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Kappa-Slender Modules

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Abstract. For an arbitrary infinite cardinal κ, we define classes of κ-cslender and κ-tslender modules as well as related classes of κ-hmodules and initiate a study of these classes.

1 Preliminaries

The notion of a slender Abelian group was introduced by Jerzy Łoś and it was initially explored by the Polish school (cf. Fuchs [4]). This notion has deep and wide ramifications in algebra as well as set theory and has evolved considerably since its inception. The expansive treatment of this and related theories is to be found in Dimitric [2], [3].

In this note, we generalize the notion by expanding the cardinality of the generator set of the image, thus grading the category we work within, by way of cardinality.[1]

We will work within a category of (left) R-modules $R\mathbf{Mod}$ and our functions will be morphisms in that category; in particular, R is seen as an object in this category, not in the category of rings. To simplify the discussion, we may assume, if need be, that rings R are domains with unities and that all modules are unitary. We will identify cardinals κ with the initial ordinals of the same cardinality, and consequently may assume that (in the presence of the Axiom of Choice), κ is well-ordered. ω denotes the smallest infinite ordinal. For a cardinal κ, we denote by κ^+ the smallest cardinal in the interval $(\kappa, \rightarrow)$. For an ordinal κ, $\mathrm{cf}(\kappa) \leq \kappa$

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[1]This is a slightly revised TEX-ed version of a handwritten note dating back to the early 1980s when we begun the program of general slenderness. At that time, we completely characterized κ-slenderness in Abelian categories for $\kappa = \omega$ (see Dimitric [1], [2]). Characterization of κ-slenderness, for arbitrary κ, remains open.

denotes the smallest ordinal cofinal with κ; for example, $\mathrm{cf}(\omega_\omega) = \omega = \mathrm{cf}(\omega^\omega)$. If $\mathrm{cf}(\kappa) = \kappa$, then κ is called *regular*, otherwise it is *singular*. A set I is said to be non-measurable if there is no non-trivial 2-valued measure on I, i.e. no σ-additive function $\mu\colon \mathcal{P}(I) \to \{0,1\}$ with $\mu(I) = 1$, and $\mu(\{i\}) = 0$, for every $i \in I$, and for every countable family $\{I_n : n \in \mathbb{N}\}$ of pairwise disjoint subsets of I,

$$\mu\Big(\dot{\bigcup} I_n\Big) = \sum \mu(I_n)\,.$$

Otherwise, I is said to be *measurable*. We may assume that cardinalities of our index sets are less than some large cardinal $\mathfrak{m}$, such as a measurable cardinal, or an inaccessible cardinal. For set theoretic notions the reader may consult some of the classical treatises on set theory, such as Hrbacek & Jech [5].

For $x = (x_i)_{i\in I} \in \prod_{i\in I} A_i$ denote the zero set

$$\mathrm{zero}(x) = \{i \in I : x_i = 0\}$$

and the complement non-zero set

$$\mathrm{supp}(x) = \{i \in I : x_i \neq 0\}\ .$$

Given a filter $\mathcal{F}$ on a non-empty index set I, the $\mathcal{F}$ subproduct is

$$\Pi(\mathcal{F}) = \prod_{i\in I}^{\mathcal{F}} M_i = \Big\{x \in \prod M_i : \mathrm{zero}(x) \in \mathcal{F}\Big\}\ .$$

If $\mathcal{F}$ is the co-κ-filter, then the corresponding subproduct will be denoted by $\prod_{i\in I}^{\kappa} M_i$; namely it consists of all the vectors $x = (x_i)_{i\in I}$ with support of cardinality $< \kappa$.

The coordinate vectors $\mathbf{e}_i\colon I \to R$ are defined by

$$\begin{cases} \mathbf{e}_i(j) = 0\,, & \text{if } i \neq j,\\ \mathbf{e}_i(i) = 1 \in R \end{cases}$$

If $J \subseteq I$, denote by $\pi_J\colon \prod_{i\in I} A_i \to \prod_{i\in J} A_i$ and $p_J\colon \prod_{i\in J} A_i \to \prod_{i\in I} A_i$ the coordinate projection and injection respectively. Note that π_J and p_J are isomorphisms

$$\prod_{i\in I} A_i \cong \prod_{j\in J} A_j\,,$$

if $A_i = 0$ for all $i \in I \setminus J$.

