 1 - D(\alpha)\,.$$

Since ε may be taken arbitrarily small, this inequality contradicts (37).

The proof of the theorem is complete. □

Theorem 3. *Assume that $h(t)$ is defined by* (9) *and $\alpha \in (0, 1/\mathrm{e}]$. If for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s > \frac{1}{D(\alpha)} - 1\,, \quad (39)$$

where g_ℓ is defined by (25) *and $D(\alpha)$ by* (4), *then all solutions of* (1) *are oscillatory.*

Proof. Assume for the sake of contradiction that there exists a nonoscillatory solution x of (1) and that x is eventually positive. Then, as in the proof of Theorem 1, for sufficiently large t we have

$$x(\tau(s)) \ge x(t) \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right). \quad (40)$$

Integrating (1) from $h(t)$ to t, we have

$$x(t) - x(h(t)) + \int_{h(t)}^{t} p(s)x(\tau(s))\,\mathrm{d}s = 0\,,$$

which, in view of (40), gives

$$x(t) - x(h(t)) + \int_{h(t)}^{t} p(s)x(t)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s \le 0$$

or

$$\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s < \frac{x(h(t))}{x(t)} - 1\,.$$

Therefore,

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s \le \limsup_{t\to\infty}\frac{x(h(t))}{x(t)} - 1\,. \quad (41)$$

By Lemma 2, it is obvious that inequality (22) is fulfilled. So, (41) leads to

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s \le \frac{1}{D(\alpha)} - 1\,.$$

Since ε may be taken arbitrarily small, this inequality contradicts (39).

The proof of the theorem is complete. □

Theorem 4. *Assume that $h(t)$ is defined by* (9) *and $\alpha \in (0, 1/\mathrm{e}]$. If for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(s)} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s > \frac{1+\ln\lambda_0}{\lambda_0} - D(\alpha)\,, \quad (42)$$

where g_ℓ is defined by (25), *$D(\alpha)$ by* (4) *and λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\alpha\lambda}$, then all solutions of* (1) *are oscillatory.*

Proof. Let x be an eventually positive solution and obtain (40) as in Theorem 3, i.e.,

$$x(\tau(s)) \ge x(t)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right).$$

Since $\tau(s) \le h(s)$, the above inequality gives

$$x(\tau(s)) \ge x(h(s)) \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right). \tag{43}$$

Observe that (23) implies that for each $\varepsilon > 0$ there exists a t_ε such that

$$\frac{x(h(t))}{x(t)} > \lambda_0 - \varepsilon \quad \text{for all } t \ge t_\varepsilon \ge t_1\,. \tag{44}$$

Noting that by nondecreasing nature of the function $x(h(t))/x(s)$ in s, it holds

$$1 = \frac{x(h(t))}{x(h(t))} \le \frac{x(h(t))}{x(s)} \le \frac{x(h(t))}{x(t)}, \quad t_\varepsilon \le h(t) \le s \le t,$$

in particular for $\varepsilon \in (0, \lambda_0 - 1)$, by continuity we see that there exists a $t^* \in (h(t), t]$ such that

$$1 < \lambda_0 - \varepsilon = \frac{x(h(t))}{x(t^*)}\,. \tag{45}$$

Integrating (1) from t^* to t, we have

$$x(t) - x(t^*) + \int_{t^*}^{t} p(s)x(\tau(s))\,\mathrm{d}s = 0\,,$$

so, by using (43) along with $h(s) \le h(t)$ in combination with the nonincreasingness of x, we have

$$x(t) - x(t^*) + x(h(t)) \int_{t^*}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s \le 0\,,$$

or

$$\int_{t^*}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s \le \frac{x(t^*)}{x(h(t))} - \frac{x(t)}{x(h(t))}\,.$$

In view of (45) and Lemma 2, for the ε considered, there exists a $t'_\varepsilon \ge t_\varepsilon$ such that

$$\int_{t^*}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s < \frac{1}{\lambda_0 - \varepsilon} - D(\alpha) + \varepsilon \tag{46}$$

for $t \ge t'_\varepsilon$.

Dividing (1) by $x(t)$ and integrating from $h(t)$ to t^*, we find

$$\int_{h(t)}^{t^*} p\,(s)\,\frac{x(\tau\,(s))}{x(s)}\,\mathrm{d}s = -\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)}\,\mathrm{d}s\,,$$

and using (43), we find

$$\int_{h(t)}^{t^*} p(s)\frac{x(h(s))}{x(s)} \exp\left(\int_{\tau(s)}^{h(s)} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\mathrm{d}u\right)\mathrm{d}s \\ \leq -\int_{h(t)}^{t^*}\frac{x'(s)}{x(s)}\,\mathrm{d}s\,. \tag{47}$$

By (23), for $s \geq h(t) \geq t'_\varepsilon$, we have $x(h(s))/x(s) > \lambda_0 - \varepsilon$, so from (47) we get

$$(\lambda_0-\varepsilon)\int_{h(t)}^{t^*} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\mathrm{d}u\right)\mathrm{d}s \\ < -\int_{h(t)}^{t^*}\frac{x'(s)}{x(s)}\,\mathrm{d}s\,.$$

