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Spectral Theory of Singular Hahn Difference Equation of the Sturm-Liouville Type

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Abstract. In this work, we consider the singular Hahn difference equation of the Sturm-Liouville type. We prove the existence of the spectral function for this equation. We establish Parseval equality and an expansion formula for this equation on a semi-unbounded interval.

1 Introduction

Spectral expansion theorems have attracted mathematicians for a long time. The first results of this type go back to Weyl [37]. Additional results were obtained by Stone [34], [35], Naimark [31], Berezanskii [13] and Titchmarsh [36]. Usually, if we want to solve a partial differential equation using the Fourier method (i.e., the separation of variables) then we consider the problem of expanding an arbitrary function as a series of eigenfunctions. Hence the eigenfunction expanding problem has been studied extensively in the literature (see [2], [3], [4], [5], [6], [7], [13], [15], [16], [21], [22], [29], [30], [31], [34], [35], [36], [38], [39]).

The study of the Hahn difference operator dates back to Hahn's works [17] and [18]. In 1949, Hahn introduced the quantum difference operator $\mathrm{D}_{\omega,q}$ defined by

$$\mathrm{D}_{\omega,q} f(x) = \frac{f(\omega + qx) - f(x)}{\omega + (q-1)x},$$

where $q \in (0,1)$ and $\omega > 0$ (see [17], [18]). The Hahn difference operator $\mathrm{D}_{\omega,q}$ is a generalization of the two well-known difference operators; namely, the quantum

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q-difference operator (see [23]) and the forward difference operator (see [24], [25]). This operator has numerous applications in the construction of families of orthogonal polynomials and approximation problems (see [8], [12], [14], [27], [28], [32]). A proper inverse of $\mathrm{D}_{\omega,q}$ and the associated integral calculus were studied in [1], [9]. Next, in [19], Hamza et al. established the theory of linear Hahn difference equations. They investigated also the existence and uniqueness of the solution of initial value problems for Hahn difference equations. Moreover, they obtained Gronwall's and Bernoulli's inequalities with respect to the Hahn difference operator, and investigated the mean value theorems for this calculus. Later, Hamza and Makharesh [20] studied Leibniz's rule and Fubini's theorem associated with the Hahn difference operator. Sitthiwirattham [33] investigated the nonlocal boundary value problem for nonlinear Hahn difference equation. Recently, in [10], Annaby et al. established a Sturm-Liouville theory associated with the Hahn difference operator in the regular setting. In [11], the authors introduce a couple of sampling theorems of Lagrange-type interpolation for ω, q-integral transforms whose kernels are either solutions or Green's function of the ω, q-Hahn-Sturm-Liouville problem.

In this paper, we study Hahn difference equations of the Sturm-Liouville type. Having the solutions of such equations we define the new Hilbert space and construct on it the Fourier transform and prove the Parseval equation. Therefore, we prove the existence of a spectral function for the Hahn difference equations of the Sturm-Liouville type in Lemma 3. In Theorem 5, a Parseval equality and an expansion formula in eigenfunctions are established in terms of this spectral function.

2 Notation and basic results

In this section, our aim is to present some basic concepts concerning the theory of Hahn calculus. For more details, the reader may refer to [9], [10], [17] and [18]. Throughout the paper, let $q \in (0,1)$ and $\omega > 0$.

Let $\omega_0 := \omega/(1-q)$ and let I be a real interval containing ω_0.

Definition 1 ([17], [18]). Let $f\colon I \to \mathbb{R} := (-\infty, \infty)$ be a function. The Hahn difference operator is defined by

$$\mathrm{D}_{\omega,q} f(x) = \begin{cases} \dfrac{f(\omega+qx)-f(x)}{\omega+(q-1)x}, & x \neq \omega_0, \\ f'(\omega_0), & x = \omega_0, \end{cases} \tag{1}$$

provided that f is differentiable at ω_0. We call $\mathrm{D}_{\omega,q} f$ a ω, q-derivative of f.