2 Coordinate slenderness

Definition 1. Given an arbitrary (infinite) cardinal κ, define a (left) R-module M to be *κ-slender*, if, for every index set I of cardinality κ, every family of R-modules A_i, $i \in I$, and every morphism

$$f\colon \prod_{i\in I} A_i \to M, \quad \big|\{i \in I : f|A_i \neq 0\}\big| < \kappa\,.$$

In this way, the well-known notion of slender module is a special case, namely of an ω-slender module (see Dimitric [2], for a thorough study of classes of slender objects). Another, more appropriate name we will use is *κ-coordinatewise slenderness* or *κ-cslenderness*.

The purpose of this note is to look into κ-slenderness for uncountable κ.

Remark 1. If M is not κ-slender, then, every morphism $f\colon \prod_I A_i \to M$, such that for $J = \{i \in I : f|A_i \neq 0\}$, $|J| \geq \kappa$ will be called a *non-slender morphism*. Given a non-κ-slender module M, such a non-slender morphism always exists and, by taking the appropriate restriction to $\prod_J A_i$ we may then assume that, for a non-κ-slender module M, there is a morphism $f\colon \prod_I A_i \to M$ with $|I| = \kappa$, such that $f|A_i \neq 0$, for every $i \in I$.

We note immediately that, in the definition, we may take any index set of cardinality $> \kappa$ as well as that we may replace all A_i by cyclic modules or by the identical objects, namely the ground ring R as detailed in the following:

Theorem 1. *Given an infinite cardinal κ and an $M \in R\mathbf{Mod}$, the following are equivalent:*

(1) M is κ-slender.

(2) $\forall I$, $|I| \geq \kappa$, $\forall A_i \in R\mathbf{Mod}$, $i \in I$, for every morphism $f\colon \prod_{i\in I} A_i \to M$,

$$\big|\{i \in I : f|A_i \neq 0\}\big| < \kappa\,.$$

(3) $\forall I$, $|I| \geq \kappa$, for every morphism $f\colon \prod_{i\in I} R_i \to M$, $\forall i$, $R_i = R$,

$$\big|\{i \in I : f(\mathbf{e}_i) \neq 0\}\big| < \kappa\,.$$

(4) $\forall I$, $|I| = \kappa$, for every morphism $f\colon \prod_{i\in I} R_i \to M$, $\forall i$, $R_i = R$,

$$\big|\{i \in I : f(\mathbf{e}_i) \neq 0\}\big| < \kappa\,.$$

(5) $\forall I$, $|I| \geq \kappa$, for every morphism $f\colon \prod_{i\in I} Ra_i \to M$,

$$\big|\{i \in I : f(a_i) \neq 0\}\big| < \kappa\,.$$

(6) $\forall I$, $|I| = \kappa$, for every morphism $f\colon \prod_{i\in I} Ra_i \to M$,

$$\big|\{i \in I : f(a_i) \neq 0\}\big| < \kappa\,.$$

In case of regular κ, we also have the following equivalent statements:

(7) $\forall I$, $|I| = \kappa$, $\forall A_i \in R\mathbf{Mod}$, $i \in I$, for every morphism $f\colon \prod_{i\in I} A_i \to M$, $\exists i_0 < \kappa$ such that $\forall i > i_0$, $f|A_i = 0$.

(8) $\forall I$, $|I| = \kappa$, for every morphism $f\colon \prod_{i\in I} Ra_i \to M$, $\exists i_0 < \kappa$ such that $\forall i > i_0$, $f(a_i) = 0$.

(9) $\forall I$, $|I| = \kappa$, *for every morphism* $f\colon \prod_{i\in I} R_i \to M$, $\forall i$, $R_i = R$, $\exists i_0 < \kappa$ *such that* $\forall i > i_0$, $f(\mathbf{e}_i) = 0$.