Hence, for all sufficiently large t we have

$$\int_{h(t)}^{t^*} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\mathrm{d}u\right)\mathrm{d}s \\ < -\frac{1}{\lambda_0-\varepsilon}\int_{h(t)}^{t^*}\frac{x'(s)}{x(s)}\,\mathrm{d}s = \frac{1}{\lambda_0-\varepsilon}\ln\frac{x(h(t))}{x(t^*)} = \frac{\ln(\lambda_0-\varepsilon)}{\lambda_0-\varepsilon}\,,$$

i.e.,

$$\int_{h(t)}^{t^*} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\mathrm{d}u\right)\mathrm{d}s < \frac{\ln(\lambda_0-\varepsilon)}{\lambda_0-\varepsilon}\,. \tag{48}$$

Adding (46) and (48), and then taking the limit as $t \to \infty$, we have

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\mathrm{d}u\right)\mathrm{d}s \\ \leq \frac{1+\ln(\lambda_0-\varepsilon)}{\lambda_0-\varepsilon} - D(\alpha)+\varepsilon\,.$$

Since ε may be taken arbitrarily small, this inequality contradicts (42).

The proof of the theorem is complete. □

Next, let us proceed to an oscillation condition involving lim inf.

Theorem 5. *Assume that $h(t)$ is defined by (9) and for some $\ell \in \mathbb{N}$*

$$\liminf_{t\to\infty}\int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi)\,\mathrm{d}\xi\right)\mathrm{d}u\right)\mathrm{d}s > \frac{1}{\mathrm{e}}\,, \tag{49}$$

where g_ℓ is defined by (25). Then all solutions of (1) are oscillatory.

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (1). Since $-x(t)$ is also a solution of (1), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists a $t_1 > t_0$ such that $x(t)$ and $x(\tau(t)) > 0$ for all $t \geq t_1$. Thus, from (1) we have

$$x'(t) = -p(t)x(\tau(t)) \leq 0 \quad \text{for all } t \geq t_1,$$

which means that $x(t)$ is an eventually nonincreasing function of positive numbers. We note that we may obtain (35) as in the proof of Theorem 1. Dividing (1) by $x(t)$ and integrating from $h(t)$ to t we have

$$\ln\left(\frac{x(h(t))}{x(t)}\right) = \int_{h(t)}^{t} p(s)\frac{x(\tau(s))}{x(s)}\,\mathrm{d}s\,,$$

from which, in view of $\tau(s) \leq h(s)$ and by (35), we obtain

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq \int_{h(t)}^{t} p(s)\frac{x(h(s))}{x(s)}\exp\left(\int_{\tau(s)}^{h(s)} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\mathrm{d}u\right)\mathrm{d}s\,.$$

Taking into account that x is nonincreasing and $h(s) < s$, the last inequality leads to

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(s)} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\mathrm{d}u\right)\mathrm{d}s\,. \quad (50)$$

From (49), it follows that there exists a constant $c > 0$ such that for a sufficiently large t holds

$$\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(s)} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi)\,\mathrm{d}\xi\right)\mathrm{d}u\right)\mathrm{d}s \geq c > \frac{1}{\mathrm{e}}\,.$$

Choose c' such that $c > c' > 1/\mathrm{e}$. For every $\varepsilon > 0$ such that $c - \varepsilon > c'$ we have

$$\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(s)} p(u)\exp\left(\int_{\tau(u)}^{u} g_\ell(\xi,\varepsilon)\,\mathrm{d}\xi\right)\mathrm{d}u\right)\mathrm{d}s \geq c - \varepsilon > c' > \frac{1}{\mathrm{e}}\,. \quad (51)$$

Combining inequalities (50) and (51), we obtain

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq c'\,,$$

or

$$\frac{x(h(t))}{x(t)} \geq \mathrm{e}^{c'} \geq \mathrm{e}c' > 1\,,$$

which implies

$$x(h(t)) \geq (\mathrm{e}c')x(t)\,.$$

Repeating the above procedure, it follows by induction that for any positive integer k,

$$\frac{x(h(t))}{x(t)} \geq (\mathrm{e}c')^k, \quad \text{for sufficiently large } t.$$

Since $\mathrm{e}c' > 1$, there is $k \in \mathbb{N}$ satisfying $k > 2\left(\ln 2 - \ln c'\right) / \left(1 + \ln c'\right)$ such that for t sufficiently large

$$\frac{x(h(t))}{x(t)} \geq (\mathrm{e}c')^k > \left(\frac{2}{c'}\right)^2. \tag{52}$$

Taking the integral on $[h(t), t]$, which is not less than c', we split the interval into two parts where integrals are not less than $c'/2$, let $t_m \in (h(t), t)$ be the splitting point:

$$\begin{aligned} &\int_{h(t)}^{t_m} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi, \varepsilon)\, \mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s \geq \frac{c'}{2}, \\ &\int_{t_m}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi, \varepsilon)\, \mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s \geq \frac{c'}{2}. \end{aligned} \tag{53}$$