Remark 1. The Hahn difference operator unifies two well known operators. When $q \to 1$, we get the forward difference operator which is defined by

$$\Delta_\omega f(x) := \frac{f(\omega+x)-f(x)}{(\omega+x)-x}, \quad x \in \mathbb{R}.$$

When $\omega \to 0$, we get the Jackson q-difference operator which is defined by

$$\mathrm{D}_q f(x) := \frac{f(qx)-f(x)}{(qx)-x}, \quad x \neq 0.$$

Furthermore, under appropriate conditions, we have

$$\lim_{\substack{q\to 1\\ \omega\to 0}} \mathrm{D}_{\omega,q} f(x) = f'(x).$$

In what follows, we present several important properties of the ω, q-derivative.

Theorem 1 ([9]). *Let $f, g\colon I \to \mathbb{R}$ be ω, q-differentiable at $x \in I$ and $h(x) := \omega + qx$. Then, for all $x \in I$ we have*

i) $\mathrm{D}_{\omega,q}(af + bg)(x) = a\mathrm{D}_{\omega,q} f(x) + b\mathrm{D}_{\omega,q} g(x), \quad a, b \in I,$

ii) $\mathrm{D}_{\omega,q}(fg)(x) = \mathrm{D}_{\omega,q}(f(x))g(x) + f(\omega + xq)\mathrm{D}_{\omega,q} g(x),$

iii) $\mathrm{D}_{\omega,q}\Big(\dfrac{f}{g}\Big)(x) = \dfrac{\mathrm{D}_{\omega,q}(f(x))g(x) - f(x)\mathrm{D}_{\omega,q} g(x)}{g(x)g(\omega + xq)},$

iv) $\mathrm{D}_{\omega,q}(h^{-1}(x)) = \mathrm{D}_{-\omega q^{-1}, q^{-1}} f(x).$

The concept of the ω, q-integral of the function f can be defined as follows.

Definition 2 (Jackson-Nörlund Integral [9]). Let $f\colon I \to \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. We define the ω, q-integral of the function f from a to b by

$$\int_a^b f(x)\,\mathrm{d}_{\omega,q}(x) := \int_{\omega_0}^b f(x)\,\mathrm{d}_{\omega,q}(x) - \int_{\omega_0}^a f(x)\,\mathrm{d}_{\omega,q}(x),$$

where

$$\int_{\omega_0}^x f(t)\,\mathrm{d}_{\omega,q}(t) := ((1-q)x - \omega)\sum_{n=0}^{\infty} q^n f\Big(\omega\frac{1-q^n}{1-q} + xq^n\Big), \quad x \in I$$

provided that the series converges at $x = a$ and $x = b$. In this case, f is called ω, q-integrable on $[a, b]$.

Similarly, one can define the ω, q-integral of a function f over (ω_0, ∞) by

$$\int_{\omega_0}^{\infty} f(x)\,\mathrm{d}_{\omega,q}(x) := \lim_{b\to\infty} \int_{\omega_0}^b f(x)\,\mathrm{d}_{\omega,q}(x)\,.$$

The following properties of ω, q-integration can be found in [9].

Lemma 1 ([9]). *Let $f, g\colon I \to \mathbb{R}$ be ω, q-integrable on I and let $a, b, c \in I$ where $a < c < b$ and $\alpha, \beta \in \mathbb{R}$. Then the following formulae hold:*

i) $\displaystyle\int_a^b (\alpha f(x) + \beta g(x))\,\mathrm{d}_{\omega,q}(x) = \alpha\int_a^b f(x)\,\mathrm{d}_{\omega,q}(x) + \beta\int_a^b g(x)\,\mathrm{d}_{\omega,q}(x),$

ii) $\displaystyle\int_a^a f(x)\,\mathrm{d}_{\omega,q}(x) = 0,$

iii) $\int_a^b f(x)\,\mathrm{d}_{\omega,q}(x) = \int_a^c f(x)\,\mathrm{d}_{\omega,q}(x) + \int_c^b f(x)\,\mathrm{d}_{\omega,q}(x)$,

iv) $\int_a^b f(x)\,\mathrm{d}_{\omega,q}(x) = -\int_b^a f(x)\,\mathrm{d}_{\omega,q}(x)$.

Next, we present the ω, q-integration by parts.