(10) $\forall I$, $\operatorname{cf}|I| \geq \kappa$, $\forall A_i \in R\mathbf{Mod}$, $i \in I$, *for every morphism* $f\colon \prod_{i\in I} A_i \to M$, $\exists i_0 < \kappa$ *such that* $\forall i > i_0$, $f|A_i = 0$.

(11) $\forall I$, $\operatorname{cf}|I| \geq \kappa$, *for every morphism* $f\colon \prod_{i\in I} Ra_i \to M$, $\exists i_0 < \kappa$ *such that* $\forall i > i_0$, $f(a_i) = 0$.

(12) $\forall I$, $\operatorname{cf}|I| \geq \kappa$, *for every morphism* $f\colon \prod_{i\in I} R_i \to M$, $\forall i$, $R_i = R$, $\exists i_0 < \kappa$ *such that* $\forall i > i_0$, $f(\mathbf{e}_i) = 0$.

Proof. (1)⇒(2): Let $|I| > \kappa$ and $f\colon \prod_{i\in I} A_i \to M$. If, on the contrary, $\exists J \subseteq I$, $|J| = \kappa$, such that $\forall j \in J$, $f|A_j \neq 0$, then we can take the restriction $f' = f|\prod_{j\in J} A_j$ with the same coordinate property. This would then contradict (1).

(2)⇒(3), (3)⇒(4), (5)⇒(6) hold because, respectively, (2) is nominally more general than (3) and (3) is nominally more general than (4) just as (5) is nominally more general than (6).

(4)⇒(5): Let, on the contrary, $\exists J \subseteq I$, $|J| = \kappa$, such that $\forall j \in J$, $f(a_j) \neq 0$. We have the quotient maps

$$q_j\colon R \to R \,/\, \operatorname{Ann}(a_j) \cong Ra_j$$

and the product map

$$q = \prod q_j\colon \prod_{j\in J} R_j \to \prod_{j\in J} Ra_j\,.$$

Consider $f' = fq\colon \prod_{j\in J} R_j \to M$. We have $f'(\mathbf{e}_j) = f(a_j) \neq 0$, $\forall j \in J$, which would contradict (4).

(6)⇒(1): Let $|I| = \kappa$ and $f\colon \prod_{i\in I} A_i \to M$ be such that, on the contrary, $\exists J \subseteq I$, $|J| = \kappa$, with $\forall j \in J$, $f|A_j \neq 0$; in other words $\exists a_j \in A_j$ with $f|Ra_j \neq 0$. Consider the restriction of f, namely $f'\colon \prod_{j\in J} Ra_j \to M$. But then $f'|Ra_j$ on a set J of cardinality κ which would contradict (6).

The equivalences (7)–(12) are proved in a like manner as (1)–(6). We only need to connect the two batches:

(7)⇒(1) follows, once we note that $\forall i_0 < \kappa$, $|(\leftarrow, i_0)| < \kappa$.

As for (1)⇒(7), given a morphism $f\colon \prod_{i\in I} A_i \to M$, the cardinality

$$|S_i| = \bigl|\{i < \kappa : f|A_i \neq 0\}\bigr| < \kappa\,.$$

This implies that $\sup S_i = i_0 < \kappa$, since κ was assumed to be regular. □

Remark 2. We note that regularity of κ is not needed for implications from the second batch of statements to the first. Furthermore, without assumption of regularity of κ, statements (7)–(12) are equivalent to the following statement:

(13) $\forall J$, $|J| = \operatorname{cf}(\kappa)$, $\forall A_j \in R\mathbf{Mod}$, $j \in J$, for every morphism $f\colon \prod_{j\in J} A_j \to M$, $\exists j_0 < \operatorname{cf}(\kappa)$ such that $\forall j > j_0$, $f|A_j = 0$.