Integrating (1) from $h(t)$ to t_m, using (35) and the fact that $x(t_m) > 0$, we obtain

$$x(h(t)) > x(h(t_m)) \int_{h(t)}^{t_m} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi, \varepsilon)\, \mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s,$$

which, in view of the first inequality in (53), implies

$$x(h(t)) > \frac{c'}{2} x(h(t_m)). \tag{54}$$

Similarly, integrating (1) from t_m to t, using (35) and the fact that $x(t) > 0$, we have

$$x(t_m) > x(h(t)) \int_{t_m}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi, \varepsilon)\, \mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s,$$

which, in view of the second inequality in (53), implies

$$x(t_m) > \frac{c'}{2} x(h(t)). \tag{55}$$

Combining the inequalities (54) and (55), we obtain

$$x(h(t_m)) < \frac{2}{c'} x(h(t)) < \left(\frac{2}{c'}\right)^2 x(t_m),$$

which contradicts (52).

The proof of the theorem is complete. $\square$

3.2 ADEs

Oscillation conditions analogous to those obtained for the delay equation (1) can be derived for the (dual) advanced differential equation (2) by following similar arguments with the ones employed for obtaining Theorems 1, 2, 3, 4, 5. The corresponding theorems are stated below while their proofs are omitted, as they are quite similar to those for Theorems 1, 2, 3, 4, 5.

Theorem 6. *Assume that $\rho(t)$ is defined by (18) and for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q(u) \exp\left(\int_u^{\sigma(u)} r_\ell(\xi)\, \mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s > 1\,, \tag{56}$$

where

$$r_\ell(t) = q(t)\left[1 + \int_t^{\sigma(t)} q(s) \exp\left(\int_t^{\sigma(s)} q(u) \exp\left(\int_u^{\sigma(u)} r_{\ell-1}(\xi)\, \mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s \right], \tag{57}$$

with

$$r_0(t) = q(t)\left[1 + \int_t^{\sigma(t)} q(s) \exp\left(\lambda_0 \int_t^{\sigma(s)} q(u)\, \mathrm{d}u \right) \mathrm{d}s \right]$$

and λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\beta\lambda}$. Then all solutions of (2) are oscillatory.

Theorem 7. *Assume that $\rho(t)$ is defined by (18) and $\beta \in (0, 1/\mathrm{e}]$. If for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q(u) \exp\left(\int_u^{\sigma(u)} r_\ell(\xi)\, \mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s > 1 - D(\beta)\,, \tag{58}$$

where r_ℓ is defined by (57) and $D(\beta)$ by (4), then all solutions of (2) are oscillatory.

Theorem 8. *Assume that $\rho(t)$ is defined by (18) and $\beta \in (0, 1/\mathrm{e}]$. If for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_t^{\sigma(s)} q(u) \exp\left(\int_u^{\sigma(u)} r_\ell(\xi)\, \mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s > \frac{1}{D(\beta)} - 1\,, \tag{59}$$

where r_ℓ is defined by (57) and $D(\beta)$ by (4), then all solutions of (2) are oscillatory.

Theorem 9. *Assume that $\rho(t)$ is defined by* (18), *$D(\beta)$ by* (4) *and $\beta \in (0, 1/\mathrm{e}]$. If for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(s)}^{\sigma(s)} q(u) \exp\left(\int_u^{\sigma(u)} r_\ell(\xi)\,\mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s > \frac{1+\ln\lambda_0}{\lambda_0} - D(\beta)\,, \quad (60)$$

where r_ℓ is defined by (57) *and λ_0 is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\beta\lambda}$, then all solutions of* (2) *are oscillatory.*

Theorem 10. *Assume that $\rho(t)$ is defined by* (18) *and for some $\ell \in \mathbb{N}$*

$$\liminf_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(s)}^{\sigma(s)} q(u) \exp\left(\int_u^{\sigma(u)} r_\ell(\xi)\,\mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s > \frac{1}{\mathrm{e}}\,, \quad (61)$$

where r_ℓ is defined by (57). *Then all solutions of* (2) *are oscillatory.*

4 Examples and comments

The examples below illustrate the significance of our results and indicate high level of improvement in the oscillation criteria. The calculations were made by the use of MATLAB software.

Example 1 (taken and adapted from [6]). Consider the DDE

$$x'(t) + \frac{229}{2000} x(\tau(t)) = 0\,, \quad t \geq 0, \quad (62)$$

with (see Fig. 1, (a))

$$\tau(t) = \begin{cases} t-1, & \text{if } t \in [8k, 8k+2]\,, \\ -4t+40k+9, & \text{if } t \in [8k+2, 8k+3]\,, \\ 5t-32k-18, & \text{if } t \in [8k+3, 8k+4]\,, \\ -4t+40k+18, & \text{if } t \in [8k+4, 8k+5]\,, \\ 5t-32k-27, & \text{if } t \in [8k+5, 8k+6]\,, \\ -2t+24k+15, & \text{if } t \in [8k+6, 8k+7]\,, \\ 6t-40k-41, & \text{if } t \in [8k+7, 8k+8]\,, \end{cases}$$

where $k \in \mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

By (9), we see (Fig. 1, (b)) that

$$h(t) = \begin{cases} t-1, & \text{if } t \in [8k, 8k+2]\,, \\ 8k+1, & \text{if } t \in [8k+2, 8k+19/5]\,, \\ 5t-32k-18, & \text{if } t \in [8k+19/5, 8k+4]\,, \\ 8k+2, & \text{if } t \in [8k+4, 8k+29/5]\,, \\ 5t-32k-27, & \text{if } t \in [8k+29/5, 8k+6]\,, \\ 8k+3, & \text{if } t \in [8k+6, 8k+44/6]\,, \\ 6t-40k-41, & \text{if } t \in [8k+44/6, 8k+8]\,. \end{cases}$$