Lemma 2 ([9]). *Let $f, g\colon I \to \mathbb{R}$ be ω, q-integrable on I and let $a, b \in I$ where $a < b$. Then the following formula holds:*

$$\int_a^b f(x)\mathrm{D}_{\omega,q}g(x)\,\mathrm{d}_{\omega,q}(x) + \int_a^b g(\omega+qx)\mathrm{D}_{\omega,q}f(x)\,\mathrm{d}_{\omega,q}(x) = f(b)g(b) - f(a)g(a)\,.$$

The next result is the fundamental theorem of Hahn calculus.

Theorem 2 ([9]). *Let $f\colon I \to \mathbb{R}$ be continuous at ω_0. Define*

$$F(x) := \int_{\omega_0}^x f(t)\,\mathrm{d}_{\omega,q}(t), \quad x \in I.$$

Then F is continuous at ω_0. Moreover, $\mathrm{D}_{\omega,q}F(x)$ exists for every $x \in I$ and $\mathrm{D}_{\omega,q}F(x) = f(x)$. Conversely,

$$\int_a^b \mathrm{D}_{\omega,q}F(x)\,\mathrm{d}_{\omega,q}(x) = F(b) - F(a).$$

Let $L^2_{\omega,q}(\omega_0,\infty)$ be the space of all complex-valued functions defined on $[\omega_0,\infty)$ such that

$$\|f\| := \Big(\int_{\omega_0}^\infty |f(x)|^2\,\mathrm{d}_{\omega,q}x\Big)^{1/2} < \infty.$$

The space $L^2_{\omega,q}(\omega_0,\infty)$ is a separable Hilbert space with the inner product

$$(f,g) := \int_{\omega_0}^\infty f(x)\overline{g(x)}\,\mathrm{d}_{\omega,q}x, \quad f,g \in L^2_{\omega,q}(\omega_0,\infty)$$

(see [9]).

The ω, q-Wronskian of $y(\cdot)$, $z(\cdot)$ is defined by

$$W_{\omega,q}(y,z)(x) := y(x)\mathrm{D}_{\omega,q}z(x) - z(x)\mathrm{D}_{\omega,q}y(x), \quad x \in [\omega_0,\infty). \tag{2}$$

Now we recall the following well-known theorems of Helly.

Theorem 3 ([26]). *Let $(w_n)_{n\in\mathbb{N}}$ ($\mathbb{N} := \{1,2,\dots\}$) be a uniformly bounded sequence of real nondecreasing functions on a finite interval $a \le \lambda \le b$. Then there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and a nondecreasing function w such that*

$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda), \quad a \le \lambda \le b.$$

Theorem 4 ([26]). *Assume $(w_n)_{n\in\mathbb{N}}$ is a real, uniformly bounded sequence of non-decreasing functions on a finite interval $a \le \lambda \le b$, and suppose*

$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda), \quad a \le \lambda \le b.$$

If f is any continuous function on $a \le \lambda \le b$, then

$$\lim_{n\to\infty} \int_a^b f(\lambda)\,\mathrm{d}w_n(\lambda) = \int_a^b f(\lambda)\,\mathrm{d}w(\lambda)\,.$$

3 Main Results

Let us consider the Hahn difference equations of the Sturm-Liouville type given by

$$\Gamma(y) := -q^{-1}\mathrm{D}_{-\omega q^{-1},q^{-1}}\mathrm{D}_{\omega,q}y(x) + v(x)y(x) = \lambda y(x), \quad x \in (\omega_0,\infty) \tag{3}$$

with the boundary condition

$$\mathrm{D}_{-\omega q^{-1},q^{-1}}y(\omega_0)\sin\beta + y(\omega_0)\cos\beta = 0, \quad \beta \in \mathbb{R}, \tag{4}$$

where λ is a complex eigenvalue parameter, v is a real-valued continuous function at ω_0 defined on $[\omega_0,\infty)$.

If we endow the problem (3)–(4) with the boundary condition

$$\mathrm{D}_{-\omega q^{-1},q^{-1}}y(q^{-n})\sin\alpha + y(q^{-n})\cos\alpha = 0, \quad \alpha \in \mathbb{R}, n \in \mathbb{N}, \tag{5}$$

then we deduce that the problem given by (3), (4) and (5) is a regular problem.