Indeed, if we assume (7), $|I| = \kappa$ and $\mathrm{cf}(\kappa) = J \subseteq I$, given a morphism

$$f\colon \prod_{j\in J} A_j \to M\,,$$

$A_i = 0$ for $i \in I \setminus J$, we have a morphism

$$\overline{f} = f\pi_J\colon \prod_{i\in I} A_i \to M$$

whereas there is an $i_0 < \kappa$ such that $\forall i > i_0$, $\overline{f}|A_i = 0$. By cofinality, there exists a $j_0 \in J$, $j_0 > i_0$ with $\overline{f}|A_j = 0$ for $j > j_0$ and the claim follows once we note that $\overline{f}|A_j = f|A_j$ for $j \in J$.

The implication (13)⇒(7) is a consequence of the fact that $\mathrm{cf}(\kappa) < \kappa$.

It appears that κ-slenderness is a characteristic of the lattice of submodules as may be seen from the following:

Proposition 1. *(1) The trivial module* 0 *is κ-slender, for every κ.*

(2) M is κ-slender, iff $\forall N \leq M$, N is κ-slender.

(3) For every infinite cardinal κ, every slender R-module is κ-slender.

(4) Let $\lambda \leq \kappa$; then every λ-slender module is also κ-slender.

(5) R^κ is not κ-slender.

(6) For all cardinals $\lambda < \mathrm{cf}(\kappa)$, and $B_j \in R\mathbf{Mod}, j \in J, |J| < \lambda$, $\prod_{j\in J} B_j$ is κ-slender, if and only if every B_j is κ-slender. In particular, R^λ is κ-slender if and only if R is κ-slender. Furthermore, $\bigoplus_{j\in J} B_j$ is κ-slender if and only if every B_j is κ-slender.

(7) If R is slender, then R^κ is κ^+-slender.

Proof. (1) The definition verifies trivially.

(2) If $f\colon R^\kappa \to N \hookrightarrow M$, then use Theorem 1 (4) to conclude that

$$\{i \in I : f(\mathbf{e}_i) \neq 0\}\,| < \kappa\,,$$

which establishes κ-slenderness of N. The other direction is a tautology.

(3) By a known result (see e.g. Dimitric [2], Theorem 3.10), an object M is slender iff for every index set I and every morphism $f\colon \prod_{i\in I} A_i \to M$,

$$\{i \in I : f|A_i \neq 0\}$$

is finite (hence $< \kappa$).

(4) Let M be λ-slender and let $f\colon \prod_{i\in I} Ra_i \to M$, ($|I| = \kappa$). By Theorem 1 (5)

$$\big|\{i \in I : f(a_i) \neq 0\}\big| < \lambda < \kappa\,,$$

which establishes κ-slenderness of M.

(5) The identity map id: $R^\kappa \to \prod_{j<\kappa} R_j$ is such that

$$\left|\{i \in I : \mathrm{id}(\mathbf{e}_i) \neq 0\}\right| = \kappa\,,$$

which shows, by Theorem 1, that R^κ is not κ-slender.

(6) If $\prod_{j\in J} B_j$ is κ-slender, then, by (2), every submodule is κ-slender, hence that applies to each B_j as well. Now assume that every B_j is κ-slender. If

$$f\colon \prod_{i<\kappa} Ra_i \to \prod_{j\in J} B_j\,,$$

we know that then

$$f = \left(f_j\colon \prod_{i<\kappa} Ra_i \to B_j\right)_{j\in J}\,,$$

$|J| = \lambda$. We know that $\forall j \in J$, for

$$S_j = \{i \in I : f_j(a_i) \neq 0\}\,,$$

$|S_j| < \kappa$, since every B_j is κ-slender. We have

$$\{i \in I : f(a_i) \neq 0\} = \bigcup_{j\in J} S_j\,.$$

Assume first that κ is a regular cardinal. Then

$$\left|\bigcup_{j\in J} S_j\right| \leq \sum_{j\in J} |S_j| < \lambda\kappa = \kappa$$

(the latter strict inequality holds because κ is regular). Thus, indeed $\prod_{j\in J} B_j$ is κ-slender. Hence, this statement is true for regular cardinal $\kappa = \lambda^+$, namely $\prod_{j\in J} B_j$ is λ^+-slender. By (4), $\prod_{j\in J} B_j$ is κ-slender, for every $\kappa \geq \lambda^+ > \lambda$. The remaining claims are a special case and the fact that the direct sum is submodule of the direct product, hence by (2) has to be slender.

