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A new parameterized logarithmic kernel function for linear optimization with a double barrier term yielding the best known iteration bound

Benhadid Ayache, Saoudi Khaled

Abstract. In this paper, we propose a large-update primal-dual interior point algorithm for linear optimization. The method is based on a new class of kernel functions which differs from the existing kernel functions in which it has a double barrier term. The investigation according to it yields the best known iteration bound $O(\sqrt{n}\log(n)\log(\frac{n}{\varepsilon}))$ for large-update algorithm with the special choice of its parameter m and thus improves the iteration bound obtained in Bai et al. [2] for large-update algorithm.

1 Introduction

After the groundbreaking paper of Karmarkar [7], Kernel functions play an important role in the complexity analysis of the interior point methods (IPMs) for linear optimization (LO).

In 2001, Peng et al. [9] designed a new paradigm of primal-dual algorithms based on the so-called self-regular proximity functions for LO. They improved iteration bound and achieved the best known complexity results for large and small-update methods. Subsequently, in 2004 Bai et al. [2] proposed new kernel function with an exponential barrier term, and introduced the first new kernel function with a trigonometric barrier term. These functions enjoy useful properties and determine new search directions for primal-dual interior point algorithms. Based on these functions, they obtained the best known complexity results for large-update methods, namely, $O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ and good numerical results.

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Affiliation:
Benhadid Ayache – Department of mathematics, Université de Batna 2 – Mostefa Ben Boulaïd, 53, Route de Constantine, Fésdis, Batna 05078, Algeria
E-mail: benhadidayache@yahoo.fr
Saoudi Khaled – Department of Mathematics and Computer Science, Abbes Laghrour University, BP 1252 Route de Batna, Khenchela 40004, Algeria
E-mail: saoudikhaled@hotmail.com

In 2008, El Ghami et al. [6] proposed parameterized kernel function with a logarithmic barrier term. This function generalized the kernel functions given in [5], [12].

In 2018, Bouafia et al. [4] proposed a parameterized logarithmic kernel function for primal-dual IPMs. They obtained the best known complexity results for large and small-update methods, they took the middle between Peng [9] and El Ghami's [6] barrier as a barrier term. The objective of this paper is to introduce a new class of kernel functions which differs from the existing kernel functions in which it has a double barrier term (logarithmic-exponential barrier term).

The paper is organized as follows. In Section 2, we recall the preliminaries. In Sections 3 and 4, we define a new kernel function and give its properties which are essential for the complexity analysis. The estimate of the step size and the decrease behavior of the new barrier function are discussed in Section 5. Also we derive the complexity result for both large-update and small-update methods. Some numerical results are provided in Section 6. Finally, we end up the paper by a conclusion.

2 Preliminaries

In this section we recall some basic concepts and the generic IPMs, we consider linear optimization (LO) problem in the standard format:

$$\min \langle c, x\rangle : Ax = b, x \geq 0, \tag{P}$$

where $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A) = m$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, and its dual problem

$$\max \langle b, y\rangle : A^T y + s = c, s \geq 0. \tag{D}$$

A new polynomial-time method for solving LO is proposed by Karmarkar [7]. After that, this method was developped in the literature which play an important role for solving linear optimization problem and its variants are now called IPMs. For more details about the subject, we can refer to Bai et al. [1], Peng et al. [11], Roos et al. [12] and Ye [14]. Without loss of generality, we assume that (P) and (D) satisfy the interior point condition (IPC), i.e., there exist (x^0, y^0, s^0) such that

$$Ax^0 = b, \quad x^0 > 0, \quad A^T y^0 + s^0 = c, \quad s^0 > 0. \tag{1}$$

It is well known that finding an optimal solution of (P) and (D) is equivalent to solving the following system

$$Ax = b, \quad x \geq 0, \quad A^T y + s = c, \quad s \geq 0, \quad xs = 0. \tag{2}$$

The basic idea of primal-dual IPMs is to replace the third equation in (2), the so-called complementarity condition for (P) and (D), by the parameterized equation $xs = \mu \mathrm{e}$, with $\mu \geq 0$. Thus we consider the system

$$Ax = b, \quad x \geq 0, \quad A^T y + s = c, \quad s \geq 0, \quad xs = \mu \mathrm{e}. \tag{3}$$

Surprisingly enough, if the IPC is satisfied, then there exists a solution, for each $\mu > 0$, and this solution is unique. It is denoted as $(x(\mu), y(\mu), s(\mu))$, and we call