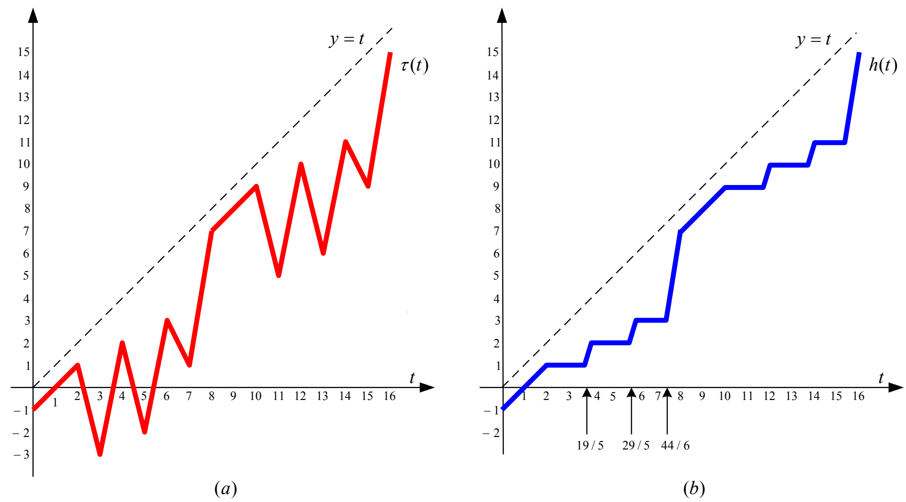

Figure 1: The graphs of $\tau(t)$ and $h(t)$

It is easy to see that

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s = \liminf_{k\to\infty} \int_{8k+1}^{8k+2} \frac{229}{2000}\,\mathrm{d}s = 0.1145$$

and therefore, the smaller root of $\mathrm{e}^{0.1145\lambda} = \lambda$ is $\lambda_0 = 1.13935$.

Observe that the function $F_j : \mathbb{R}_0 \to \mathbb{R}_+$ defined as

$$F_\ell(t) = \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi)\,\mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s$$

attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$, for every $\ell \in \mathbb{N}$. Specifically

$$F_1(t = 8k + 44/6) = \int_{8k+3}^{8k+44/6} p(s) \exp\left(\int_{\tau(s)}^{8k+3} p(u) \exp\left(\int_{\tau(u)}^{u} g_1(\xi)\,\mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s$$

with

$$g_1(\xi) = p(\xi) \left[1 + \int_{\tau(\xi)}^{\xi} p(v) \exp\left(\int_{\tau(v)}^{\xi} p(w) \exp\left(\int_{\tau(w)}^{w} g_0(z)\,\mathrm{d}z \right) \mathrm{d}w \right) \mathrm{d}v \right]$$

and

$$g_0(z) = p(z) \left[1 + \int_{\tau(z)}^{z} p(\omega) \exp\left(\lambda_0 \int_{\tau(\omega)}^{z} p(\varphi)\,\mathrm{d}\varphi \right) \mathrm{d}\omega \right].$$

By using an algorithm on MATLAB software, we obtain

$$F_1(t = 8k + 44/6) \simeq 1.023$$

and therefore

$$\limsup_{t\to\infty} F_1(t) \simeq 1.023 > 1\,.$$

That is, condition (24) of Theorem 1 is satisfied for $\ell = 1$, and therefore all solutions of 62 are oscillatory.

Observe, however, that

$$LD = \limsup_{k\to\infty} \int_{8k+3}^{8k+44/6} \frac{229}{2000} \mathrm{d}s \simeq 0.4962 < 1\,,$$

$$\alpha = 0.1145 < \frac{1}{\mathrm{e}}$$

and

$$0.4962 < \frac{1+\ln\lambda_0}{\lambda_0} - D(\alpha) \simeq 0.9848.$$

Noting that the function Φ_j defined by

$$\Phi_j(t) = \int_{h(t)}^{t} p(s)\exp\left(\int_{h(s)}^{h(t)} p(u)\psi_j(u)\,\mathrm{d}u\right)\mathrm{d}s\,, \quad (j \geq 2)\,,$$

attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$ for every $j \geq 2$. Specifically,