(7) Replace κ by κ^+ and λ by κ in (6) and use (3). □

Modification of Theorem 3.40 in Dimitric [2] would establish the fact that $\bigoplus_{j\in J} B_j$ is κ-slender iff each B_j is k-slender, regardless of the cardinality of the index set J (see Dimitric [3]).

Given an infinite cardinal κ, then a submodule $N \leq M \in R\mathbf{Mod}$ is said to be *κ-pure* in M, if every system of ($|I| < \kappa$) equations of the form

$$\sum_{j\in J} r_{ij} x_j = n_i \in N, \quad i \in I, \quad r_{ij} \in R,$$

with $< \kappa$ unknowns x_j, $j \in J$, $|J| < \kappa$ that has a solution in M^J, also has a solution in N^J. Notation for this is $N \leq_{\kappa *} M$. Thus purity is then same as

$\aleph_0$-purity. The derivative notion of a κ-pure exact sequence is straightforward. As κ is increased, the classes (sets) of κ-pure exact sequences get smaller, in general. A module is κ-pure injective if it has injective property with respect to all κ-pure exact sequences. A module M is equationally (algebraically) *κ-compact*, if every system of $\leq \kappa$ linear equations:

$$\sum_{j\in J} r_{ij}x_j = m_i \in M, \quad i \in I, \quad r_{ij} \in R$$

with the property that every finite subsystem has a solution, then has a global solution. A module is algebraically compact iff it is κ-compact, for every cardinal κ. Given $\kappa < |R|$ one can construct examples of κ-compact modules that are not algebraically compact. However, if $M \in R\mathbf{Mod}$ is κ-algebraically compact, for some $\kappa \geq |R|$, then M is algebraically compact.

We have mimicked Łoś [6] to produce the following result, needed in the sequel:

Theorem 2. *Let A be an index set, $\mathcal{F}$ a κ-complete filter on A and $M_\alpha \in R\mathbf{Mod}$, $\alpha \in A$; then $\prod^{\mathcal{F}}_{\alpha\in A} M_\alpha$ is κ-pure in $\prod_{\alpha\in A} M_\alpha$. Specially, for $\mathcal{F} = \mathcal{F}_0$, the coproduct $\bigoplus M_\alpha$ is pure in $\prod M_\alpha$.*

Proof. Assume that the system of linear equations

$$\sum_{j\in J} r_{ij}x_j = n_i \in \prod_{\alpha\in A}^{\mathcal{F}} M_\alpha, \quad i \in I, \quad (r_{ij})_{I\times J} \text{ row finite}, \quad |I|, |J| < \kappa$$

has a solution $m_j \in \prod_{\alpha\in A} M_\alpha, j \in J$; this then translates into the componentwise equalities:

$$\sum_{j\in J} r_{ij}m_{j\alpha} = n_{i\alpha} \quad \alpha \in A\,. \tag{$*$}$$

By definition $\mathrm{zero}(n_i) \in \mathcal{F}$, for all $i \in I$, and since $|I| < \kappa$, we get, by κ-completeness of $\mathcal{F}$, that

$$Z = \bigcap_{i\in I} \mathrm{zero}(n_i) \in \mathcal{F}\,.$$

Now define $y_j \in \prod_{\alpha\in A} M_\alpha$ componentwise: $y_{j\alpha} = m_{j\alpha}$, if $\alpha \notin Z$ and $y_{j\alpha} = 0$, if $\alpha \in Z$. Every

$$\mathrm{zero}(y_j) \supseteq Z \in \mathcal{F}\,,$$

thus all $y_j \in \prod^{\mathcal{F}} M_\alpha$; but the y_j also provide a solution of the original system of equations, by the way we defined them, by $(*)$ and by the fact that for $\alpha \in Z$ we have $n_{i\alpha} = 0$, for all i. □

Denote by $\mathcal{S}_\kappa$ the class of κ-slender modules, where $\mathcal{S}$ denotes, for brevity, the class of slender modules.