$x(\mu)$ the μ-center of (P) and $(y(\mu), s(\mu))$ the μ-center of (D). The set of μ-centers (with μ running through all positive real numbers) gives a homotype path, which is called the central path of (P) and (D). The relevance of the central path for LO was recognized first by Megiddo [8] and Sonnevend [13]. If $\mu \to 0$, then the limit of the central path exists, and since the limit points satisfy the complementarity condition, the limit yields optimal solutions for (P) and (D). From a theoretical point of view, the IPC can be assumed without loss of generality. In fact, we may, and will, assume that $x^0 = s^0 = \mathrm{e}$. In practice, this can be realized by embedding the given problems (P) and (D) into a homogeneous self-dual problem, which has two additional variables and two additional constraints. For this and the other properties mentioned above, see [12].

The IPMs follow the central path approximately. We briefly describe the usual approach. Without loss of generality, we assume that $(x(\mu), y(\mu), s(\mu))$ is known for some positive μ. For example, due to the above assumption, we may assume this for $\mu = 1$, with $x(1) = s(1) = \mathrm{e}$. We then decrease μ to $\mu = (1-\theta)\mu$ for some fixed $\theta \in (0, 1)$, and we solve the following Newton system:

$$A\Delta x = 0, \quad A^T\Delta y + \Delta s = 0, \quad s\Delta x + x\Delta s = \mu\mathrm{e} - xs. \tag{4}$$

This system uniquely defines a search direction $(\Delta x, \Delta y, \Delta s)$. By taking a step along the search direction, with the step size defined by some line search rules, we construct a new triple (x, y, s). If necessary, we repeat the procedure until we find iterates that are "close" to $(x(\mu), y(\mu), s(\mu))$. Then μ is again reduced by the factor $1-\theta$, and we apply Newton's method targeting the new μ-centers, and so on. This process is repeated until μ is small enough, say until $n\mu \le \varepsilon$, at this stage, we have found an ε-solution of problems (P) and (D). The result of a Newton step with step size α is denoted as

$$x_+ = x + \alpha\Delta x, \quad s_+ = s + \alpha\Delta s, \quad y_+ = y + \alpha\Delta y, \tag{5}$$

where the step size α satisfies $0 < \alpha \le 1$. Now we introduce the scaled vector v and the scaled search directions d_x and d_s as follows:

$$v = \sqrt{\frac{xs}{\mu}}, \quad d_x = \frac{v\Delta x}{x}, \quad d_s = \frac{v\Delta s}{s}. \tag{6}$$

System (4) can be rewritten as follows:

$$\overline{A}d_x = 0, \quad \overline{A}^T\Delta y + d_s = 0, \quad d_x + d_s = v^{-1} - v, \tag{7}$$

where $\overline{A} = \frac{1}{\mu}AV^{-1}X$, $V = \mathrm{diag}(v)$, $X = \mathrm{diag}(x)$. Note that the right-hand side of the third equation in (7) is equal to the negative gradient of the logarithmic barrier function $\Phi(v)$, i.e., $d_x + d_s = -\nabla\Phi(v)$, system (7) can be rewritten as follows:

$$\overline{A}d_x = 0, \quad \overline{A}^T\Delta y + d_s = 0, \quad d_x + d_s = -\nabla\Phi(v), \tag{8}$$

where the barrier function $\Phi(v)\colon \mathbb{R}^n_{++} \to \mathbb{R}_+$ is defined as follows:

$$\Phi(v) = \Phi(x, s; \mu) = \sum_{i=1}^{n} \psi(v_i), \tag{9}$$

$$\psi(v_i) = \frac{v_i^2 - 1}{2} - \log v_i\,. \tag{10}$$

We use $\Phi(v)$ as the proximity function to measure the distance between the current iterate and the μ-center for given $\mu > 0$. We also define the norm-based proximity measure, $\delta(v)\colon \mathbb{R}^n_{++} \to \mathbb{R}_+$, as follows

$$\delta(v) = \frac{1}{2}\|\nabla\Phi(v)\| = \frac{1}{2}\|d_x + d_s\|, \tag{11}$$