In [10], Annaby et al. showed that the boundary value problem (3) with the boundary conditions (4) and (5) has a compact resolvent, thus this problem has a purely discrete spectrum.

Let $\varphi(x)$ be a solution of the system (3) satisfying the initial conditions

$$\varphi(\omega_0) = \sin\beta, \quad \mathrm{D}_{-\omega q^{-1},q^{-1}}\varphi(\omega_0) = -\cos\beta. \tag{6}$$

Let ϱ be any nondecreasing function on $\mathbb{R}$. Denote by $L^2_\varrho(\mathbb{R})$ the Hilbert space of all functions $f\colon \mathbb{R}\to\mathbb{R}$ measurable with respect to the Lebesgue-Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^{\infty} f^2(\lambda)\,\mathrm{d}\varrho(\lambda) < \infty,$$

with the inner product

$$(f,g)_\varrho := \int_{-\infty}^{\infty} f(\lambda)g(\lambda)\,\mathrm{d}\varrho(\lambda).$$

The main result of this section is the following.

Theorem 5. *There exists a nondecreasing function $\varrho(\lambda)$ on $-\infty < \lambda < \infty$, a spectral function for the boundary value problem given by* (3)–(4), *with the following properties.*

i) If f is a real-valued function and $f \in L^2_{\omega,q}(\omega_0, q^{-n})$, then there exists a function $F \in L^2_\varrho(\mathbb{R})$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left\{ F(\lambda) - \int_{\omega_0}^{q^{-n}} f(x)\varphi(x,\lambda)\, \mathrm{d}_{\omega,q}x \right\} \mathrm{d}\varrho(\lambda) = 0, \tag{7}$$

and the Parseval equality

$$\int_{\omega_0}^{\infty} f^2(x)\, \mathrm{d}_{\omega,q}x = \int_{-\infty}^{\infty} F^2(\lambda)\, \mathrm{d}\varrho(\lambda) \tag{8}$$

holds.

ii) The integral

$$\int_{-\infty}^{\infty} F(\lambda)\varphi(x,\lambda)\, \mathrm{d}\varrho(\lambda)$$

converges to f in $L^2_{\omega,q}(\omega_0, \infty)$. That is,

$$\lim_{n\to\infty} \int_{\omega_0}^{q^{-n}} \left\{ f(x) - \int_{-n}^{n} F(\lambda)\varphi(x,\lambda)\, \mathrm{d}\varrho(\lambda) \right\}^2 \mathrm{d}_{\omega,q}x = 0. \tag{9}$$

Remark 2. The expression

$$f(x) = \int_{-\infty}^{\infty} F(\lambda)\varphi(x,\lambda)\, \mathrm{d}\varrho(\lambda),$$

where the equality is in the sense of (9), is called the expansion theorem.

Let $\lambda_{m,q^{-n}}$ (where $m, n \in \mathbb{N}$) denote the eigenvalues of the regular problem given by (3)–(5), and $\varphi(x,\lambda)$ be the solution of the equation (3) satisfying the initial conditions in (6). The function $\varphi_{m,q^{-n}}(x) = \varphi(x, \lambda_{m,q^{-n}})$ will be an eigenfunction corresponding to the eigenvalue $\lambda_{m,q^{-n}}$.

Let $f(\cdot)$ be an arbitrary real-valued function on $L^2_{\omega,q}(\omega_0, q^{-n})$ and

$$\alpha^2_{m,q^{-n}} = \int_{\omega_0}^{q^{-n}} \varphi^2_{m,q^{-n}}(x)\, \mathrm{d}_{\omega,q}x$$

where $m \in \mathbb{N}$. Then we have

$$\int_{\omega_0}^{q^{-n}} f^2(x)\, \mathrm{d}_{\omega,q}x = \sum_{m=1}^{\infty} \frac{1}{\alpha^2_{m,q^{-n}}} \left\{ \int_{\omega_0}^{q^{-n}} f(x)\varphi_{m,q^{-n}}(x)\, \mathrm{d}_{\omega,q}x \right\}^2, \tag{10}$$

which is called the *Parseval equality* (see [10]).