Proposition 2. *(1) We have an ascending chain of non-empty classes:*

$$\mathcal{S} \subset \cdots \subset \mathcal{S}_\kappa \subset \mathcal{S}_{\kappa^+} \subset \cdots \subset \mathcal{S}_\lambda \subset \cdots \subset R\mathbf{Mod}, \quad \kappa < \lambda\,.$$

The chain is strictly ascending, if R is slender.

(2) *The union of this chain is* $\neq R\mathbf{Mod}$, *since non-zero algebraically compact modules are not* κ*-slender (hence cannot be contained in a* κ*-slender module), for any* κ.[2]

Proof. (1) is a consequence of Proposition 1.

For (2), given a cardinal κ, assume that $M \in R\mathbf{Mod}$ is algebraically compact and let $0 \neq a \in M$. By Theorem 2, we have a pure exact sequence

$$0 \longrightarrow \bigoplus_I R_i \longrightarrow \prod_I R_i \longrightarrow \prod_I R_i \Big/ \bigoplus_I R_i \longrightarrow 0\,, \quad R_i = R$$

Define $f_0\colon \bigoplus R_i \to M$ coordinatewise: $\forall i \in I$, $f_0(\mathbf{e}_i) = a$. Since M is algebraically compact, we can extend f_0 to the morphism $f\colon \prod_I R_i \to M$, for which we have $\forall i$, $f(\mathbf{e}_i) = a \neq 0$, which shows that M is not κ-slender, for any κ. □

Consequently, if R is algebraically compact (pure injective), then, by Proposition 2 (2), R is not λ-slender, for any λ and then the product R^κ, being algebraically compact, is not λ-slender, for any λ, κ.

3 Tailwise slenderness

Definition 2. Given a regular cardinal $\kappa = |I|$, an $M \in R\mathbf{Mod}$ is said to be *κ-tailwise slender*, or *κ-tslender* for short, if for every morphism

$$f\colon \prod_{i<\kappa} Ra_i \to M\,,$$

there exists an $i_0 < \kappa$, such that $f(\prod_{i\geq i_0} Ra_i) = 0$. This is equivalent to the requirement that, for every morphism

$$f\colon \prod_{i<\kappa} R_i \to M\,,$$

$R_i = R$, there exists an $i_0 < \kappa$ such that $f(\prod_{i\geq i_0} R_i) = 0$.

Without the regularity condition on κ, one can show, just as in Remark 2, that κ-tslenderness would be equivalent to $\mathrm{cf}(\kappa)$-tslenderness. In order to avoid this ambiguity, the referee suggests defining tailwise slenderness for a (singular) cardinal κ as follows: For every morphism $f\colon \prod_{i\in I} Ra_i \to M$, there exists an $I_0 \subseteq I$, $|I_0| < \kappa$, such that $f(\prod_{i\in I\setminus I_0} Ra_i) = 0$.

We note a straightforward but important fact as follows:

Proposition 3. (1) *If* M *is* κ*-tslender, then it is* κ*-cslender.*

(2) *If* M *is* κ*-tslender, then, for all cyclic modules* Ra_i, $i \in I$, $|I| = \kappa$*:*

$$\mathrm{Hom}_R\left(\prod_{i\in I} Ra_i \Big/ \prod_{i\in I}^{\kappa} Ra_i, M\right) = 0.$$

[2] The referee remarked that, moreover, $R\mathbf{Mod} = \bigcup_{\kappa<\lambda} S_\kappa \cup \mathcal{AC}$ (here we denote by $\mathcal{AC}$ the class of algebraically compact modules).

Proof. (1) $S_i = \{i < \kappa : f|A_i \neq 0\} \subseteq (\leftarrow, i_0)$ and $|(\leftarrow, i_0)| < \kappa$, since κ is a cardinal.