We call $\psi(t)$ the kernel function of the logarithmic barrier function $\Phi(v)$. In this paper, we replace $\psi(t)$ by a new kernel function $\psi_{\mathrm{New}}(t)$ and $\Phi(v)$ by a new barrier function $\Phi_{\mathrm{New}}(v)$, which will be defined in Section 3. Note that the pair (x, s) coincides with the μ-center $(x(\mu), s(\mu))$ if and only if $v = \mathrm{e}$. It is clear from the above description that the closeness of (x, s) to $(x(\mu), s(\mu))$ is measured by the value of $\Phi(v)$ with $\tau > 0$ as a threshold value. If $\Phi(v) \le \tau$, then we start a new outer iteration by performing a μ-update; otherwise, we enter an inner iteration by computing the search directions at the current iterates with respect to the current value of μ and apply (5) to get new iterates. If necessary, we repeat the procedure until we find iterates that are in the neighborhood of $(x(\mu), s(\mu))$. Then μ is again reduced by the factor $1 - \theta$ with $0 < \theta < 1$, and we apply Newton's method targeting the new μ-centers, and so on. This process is repeated until μ is small enough, say until $n\mu < \varepsilon$; at this stage, we have found an ε-approximate solution of LO. The parameters τ, θ and the step size α should be chosen in such a way that the algorithm is optimized in the sense that the number of iterations required by algorithm is as small as possible. The choice of the so-called barrier update parameter θ plays an important role in both theory and practice of IPMs. Usually, if θ is a constant independent of the dimension n of the problem, for instance, $\theta = \frac{1}{2}$, then we call the algorithm a large-update (or long-step) method. If θ depends on the dimension of the problem, such as $\theta = \frac{1}{\sqrt{n}}$, then the algorithm is called a small-update (or short-step) method. The generic form of the algorithm is shown in Table 1.

In most cases, the best complexity result obtained for small-update IPMs is $O(\sqrt{n}\log\frac{n}{\varepsilon})$. For large-update methods the best obtained bound is

$$O\Big(\sqrt{n}\log n \log\frac{n}{\varepsilon}\Big),$$

which until now has been the best known bound for such methods [2], [9].

In this paper, we define a new kernel function and propose primal-dual interior point methods which improve all the results of the complexity bound for large-update methods based on a logarithmic-exponential kernel function for LO. Another interesting choice is m dependent with n, which minimizes the iteration

Generic Primal-dual IPMs for LO

Input: A proximity the function $\Phi_{\text{New}}(v)$,
a threshold parameter $\tau > 1$
an accuracy parameter 0,
a fixed barrier update parameter $\theta, 0 < \theta < 1$,
begin
$x = \mathrm{e}, \quad s = \mathrm{e}, \quad \mu = 1, \quad v = \mathrm{e}.$
while $n\mu \geq \varepsilon$ do
begin (outer iteration)
$\mu = (1-\theta)\mu$,
while $\Phi(x, s; \mu) > \tau$ do
begin (inner iteration)
solve the system (8), $\Phi(v)$ replaced by $\Phi_{\text{New}}(v)$ to obtain $(\Delta x, \Delta y, \Delta s)$,
choose a suitable step size α,
$x = x + \alpha\Delta x, y = y + \alpha\Delta y, s = s + \alpha\Delta s$
$v = \sqrt{\frac{xs}{\mu}}$,
end (inner iteration)
end (outer iteration)
end.

Table 1: Generic algorithm

complexity bound. In fact, if we take $m = \log n$, we obtain the best known complexity bound for large-update methods namely $O(\sqrt{n}\log(n)\log(\frac{n}{\varepsilon}))$. This bound improves the so far obtained complexity results for large-update methods based on a logarithmic kernel function given by El Ghami et al. [2].

3 The New Kernel Function and Its Properties

This section is devoted to introduce our new kernel function, which are used in the Generic Algorithm 1.

$$\psi(t) = t^2 - 1 - \log(t) + \frac{\mathrm{e}^{m(\frac{1}{t}-1)} - 1}{m}, \quad m \geq 1. \tag{12}$$

It can be easily seen that as $t \to 0^+$ or $t \to +\infty$, then $\psi(t) \to +\infty$. Therefore, $\psi(t)$ is indeed a kernel function. As we need the first three derivatives of $\psi(t)$, we list them here:

$$\psi'(t) = 2t - t^{-1} - t^{-2}\mathrm{e}^{m(\frac{1}{t}-1)}, \tag{13}$$

$$\psi''(t) = 2 + t^{-2} + (m+2t)t^{-4}\mathrm{e}^{m(\frac{1}{t}-1)}, \tag{14}$$

$$\psi'''(t) = -[2t^{-3} + (6t^2 + 6mt + m^2)t^{-6}\mathrm{e}^{m(\frac{1}{t}-1)}]. \tag{15}$$