$$\begin{aligned}
\Phi_2(t = 8k+44/6) &= \int_{8k+3}^{8k+44/6} p(s)\exp\left(\int_{h(s)}^{8k+3} p(s)\psi_2(u)\,\mathrm{d}u\right)\mathrm{d}s \\
&= \int_{8k+3}^{8k+44/6} \frac{229}{2000}\exp\left(\int_{h(s)}^{8k+3} \frac{229}{2000}\exp\left(\int_{\tau(u)}^{u} \frac{229}{2000}\cdot 0\,\mathrm{d}w\right)\mathrm{d}u\right)\mathrm{d}s \\
&= \frac{229}{2000}\cdot\left[\int_{8k+3}^{8k+19/5} \exp\left(\frac{229}{2000}\int_{8k+1}^{8k+3} \mathrm{d}u\right)\mathrm{d}s\right. \\
&\quad + \int_{8k+19/5}^{8k+4} \exp\left(\frac{229}{2000}\int_{5s-32k-18}^{8k+3} \mathrm{d}u\right)\mathrm{d}s \\
&\quad + \int_{8k+4}^{8k+29/5} \exp\left(\frac{229}{2000}\int_{8k+2}^{8k+3} \mathrm{d}u\right)\mathrm{d}s \\
&\quad + \int_{8k+29/5}^{8k+6} \exp\left(\frac{229}{2000}\int_{5s-32k-27}^{8k+3} \mathrm{d}u\right)\mathrm{d}s \\
&\quad \left. + \int_{8k+6}^{8k+44/6} \exp\left(\frac{229}{2000}\int_{8k+3}^{8k+3} \mathrm{d}u\right)\mathrm{d}s\right] \\
&\simeq 0.5504\,.
\end{aligned}$$

Thus

$$\limsup_{t\to\infty} \Phi_2(t) \simeq 0.5504 < 1 - D(\alpha) \simeq 0.9925\,.$$

Also

$$\begin{aligned}
&\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u)\,\mathrm{d}u\right) \mathrm{d}s \\
&\quad = \limsup_{k\to\infty} \int_{8k+3}^{8k+44/6} \frac{229}{2000} \exp\left(\int_{\tau(s)}^{8k+3} \frac{229}{2000}\,\mathrm{d}u\right) \mathrm{d}s \\
&\quad = \frac{229}{2000} \limsup_{k\to\infty} \left[\int_{8k+3}^{8k+4} \exp\left(\frac{229}{2000}\int_{5s-32k-18}^{8k+3} \mathrm{d}u\right) \mathrm{d}s\right. \\
&\qquad + \int_{8k+4}^{8k+5} \exp\left(\frac{229}{2000}\int_{-4s+40k+18}^{8k+3} \mathrm{d}u\right) \mathrm{d}s \\
&\qquad + \int_{8k+5}^{8k+6} \exp\left(\frac{229}{2000}\int_{5s-32k-27}^{8k+3} \mathrm{d}u\right) \mathrm{d}s \\
&\qquad + \int_{8k+6}^{8k+7} \exp\left(\frac{229}{2000}\int_{-2s+24k+15}^{8k+3} \mathrm{d}u\right) \mathrm{d}s \\
&\qquad \left. + \int_{8k+7}^{8k+44/6} \exp\left(\frac{229}{2000}\int_{6s-40k-41}^{8k+3} \mathrm{d}u\right) \mathrm{d}s\right] \\
&\quad \simeq 0.6622 < 1\,,
\end{aligned}$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p_1(u)\,\mathrm{d}u\right) \mathrm{d}s \simeq 0.7459 < 1$$

and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} R_1(\xi)\,\mathrm{d}\xi\right) \mathrm{d}u\right) \mathrm{d}s \simeq 0.9305 < 1\,.$$

That is, none of the conditions (6), (7), (8), (10) (for $j = 2$), (11), (13) (for $j = 1$) and (15) (for $\ell = 1$) is satisfied.

Also, the condition (37) of Theorem 6 [6] is not satisfied since the value is: $\simeq 0.8892 < 1$.

Hence, the improvement of condition (3.3) to the condition (37) of Theorem 6 [6] is very satisfactory, approximately $(1.023 - 0.8892)/0.8892 \simeq 15.05\%$.

Comments 1. It is worth noting that the improvement of condition (24) to the corresponding condition (6) is significant, approximately $(1.023 - 0.4962)/0.4962 \simeq 106.2\%$, if we compare the values on the left-side of these conditions. Also, the improvement compared to conditions (10), (11), (13) and (15) is very satisfactory, around $(1.023-0.5504)/0.5504 \simeq 85.86\%$, $(1.023-0.6622)/0.6622 \simeq 54.49\%$, $(1.023-0.7459)/0.7459 \simeq 37.15\%$ and $(1.023-0.9305)/0.9305 \simeq 10\%$, respectively. In addition, observe that conditions (10), (13) and (15) do not lead to oscillation for first iteration. On the contrary, condition (24) is satisfied from the first iteration. This means that our condition is better and much faster than (10), (13) and (15).