Now we will introduce the nondecreasing step function $\varrho_{q^{-n}}$ on $[0,\infty)$ by

$$\varrho_{q^{-n}}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_{m,q^{-n}}<0} \dfrac{1}{\alpha^2_{m,q^{-n}}}, & \text{for } \lambda \le 0, \\ \sum\limits_{0\le\lambda_{m,q^{-n}}<\lambda} \dfrac{1}{\alpha^2_{m,q^{-n}}}, & \text{for } \lambda \ge 0. \end{cases}$$

Then the equality (10) can be written as

$$\int_{\omega_0}^{q^{-n}} f^2(x)\,\mathrm{d}_{\omega,q}x = \int_{-\infty}^{\infty} F^2(\lambda)\,\mathrm{d}\varrho_{q^{-n}}(\lambda), \tag{11}$$

where

$$F(\lambda) = \int_{\omega_0}^{q^{-n}} f(x)\varphi(x,\lambda)\,\mathrm{d}_{\omega,q}x.$$

We will show that the Parseval equality for the problem given by (3) and (4) can be obtained from (11) by letting $n \to \infty$. To this end, we shall prove the following lemma.

Lemma 3. *The total variation of the functions $\varrho_{q^{-n}}(\lambda)$ is uniformly bounded with respect to q^{-n} in each finite interval in the domain λ, i.e., for $s > 0$, there exists a constant $M = M(s) > 0$ not depending on q^{-n} such that*

$$\bigvee_{-s}^{s}\{\varrho_{q^{-n}}(\lambda)\} = \sum_{-s\le\lambda_{m,q^{-n}}<s} \frac{1}{\alpha^2_{m,q^{-n}}} = \varrho_{q^{-n}}(s) - \varrho_{q^{-n}}(-s) < M. \tag{12}$$

Proof. Let $\sin\beta \neq 0$. Since $\varphi(x,\lambda)$ is continuous at ω_0, we deduce from the condition $\varphi(\omega_0,\lambda) = \sin\beta$ that there exists a positive number k such that $k-\omega_0$ is so small and

$$\frac{1}{k}\Big(\int_{\omega_0}^{k} \varphi(x,\lambda)\,\mathrm{d}_{\omega,q}x\Big)^2 > \frac{1}{2}\sin^2\beta. \tag{13}$$

Let us define $f_k(x)$ by the formula

$$f_k(x) = \begin{cases} \dfrac{1}{k}, & \omega_0 \le x < k \\ 0, & x > k. \end{cases}$$

By virtue of (11) and (13), we conclude that

$$\begin{aligned}\int_{\omega_0}^{k} f_k^2(x)\,\mathrm{d}_{\omega,q}x = \frac{1}{k} &= \int_{-\infty}^{\infty}\Big(\frac{1}{k}\int_{\omega_0}^{k}\varphi(x,\lambda)\,\mathrm{d}_{\omega,q}x\Big)^2\,\mathrm{d}\varrho_{q^{-n}}(\lambda)\\ &\ge \int_{-s}^{s}\Big(\frac{1}{k}\int_{\omega_0}^{k}\varphi(x,\lambda)\,\mathrm{d}_{\omega,q}x\Big)^2\,\mathrm{d}\varrho_{q^{-n}}(\lambda)\\ &> \frac{1}{2}\sin^2\beta\int_{-s}^{s}\mathrm{d}\varrho_{q^{-n}}(\lambda)\\ &= \frac{1}{2}\sin^2\beta\{\varrho_{q^{-n}}(s) - \varrho_{q^{-n}}(-s)\},\end{aligned}$$

which is the desired result.

If $\sin\beta = 0$, then we can define the function $f_k(x)$ by the formula

$$f_k(x) = \begin{cases} \dfrac{1}{k^2}, & \omega_0 \le x < k \\ 0, & x > k. \end{cases}$$

By virtue of (10), we get the inequality (12). □

Proof of Theorem 5. Suppose that the function $f_\xi(x)$ satisfies the following conditions:

i) $f_\xi(x)$ vanishes outside the interval $[\omega_0, q^{-\xi}], q^{-\xi} < q^{-n}$.

ii) The functions $f_\xi(x)$ and $\mathrm{D}_{\omega,q} f_\xi(x)$ are continuous on $[\omega_0, \infty)$.

iii) $f_\xi(x)$ satisfies the boundary condition (4).