Kernel function	Large update	References
$\frac{1}{2}(t^2-1)-\log(t)$	$O(n\log\frac{n}{\varepsilon})$	[2]
$\frac{1}{2}\left(t-\frac{1}{t}\right)^2$	$O(n^{\frac{2}{3}}\log\frac{n}{\varepsilon})$	[10]
$\frac{1}{1+p}(t^{1+p}-1)-\log(t), p\in[0,1]$	$O(n\log\frac{n}{\varepsilon})$	[6]
$\frac{1}{2}(t^2-1)+\mathrm{e}^{(\frac{1}{t}-1)}-1$	$O(\sqrt{n}(\log n)^2\log\frac{n}{\varepsilon})$	[2]
$\frac{1}{2}(t^2-1)+\frac{t^{1-q}-1}{q-1}, q>1$	$O(qn^{\frac{q+1}{2q}}\log\frac{n}{\varepsilon})$	[9]
$t-1+\frac{t^{1-q}-1}{q-1}, q>1$	$O(qn\log\frac{n}{\varepsilon})$	[1]
$\frac{1}{2}(t^2-1-\log(t))+\frac{t^{1-q}-1}{2(q-1)}, q>1$	$O(qn^{\frac{q+1}{2q}}\log\frac{n}{\varepsilon})$	[4]
$\frac{p}{2}(t^2-1)+\mathrm{e}^{p(\frac{1}{t}-1)}-1, p\geq 1$	$O(\sqrt{np^5}(\log pn)^2\log\frac{n}{\varepsilon})$	[3]
$t^2-1-\log(t)+\frac{\mathrm{e}^{m(\frac{1}{t}-1)}-1}{m}, m\geq 1$	$O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$	New

Table 2: Examples of kernel functions and its iteration bound for large-update methods.

4 Eligibility of the New Kernel Function

Next lemma serves to prove that the new kernel function (12) is efficient.

Lemma 1. *Let $\psi(t)$ be as defined in (12) and $t>0$. Then,*

$$\psi''(t) > 2, \tag{16}$$
$$\psi'''(t) < 0, \tag{17}$$
$$t\psi''(t)-\psi'(t) > 0, \tag{18}$$
$$t\psi''(t)+\psi'(t) > 0. \tag{19}$$

Proof. It is easy to see that (16) and (17) follow from (14) and (15) respectively. To prove (18) and (19), we have from (13) and (14) the following

$$t\psi''(t)-\psi'(t) = 2t^{-1}+(m+3t)t^{-3}\mathrm{e}^{m(\frac{1}{t}-1)} > 0.$$

and

$$t\psi''(t)+\psi'(t) = 4t+(m+t)t^{-3}\mathrm{e}^{m(\frac{1}{t}-1)} > 0,$$

the right-hand side of the above equality is positive, which proves (19). □

The last property (19) in Lemma 1 is equivalent to convexity of composed functions $t\to\psi(\mathrm{e}^t)$ and this holds if only if

$$\psi(\sqrt{t_1t_2}) \leq \frac{1}{2}(\psi(t_2)+\psi(t_2)), \text{ for any } t_1,t_2\geq 0. \tag{20}$$

This property is known in the literature, and it was demonstrated by several researchers (see [7], [14]).

Lemma 2. *For $\psi(t)$, we have*

$$(t-1)^2 \le \psi(t) \le \frac{1}{4}[\psi'(t)]^2, \quad t>0. \tag{21}$$

$$\psi(t) \le \frac{1}{2}[5+m](t-1)^2, \quad t>1. \tag{22}$$

Proof. For (21), using (16), we have

$$\psi(t) = \int_1^t \int_1^x \psi''(y)\,\mathrm{d}y\,\mathrm{d}x \ge \int_1^t \int_1^x 2\,\mathrm{d}y\,\mathrm{d}x = (t-1)^2.$$

$$\begin{aligned}\psi(t) &= \int_1^t \int_1^x \psi''(y)\,\mathrm{d}y\,\mathrm{d}x \le \int_1^t \int_1^x \frac{1}{2}\psi''(x)\psi''(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{2}\int_1^t \psi''(x)\psi'(x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_1^t \psi'(x)\,\mathrm{d}\psi'(x) = \frac{1}{4}[\psi'(t)]^2.\end{aligned}$$

Since $\psi(1)=\psi'(1)=0$, $\psi'''(t)<0$, $\psi''(1)=5+m$, and by using Taylor's Theorem, we have

$$\begin{aligned}\psi(t) &= \psi(1)+\psi'(1)(t-1)+\frac{1}{2}\psi''(1)(t-1)^2+\frac{1}{6}\psi'''(\xi)(t-1)^3 \\ &\le \frac{1}{2}\psi''(1)(t-1)^2 \\ &= \frac{1}{2}[5+m](t-1)^2,\end{aligned}$$

for some ξ, $1\le\xi\le t$. This completes the proof. □

Now, we analyze the generic algorithm by following the steps presented in [2].