Example 2. Consider the DDE

$$x'(t) + \frac{141}{500} x(\tau(t)) = 0, \quad t \geq 0, \tag{63}$$

with (see Fig. 2, blue line)

$$\tau(t) = t - 1.5 + \sin(2t), \quad t \geq 0.$$

By (9), we see (Fig. 2, red line) that

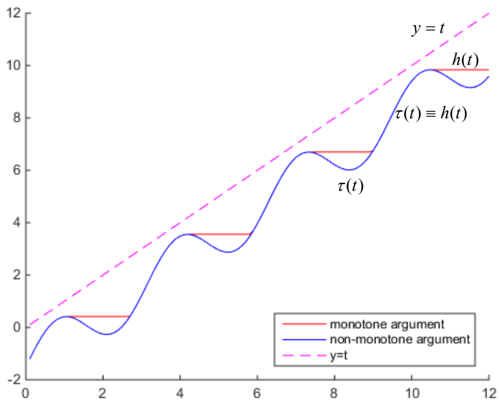

Figure 2: The graphs of $\tau(t)$ and $h(t)$

$$h(t) = \begin{cases} t - 1.5 + \sin(2t), & \text{if } t \in [0, \pi/3] \cup \bigcup_{k=0}^{\infty} [2.6938 + k\pi, (k+1)\pi + \pi/3], \\ \frac{2\pi - 9 + 3\sqrt{3}}{6} + k\pi & \text{if } t \in \bigcup_{k=0}^{\infty} [k\pi + \pi/3, 2.6938 + k\pi]. \end{cases}$$

It is easy to see that

$$\alpha = \liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \mathrm{d}s = \liminf_{k \to \infty} \int_{\pi/4 + k\pi - 0.5}^{\pi/4 + k\pi} \frac{141}{500} \, \mathrm{d}s \simeq 0.141 < \frac{1}{\mathrm{e}}$$

and therefore, the smaller root of $\mathrm{e}^{0.141\lambda} = \lambda$ is $\lambda_0 = 1.18123$.

Also

$$\limsup_{t \to \infty} \int_{h(t)}^{t} p(s) \mathrm{d}s = \limsup_{k \to \infty} \int_{\frac{2\pi - 9 + 3\sqrt{3}}{6} + k\pi}^{2.6938 + k\pi} \frac{141}{500} \, \mathrm{d}s \simeq 0.6431 < 1.$$

That is, conditions (6) and (7) are not satisfied.

Observe, however, that the function $F_\ell : \mathbb{R}_0 \to \mathbb{R}_+$ defined as

$$F_\ell(t) = \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} g_\ell(\xi) \, \mathrm{d}\xi \right) \mathrm{d}u \right) \mathrm{d}s$$

attains its maximum at $t = 2.6938 + k\pi$, $k \in \mathbb{N}_0$, for every $\ell \in \mathbb{N}$. Specifically, by using an algorithm on MATLAB software, we obtain

$$\limsup_{t\to\infty} F_1(t) \simeq 0.9956 > 1 - D(\alpha) \simeq 0.9883\,.$$

That is, condition (37) of Theorem 2 is satisfied for $\ell = 1$, and therefore all solutions of (63) oscillate.

Example 3. Consider the ADE

$$x'(t) - \frac{121}{1000} x(\sigma(t)) = 0\,, \quad t \geq 0, \tag{64}$$

where (see Fig. 3, a)

$$\sigma(t) = \begin{cases} 6t - 35k - 4, & \text{if } t \in [7k+1, 7k+2]\,, \\ -2t + 21k + 12, & \text{if } t \in [7k+2, 7k+3]\,, \\ 5t - 28k - 9, & \text{if } t \in [7k+3, 7k+4]\,, \\ -3t + 28k + 23, & \text{if } t \in [7k+4, 7k+5]\,, \\ 7k + 8, & \text{if } t \in [7k+5, 7k+6]\,, \\ t + 2, & \text{if } t \in [7k+6, 7k+7]\,, \\ 7k + 9, & \text{if } t \in [7k+7, 7k+8]\,. \end{cases}$$

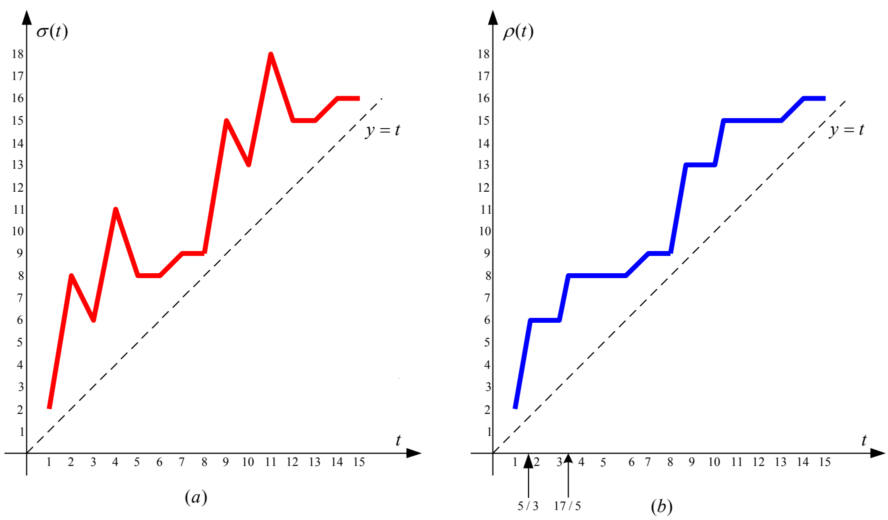

Figure 3: The graphs of $\sigma(t)$ and $\rho(t)$

By (18), we see (Fig. 3, b) that

$$\rho(t)=\begin{cases}6t-35k-4, & \text{if } t\in[7k+1,7k+5/3]\,,\\ 7k+6, & \text{if } t\in[7k+5/3,7k+3]\,,\\ 5t-28k-9, & \text{if } t\in[7k+3,7k+17/5]\,,\\ 7k+8, & \text{if } t\in[7k+17/5,7k+6]\,,\\ t+2, & \text{if } t\in[7k+6,7k+7]\,,\\ 7k+9, & \text{if } t\in[7k+7,7k+8]\,.\end{cases}$$