Applying (11) to $f_\xi(x)$, we deduce that

$$\int_{\omega_0}^{q^{-\xi}} f_\xi^2(x)\,\mathrm{d}_{\omega,q}x = \int_{-\infty}^{\infty} F^2(\lambda)\,\mathrm{d}\varrho(\lambda), \tag{14}$$

where

$$F(\lambda) = \int_{\omega_0}^{q^{-\xi}} f_\xi(x)\varphi(x,\lambda)\,\mathrm{d}_{\omega,q}x\,. \tag{15}$$

Since $\varphi(x,\lambda)$ is a solution of the equation (3), we have

$$\varphi(x,\lambda) = \frac{1}{\lambda}\Gamma(\varphi(x,\lambda))\,.$$

By (15), we obtain

$$F_n(\lambda) = \frac{1}{\lambda}\int_{\omega_0}^{q^{-n}} f_\xi(x)\Gamma(\varphi(x,\lambda))\,\mathrm{d}_{\omega,q}x\,.$$

Since $f_\xi(x)$ vanishes in a neighbourhood of the point q^{-n}, and since $f_\xi(x)$ and $\varphi(x,\lambda)$ satisfy the boundary condition (6), by using Lemma 2 we obtain that

$$F_n(\lambda) = \frac{1}{\lambda}\int_{\omega_0}^{q^{-n}} \varphi(x,\lambda)\Gamma(f_\xi(x))\,\mathrm{d}_{\omega,q}x\,.$$

By virtue of (11), for any finite $s > 0$, we conclude that

$$\begin{aligned}\int_{|\lambda|>s} F_n^2(\lambda)\,\mathrm{d}\varrho_{q^{-n}}(\lambda) &\le \frac{1}{s^2}\int_{|\lambda|>s}\left\{\int_{\omega_0}^{q^{-n}} \varphi(x,\lambda)\Gamma(f_\xi(x))\,\mathrm{d}_{\omega,q}x\right\}^2 \mathrm{d}\varrho_{q^{-n}}(\lambda)\\ &\le \frac{1}{s^2}\int_{-\infty}^{\infty}\left\{\int_{\omega_0}^{q^{-n}} \varphi(x,\lambda)\Gamma(f_\xi(x))\,\mathrm{d}_{\omega,q}x\right\}^2 \mathrm{d}\varrho_{q^{-n}}(\lambda)\\ &= \frac{1}{s^2}\int_{\omega_0}^{q^{-\xi}} [\Gamma(f_\xi(x))]^2\,\mathrm{d}_{\omega,q}x\,.\end{aligned}$$

By the formula (14), we deduce that

$$\begin{aligned}&\left|\int_{\omega_0}^{q^{-\xi}} f_\xi^2(x)\mathrm{d}_{\omega,q}x - \int_{-s}^{s} F_n^2(\lambda)\mathrm{d}\varrho_{q^{-n}}(\lambda)\right|\\ &\qquad < \frac{1}{s^2}\int_{\omega_0}^{q^{-\xi}} [-q^{-1}\mathrm{D}_{-\omega q^{-1},q^{-1}}\mathrm{D}_{\omega,q} f_\xi(x) + v(x)f_\xi(x)]^2\,\mathrm{d}_{\omega,q}x\,.\end{aligned} \tag{16}$$

From Lemma 3, it follows that the set $\{\varrho_{q^{-n}}(\lambda)\}$ is bounded. Using Theorems 3 and 4, one can find a sequence $\{q^{-n_k}\}$ such that the sequence $\varrho_{q^{-n_k}}(\lambda)$ converges to a monotone function $\varrho(\lambda)$ as $k \to \infty$. Passing to the limit with respect to $\{q^{-n_k}\}$ in (16), we conclude that

$$\left| \int_{\omega_0}^{q^{-\xi}} f_\xi^2(x)\, \mathrm{d}_{\omega,q}x - \int_{-s}^{s} F_n^2(\lambda)\, \mathrm{d}\varrho(\lambda) \right| < \frac{1}{s^2} \int_{\omega_0}^{q^{-\xi}} [\Gamma(f_\xi(x))]^2\, \mathrm{d}_{\omega,q}x\,.$$