(2) If $\kappa = |I|$ and $f\colon \prod_{i\in I} Ra_i \to M$ is a morphism, then, by κ-tslenderness of M, $\exists i_0 < \kappa$ such that $f(\prod_{i\geq i_0} Ra_i) = 0$. We note that

$$\prod_{i<i_0} Ra_i \subseteq \prod_{i\in I}^{\kappa} Ra_i$$

since κ is a cardinal and $|(\leftarrow, i_0)| < \kappa$. The claim will follow, once we note the obvious splitting:

$$\prod_{i<\kappa} Ra_i = \prod_{i<i_0} Ra_i \oplus \prod_{i\geq i_0} Ra_i\,. \qquad \square$$

As for the converse of implication (1) in this proposition, it may not always be true and it is related to intricate constructions of set-theoretical nature. It is well-known that, for $\kappa = \omega$, the equivalence holds, if and only the index sets of the products involved are non-measurable cardinals (see Dimitric [2], Theorem 3.10).

Proposition 4. *(1) The trivial module* 0 *is* κ*-tslender, for every* κ*.*

(2) M *is* κ*-tslender, iff* $\forall N \leq M$, N *is* κ*-tslender.*

(3) For every infinite non-measurable cardinal κ*, every slender* R*-module is* κ*-tslender (cf. Proposition 1 (4)).*

(4) R^κ *is not* κ*-tslender.*

(5) For all cardinals $\lambda < \mathrm{cf}(\kappa)$*, and* $\{B_j \in R\mathbf{Mod} : j < \lambda\}$, $\prod_{j<\lambda} B_j$ *is* κ*-tslender, if and only if every* B_j *is* κ*-tslender. In particular,* R^λ *is* κ*-tslender if and only if* R *is* κ*-tslender. Furthermore,* $\bigoplus_{j<\lambda} B_j$ *is* κ*-tslender if and only if every* B_j *is* κ*-tslender.*

(6) If $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is an exact sequence and* A, C *are* κ*-tslender, then* B *is likewise* κ*-tslender.*

Proof. (1) and (2) follow directly from the definition.

(3) follows from Dimitric [2], Theorem 3.10 (5), which states, that if $|I| = \kappa$ is a non-measurable cardinal and M is slender, then, for every morphism

$$f\colon \prod_{i\in I} A_i \to M\,,$$

there exists an $i_0 \in I$, $i_0 < \omega$, such that $f(\prod_{i\geq i_0} A_i) = 0$.

For (4), consider the non-κ-tslender identity morphism $\mathrm{id}\colon R^\kappa \to R^\kappa$.

(5) If $\prod_{j<\lambda} B_j$ is κ-tslender, then, by (2), every submodule is κ-tslender, hence that applies to each B_j as well. Now assume that every B_j is κ-tslender. If

$$f\colon \prod_{i<\kappa} Ra_i \to \prod_{j<\lambda} B_j\,,$$

we know that then

$$f = \left(f_j \colon \prod_{i<\kappa} Ra_i \to B_j \right)_{j \in J},$$

$|J| = \lambda$. We have $\forall j \in J\ \exists i(j) < \kappa$ with $f(\prod_{i \geq i(j)} Ra_i) = 0$. Let

$$i_0 = \sup \{ i(j) : j < \lambda \} .$$

By the assumption, $\mathrm{cf}(\kappa) > \lambda$, therefore $i_0 < \kappa$. Now we clearly have

$$f\Big(\prod_{i \geq i_0} Ra_i \Big) = 0 .$$

The remaining claims are a special case and the fact that the direct sum is submodule of the direct product, hence by (2) has to be κ-tslender.