It is easy to see that

$$\beta=\liminf_{t\to\infty}\int_t^{\sigma(t)}q(s)\,\mathrm{d}s=\liminf_{k\to\infty}\int_{7k+1}^{7k+2}\frac{121}{1000}\,\mathrm{d}s=0.121$$

and therefore, the smaller root of $\mathrm{e}^{0.121\lambda}=\lambda$ is $\lambda_0=1.14918$.

Observe that the function $F_j\colon\mathbb{R}_0\to\mathbb{R}_+$ defined as

$$F_\ell(t)=\int_t^{\rho(t)}q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)}q(u)\exp\left(\int_u^{\sigma(u)}r_\ell(\xi)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s$$

attains its mamimum at $t=7k+17/5$, $k\in\mathbb{N}_0$, for every $\ell\in\mathbb{N}$. Specifically,

$$F_1(t=7k+17/5)=\int_{7k+17/5}^{7k+8}q(s)\exp\left(\int_{7k+8}^{\sigma(s)}q(u)\exp\left(\int_u^{\sigma(u)}r_1(\xi)\,\mathrm{d}\xi\right)\,\mathrm{d}u\right)\,\mathrm{d}s$$

with

$$r_1(\xi)=q(\xi)\left[1+\int_\xi^{\sigma(\xi)}q(v)\exp\left(\int_\xi^{\sigma(v)}q(w)\exp\left(\int_w^{\sigma(w)}r_0(z)\,\mathrm{d}z\right)\,\mathrm{d}w\right)\,\mathrm{d}v\right]$$

and

$$r_0(z)=q(z)\left[1+\int_z^{\sigma(z)}q(\omega)\exp\left(\lambda_0\int_z^{\sigma(\omega)}q(\varphi)\,\mathrm{d}\varphi\right)\,\mathrm{d}\omega\right].$$

By using an algorithm on MATLAB software, we obtain

$$F_1(t=7k+17/5)\simeq 1.0406$$

and therefore

$$\limsup_{t\to\infty}F_1(t)\simeq 1.0406>1\,.$$

That is, condition (56) of Theorem 6 is satisfied for $\ell=1$, and therefore all solutions of (64) are oscillatory.

Observe, however, that

$$\limsup_{t\to\infty}\int_t^{\rho(t)}q(s)\mathrm{d}s=\limsup_{k\to\infty}\int_{7k+17/5}^{7k+8}\frac{121}{1000}\mathrm{d}s\simeq 0.5566<1\,,$$

$$\beta = 0.121 < \frac{1}{\mathrm{e}},$$

$$\begin{aligned}
\limsup_{t\to\infty} \int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} q(u)\,\mathrm{d}u\right)\mathrm{d}s
&= \limsup_{k\to\infty} \int_{7k+17/5}^{7k+8} q(s)\exp\left(\int_{7k+8}^{\sigma(s)} q(u)\,\mathrm{d}u\right)\mathrm{d}s \\
&= \frac{121}{1000}\cdot \limsup_{k\to\infty}\left\{\int_{7k+17/5}^{7k+4} \exp\left(\frac{121}{1000}\int_{7k+8}^{5s-28k-9} \mathrm{d}u\right)\mathrm{d}s\right. \\
&\quad + \int_{7k+4}^{7k+5} \exp\left(\frac{121}{1000}\int_{7k+8}^{-3s+28k+23} \mathrm{d}u\right)\mathrm{d}s \\
&\quad + \int_{7k+5}^{7k+6} \exp\left(\frac{121}{1000}\int_{7k+8}^{7k+8} \mathrm{d}u\right)\mathrm{d}s \\
&\quad + \int_{7k+6}^{7k+7} \exp\left(\frac{121}{1000}\int_{7k+8}^{s+2} \mathrm{d}u\right)\mathrm{d}s \\
&\quad \left. + \int_{7k+7}^{7k+8} \exp\left(\frac{121}{1000}\int_{7k+8}^{7k+9} \mathrm{d}u\right)\mathrm{d}s\right\} \\
&\simeq 0.6196 < 1,
\end{aligned}$$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} q_1(u)\,\mathrm{d}u\right)\mathrm{d}s \simeq 0.6483 < 1,$$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} q(u)\exp\left(\int_u^{\sigma(u)} G_\ell(\xi)\,\mathrm{d}\xi\right)\mathrm{d}u\right)\mathrm{d}s \simeq 0.8778 < 1.$$

That is, none of the conditions (16), (17), (19), (20) (for $j = 1$) and (21) (for $\ell = 1$) is satisfied.