Letting $s \to \infty$, we see that

$$\int_{\omega_0}^{q^{-\xi}} f_\xi^2(x)\, \mathrm{d}_{\omega,q}x = \int_{-\infty}^{\infty} F_n^2(\lambda)\, \mathrm{d}\varrho(\lambda)\,.$$

Now, let f be an arbitrary real-valued function on $L^2_{\omega,q}(\omega_0,\infty)$. It is known that there exists a sequence $\{f_\xi(x)\}$ satisfying the conditions i)–iii) and such that

$$\lim_{\xi\to\infty} \int_{\omega_0}^{\infty} (f(x) - f_\xi(x))^2\, \mathrm{d}_{\omega,q}x = 0\,.$$

Let

$$F_\xi(\lambda) = \int_{\omega_0}^{\infty} f_\xi(x)\varphi(x,\lambda)\, \mathrm{d}_{\omega,q}x\,.$$

Then, we have

$$\int_{\omega_0}^{\infty} f_\xi^2(x)\, \mathrm{d}_{\omega,q}x = \int_{-\infty}^{\infty} F_\xi^2(\lambda)\, \mathrm{d}\varrho(\lambda).$$

Since

$$\int_{\omega_0}^{\infty} (f_{\xi_1}(x) - f_{\xi_2}(x))^2\, \mathrm{d}_{\omega,q}x \to 0 \quad \text{as } \xi_1, \xi_2 \to \infty\,,$$

we have

$$\int_{-\infty}^{\infty} (F_{\xi_1}(\lambda) - F_{\xi_2}(\lambda))^2\, \mathrm{d}\varrho(\lambda) = \int_{\omega_0}^{\infty} (f_{\xi_1}(x) - f_{\xi_2}(x))^2\, \mathrm{d}_{\omega,q}x \to 0$$

as $\xi_1, \xi_2 \to \infty$. It follows from the completeness of the space $L^2_\varrho(\mathbb{R})$ that there exists a limit function F which satisfies

$$\int_{\omega_0}^{\infty} f^2(x)\, \mathrm{d}_{\omega,q}x = \int_{-\infty}^{\infty} F^2(\lambda)\, \mathrm{d}\varrho(\lambda).$$

Our next aim is to prove that the function

$$K_\xi(\lambda) = \int_{\omega_0}^{q^{-\xi}} f(x)\varphi(x,\lambda)\, \mathrm{d}_{\omega,q}x$$

converges to F as $\xi \to \infty$, in the metric of the space $L^2_\varrho(\mathbb{R})$. Let g be another function in $L^2_{\omega,q}(\omega_0,\infty)$. Similarly, let $G(\lambda)$ be defined by g. It is evident that

$$\int_{\omega_0}^{\infty} (f(x) - g(x))^2\, \mathrm{d}_{\omega,q}x = \int_{-\infty}^{\infty} \{F(\lambda) - G(\lambda)\}^2\, \mathrm{d}\varrho(\lambda).$$

We define

$$g(x)=\begin{cases} f(x), & x\in[\omega_0,q^{-\xi}]\\ 0, & x\in(q^{-\xi},\infty).\end{cases}$$

Then we obtain that

$$\int_{-\infty}^{\infty}\{F(\lambda)-K_\xi(\lambda)\}^2\,\mathrm{d}\varrho(\lambda)=\int_{q^{-\xi}}^{\infty}f^2(x)\,\mathrm{d}_{\omega,q}x\to 0\quad(\xi\to\infty),$$

i.e., K_ξ converges to F in $L^2_\varrho(\mathbb{R})$ as $\xi\to\infty$. This proves i).