(6) Let $f \colon \prod_{i<\kappa} Ra_i \to B$ be an arbitrary morphism. Then

$$\beta f \colon \prod_{i<\kappa} Ra_i \to C ,$$

hence, by κ-tslenderness of C, there is an $i' < \kappa$ such that

$$\beta f\Big(\prod_{\kappa > i \geq i'} Ra_i \Big) = 0 .$$

In other words,

$$f\Big(\prod_{\kappa > i \geq i'} Ra_i \Big) \subseteq \ker \beta = \operatorname{Im} \alpha \cong A ,$$

which implies that f maps $\prod_{\kappa > i \geq i'} Ra_i$ into $\operatorname{Im} \alpha \cong A$. On the other hand, A was assumed to be κ-tslender which then implies that there is an $i_0 < \kappa$, $i_0 \geq i'$, with $f(\prod_{\kappa > i \geq i_0} Ra_i) = 0$, which establishes slenderness of B. □

4 Classes $\mathcal{H}_\kappa$

In this section, we assume that cardinals are non-measurable.

Definition 3. Given an infinite cardinal κ, an $M \in R\mathbf{Mod}$ is called a *κ-hmodule*, if, for every index set I, and every family of R-modules $\{A_i : i \in I\}$, the following holds:

$$\mathrm{Hom}_R \left(\prod_{i \in I} A_i \Big/ \prod_{i \in I}^{\kappa} A_i, M \right) = 0.$$

For brevity, denote $D_\kappa = \prod_{i \in I} A_i \,/\, \prod_{i \in I}^{\kappa} A_i$, so that we can rewrite this condition as

$$\mathrm{Hom}_R \left(D_\kappa, M \right) = 0 .$$

The class of κ-hmodules is denoted by $\mathcal{H}_\kappa$.

Some well-known properties of the Hom functor are instrumental in obtaining some properties of κ-hmodules as follows:

Proposition 5. *(1)* $0 \in \mathcal{H}_\kappa$.

(2) $\mathcal{H}_k$ *is closed with respect to submodules.*

(3) $\prod M_j \in \mathcal{H}_k$ *if and only if, every* $M_j \in \mathcal{H}_\kappa$ *(closure with respect to products).*

(4) For a short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$*, if* $A, C \in \mathcal{H}_\kappa$*, then* $B \in \mathcal{H}_\kappa$*.*

(5) If $\kappa < \lambda$*, then* $\mathcal{H}_\kappa \subseteq \mathcal{H}_\lambda$*.*

Proof. (1) is trivial.

(2) follows from the fact that, if $B \leq A$, then $\operatorname{Hom}(D_\kappa, B) \leq \operatorname{Hom}(D_\kappa, A)$.

The natural isomorphism

$$\operatorname{Hom}\left(D_\kappa, \prod M_j\right) \cong \prod \operatorname{Hom}(D_\kappa, M_j)$$

establishes one implication of (3) and the other implication is a consequence of (2).

(4) is a consequence of left exactness of the Hom functor:

$$0 \longrightarrow \operatorname{Hom}(D_\kappa, A) \longrightarrow \operatorname{Hom}(D_\kappa, B) \longrightarrow \operatorname{Hom}(D_\kappa, C)\ .$$

For (5) assume that, on the contrary, there were an

$$a = (a_i)_{i \in I} \in \prod_{i \in I} A_i\,,$$

and a morphism $f\colon D_\kappa \to M$ with $f(\overline{a}) \neq 0$; then it would contradict Proposition 3 (2) since we would have

$$\operatorname{Hom}_R\left(\prod_{i \in I} Ra_i \Big/ \prod_{i \in I}^{\kappa} Ra_i, M\right) \neq 0\,.$$

For (6), use the fact that, for $\kappa < \lambda$,

$$\prod_{i \in I}^{\kappa} A_i \leq \prod_{i \in I}^{\lambda} A_i\,. \qquad \square$$

Properties (2)–(4) signify that $\mathcal{H}_\kappa$ is a torsion free class for a torsion theory, for every κ (cf. e.g. Stenström [7]).

A good question is whether Proposition 3 (2) holds for κ-cslender modules. It does for $\kappa = \omega$ and non-measurable index sets I (cf. Dimitric [2], Theorem 3.9). We are exploring this issue for uncountable κ.

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