Comments 2. It is worth noting that the improvement of condition (56) to the corresponding condition (16) is significant, approximately 86.96%, if we compare the values on the left-side of these conditions. Also, the improvement compared to conditions (19), (20) and (21) is very satisfactory, around 67.95%, 60.05%, and 18.54%, respectively. In addition, observe that conditions (20) and (21) do not lead to oscillation for first iteration. On the contrary, condition (56) is satisfied from the first iteration. This means that our condition is better and much faster than (20) and (21).

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References

[1] E. Braverman, G.E. Chatzarakis, I.P. Stavroulakis: Iterative oscillation tests for differential equations with several non-monotone arguments. *Adv. Difference Equ.* 87 (2016).

[2] E. Braverman, B. Karpuz: On oscillation of differential and difference equations with non-monotone delays. *Appl. Math. Comput.* 218 (7) (2011) 3880–3887.

[3] G.E. Chatzarakis: Differential equations with non-monotone arguments: Iterative Oscillation results. *J. Math. Comput. Sci.* 6 (5) (2016) 953–964.

[4] G.E. Chatzarakis: On oscillation of differential equations with non-monotone deviating arguments. *Mediterr. J. Math.* 14 (2) (2017) 82.

[5] G.E. Chatzarakis, I. Jadlovská: Improved iterative oscillation tests for firs-order deviating differential equations. *Opuscula Math.* 38 (3) (2018) 327–356.

[6] G.E. Chatzarakis, T. Li: Oscillation criteria for delay and advanced differential equations with non-monotone arguments. *Complexity* 2018 (2018) 1–18. Article ID 8237634.

[7] G.E. Chatzarakis, Ö. Öcalan: Oscillations of differential equations with several non-monotone advanced arguments. *Dynamical Systems* 30 (3) (2015) 310–323.

[8] L.H. Erbe, Qingkai Kong, B.G. Zhang: *Oscillation Theory for Functional Differential Equations*. Monographs and Textbooks in Pure and Applied Mathematics, 190. Marcel Dekker, Inc., New York (1995).

[9] L.H. Erbe, B.G. Zhang: Oscillation of first order linear differential equations with deviating arguments. *Differential Integral Equations* 1 (3) (1988) 305–314.

[10] N. Fukagai, T. Kusano: Oscillation theory of first order functional-differential equations with deviating arguments. *Ann. Mat. Pura Appl.* 136 (1) (1984) 95–117.

[11] J. Jaroš, I.P. Stavroulakis: Oscillation tests for delay equations. *Rocky Mountain J. Math.* 29 (1) (1999) 197–207.

[12] C. Jian: On the oscillation of linear differential equations with deviating arguments. *Math. in Practice and Theory* 1 (1) (1991) 32–40.

[13] M. Kon, Y.G. Sficas, I.P. Stavroulakis: Oscillation criteria for delay equations. *Proc. Amer. Math. Soc.* 128 (10) (2000) 2989–2998.

[14] R.G. Koplatadze, T.A. Chanturija: Oscillating and monotone solutions of first-order differential equations with deviating argument. *Differentsial'nye Uravneniya* 18 (8) (1982) 1463–1465. (in Russian)

[15] R.G. Koplatadze, G. Kvinikadze: On the oscillation of solutions of first order delay differential inequalities and equations. *Georgian Math. J.* 1 (6) (1994) 675–685.

[16] M.K. Kwong: Oscillation of first-order delay equations. *J. Math. Anal. Appl.* 156 (1) (1991) 274–286.

[17] G. Ladas, V. Lakshmikantham, L.S. Papadakis: Oscillations of higher-order retarded differential equations generated by the retarded arguments. In: Klaus Schmitt: *Delay and functional differential equations and their applications*. Academic Press (1972) 219–231.

[18] G.S. Ladde: Oscillations caused by retarded perturbations of first order linear ordinary differential equations. *Atti Acad. Naz. Lincei Rendiconti* 63 (5) (1977) 351–359.

[19] G.S. Ladde, V. Lakshmikantham, B.G. Zhang: *Oscillation Theory of Differential Equations with Deviating Arguments*. Monographs and Textbooks in Pure and Applied Mathematics, 110, Marcel Dekker, Inc., New York (1987).

[20] X. Li, D. Zhu: Oscillation and nonoscillation of advanced differential equations with variable coefficients. *J. Math. Anal. Appl.* 269 (2) (2002) 462–488.

[21] H.A. El-Morshedy, E.R. Attia: New oscillation criterion for delay differential equations with non-monotone arguments. *Appl. Math. Lett.* 54 (2016) 54–59.

[22] A.D. Myškis: Linear homogeneous differential equations of first order with deviating arguments. *Uspekhi Mat. Nauk* 5 (36) (1950) 160–162. (in Russian)

[23] J.S. Yu, Z.C. Wang, B.G. Zhang, X.Z. Qian: Oscillations of differential equations with deviating arguments. *Panamer. Math. J.* 2 (2) (1992) 59–78.

[24] B.G. Zhang: Oscillation of solutions of the first-order advanced type differential equations. *Science Exploration* 2 (1982) 79–82.

[25] D. Zhou: On some problems on oscillation of functional differential equations of first order. *J. Shandong University* 25 (1990) 434–442.

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