Now, we will prove ii). Suppose that the real-valued functions f,g are in $L^2_{\omega,q}(\omega_0,q^{-n})$; and $F(\lambda)$ and $G(\lambda)$ are their Fourier transforms, respectively. Then $F\pm G$ are the transforms of $f\pm g$. Consequently, in view of (8), we have

$$\int_{\omega_0}^{\infty}[f(x)+g(x)]^2\,\mathrm{d}_{\omega,q}x=\int_{-\infty}^{\infty}(F(\lambda)+G(\lambda))^2\,\mathrm{d}\varrho(\lambda),$$
$$\int_{\omega_0}^{\infty}[f(x)-g(x)]^2\,\mathrm{d}_{\omega,q}x=\int_{-\infty}^{\infty}(F(\lambda)-G(\lambda))^2\,\mathrm{d}\varrho(\lambda).$$

Subtracting the second relation from the first one, we deduce that

$$\int_{\omega_0}^{\infty}f(x)g(x)\,\mathrm{d}_{\omega,q}x=\int_{-\infty}^{\infty}F(\lambda)G(\lambda)\,\mathrm{d}\varrho(\lambda).\tag{17}$$

We set

$$f_\tau(x)=\int_{-\tau}^{\tau}F(\lambda)\varphi(x,\lambda)\,\mathrm{d}\varrho(\lambda),$$

where F is the function defined in (7). Let $g(\cdot)$ be a real-valued function which is equal to zero outside the finite interval $[\omega_0,q^{-\mu}]$. Thus we obtain

$$\begin{aligned}\int_{\omega_0}^{q^{-\mu}}f_\tau(x)g(x)\,\mathrm{d}_{\omega,q}x&=\int_{\omega_0}^{q^{-\mu}}\left\{\int_{-\tau}^{\tau}F(\lambda)\varphi(x,\lambda)\,\mathrm{d}\varrho(\lambda)\right\}g(x)\,\mathrm{d}_{\omega,q}x\\&=\int_{-\tau}^{\tau}F(\lambda)\left\{\int_{\omega_0}^{q^{-\mu}}\varphi(x,\lambda)g(x)\,\mathrm{d}_{\omega,q}x\right\}\mathrm{d}\varrho(\lambda)\\&=\int_{-\tau}^{\tau}F(\lambda)G(\lambda)\,\mathrm{d}\varrho(\lambda).\end{aligned}\tag{18}$$

By (17), we have

$$\int_{\omega_0}^{\infty}f(x)g(x)\,\mathrm{d}_{\omega,q}x=\int_{-\infty}^{\infty}F(\lambda)G(\lambda)\,\mathrm{d}\varrho(\lambda).\tag{19}$$

By (18) and (19), we obtain

$$\int_{\omega_0}^{\infty}(f(x)-f_\tau(x))g(x)\,\mathrm{d}_{\omega,q}x=\int_{|\lambda|>\tau}F(\lambda)G(\lambda)\,\mathrm{d}\varrho(\lambda).$$

By using Cauchy-Schwarz inequality, we see that

$$\begin{aligned}\left|\int_{\omega_0}^{\infty}(f(x)-f_\tau(x))g(x)\,\mathrm{d}_{\omega,q}x\right|^2 &\le \int_{|\lambda|>\tau}F^2(\lambda)\,\mathrm{d}\varrho(\lambda)\int_{|\lambda|>\tau}G^2(\lambda)\,\mathrm{d}\varrho(\lambda)\\ &\le \int_{|\lambda|>\tau}F^2(\lambda)\,\mathrm{d}\varrho(\lambda)\int_{-\infty}^{\infty}G^2(\lambda)\,\mathrm{d}\varrho(\lambda).\end{aligned}$$

Applying this inequality to the function

$$g(x)=\begin{cases}f_\tau(x)-f(x), & x\in[\omega_0,q^{-\mu}]\\ 0, & x\in(q^{-\mu},\infty),\end{cases}$$

we deduce that

$$\int_{\omega_0}^{q^{-\mu}}(f(x)-f_\tau(x))^2\,\mathrm{d}_{\omega,q}x\le\int_{|\lambda|>\tau}F^2(\lambda)\,\mathrm{d}\varrho(\lambda). \tag{20}$$

Let us mention that the right-hand side of the inequality (20) does not depend on μ. Hence passing to the limit as $\tau\to\infty$ gives the desired result. Thus Theorem 5 is proved. □

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