Step 1.

(Steps 1 and 3 in [2].) We derive some bounds for $\sigma(t)$ and $\rho(t)$.

Let $\sigma\colon [0,\infty)\to[1,+\infty)$ be the inverse function of $\psi(t)$ for $t\ge 1$ and $\rho\colon [0,\infty)\to(0,1]$ be the inverse function of $-\frac{1}{2}\psi'(t)$ for all $t\in(0,1]$. Then we have the following lemma.

Proposition 1. *For $\psi(t)$, we have*

$$1+\sqrt{\frac{2s}{5+m}} \le \sigma(s) \le 1+\sqrt{s}\,, \quad s\ge 0. \tag{23}$$

$$\rho(z) \ge \frac{1}{1+\frac{1}{m}\log(2z+1)}\,, \quad z>0. \tag{24}$$

Proof. For (23), let $s = \psi(t)$, $t \geq 1$, i.e., $\sigma(s) = t$, $t \geq 1$.
By (21), we have $s \geq (t-1)^2$, this implies that $t = \sigma(s) \leq 1 + \sqrt{s}$.
By (22), we have:

$$s = \psi(t) \leq \frac{1}{2}(5+m)(t-1)^2, \quad t \geq 1, \text{ so } t = \sigma(s) \geq 1 + \sqrt{\frac{2s}{5+m}}\,.$$

For (24), let $z = -\frac{1}{2}\psi'(t)$, $t \in (0, 1]$. By the definition of $\psi'(t)$, we have:

$$\begin{aligned} 2z &= -2t + t^{-1} + t^{-2}\mathrm{e}^{m(\frac{1}{t}-1)} \\ &\geq -2 + t^{-1} + t^{-2}\mathrm{e}^{m(\frac{1}{t}-1)} \\ &\geq -1 + t^{-2}\mathrm{e}^{m(\frac{1}{t}-1)} \\ &\geq -1 + \mathrm{e}^{m(\frac{1}{t}-1)}. \end{aligned}$$

which implies

$$t = \rho(z) \geq \frac{1}{1 + \frac{1}{m}\log(2z+1)}\,.$$

This completes the proof. □

Step 2.

Derive a lower bound for δ in term of Φ.

Proposition 2. *Let $\delta(v)$ be as defined in* (11)*. Then we have*

$$\delta(v) \geq \sqrt{\Phi(v)}\,. \tag{25}$$

Proof. Using (21), we have

$$\Phi(v) = \sum_{i=1}^{n} \psi(v_i) \leq \sum_{i=1}^{n} \frac{1}{4}[\psi'(v_i)]^2 = \frac{1}{4}\|\nabla\Phi(v)\|^2 = \delta(v)^2,$$

so $\delta(v) \geq \sqrt{\Phi(v)}$. □

5 An Estimation for the Step Size

Step 3.

In this section, we compute a default step size α, we have

$$x_+ = x + \alpha\Delta x, \quad y_+ = y + \alpha\Delta y, \quad s_+ = s + \alpha\Delta s.$$

Using (6), we have

$$\begin{aligned} x_+ &= x\Big(\mathrm{e} + \alpha\frac{\Delta x}{x}\Big) = x\Big(\mathrm{e} + \alpha\frac{d_x}{v}\Big) = \frac{x}{v}(v + \alpha d_x), \\ s_+ &= s\Big(\mathrm{e} + \alpha\frac{\Delta s}{s}\Big) = s\Big(\mathrm{e} + \alpha\frac{d_s}{v}\Big) = \frac{s}{v}(v + \alpha d_s). \end{aligned}$$

So, we have

$$v_+ = \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_s)}\,.$$

Define for $\alpha > 0$, $f(\alpha) = \Phi(v_+) - \Phi(v)$.

Then $f(\alpha)$ is the difference of proximities between a new iterate and a current iterate for fixed μ. By (19), we have

$$\Phi(v_+) = \Phi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}) \leq \frac{1}{2}(\Phi((v + \alpha d_x)) + \Phi((v + \alpha d_s))).$$

Therefore, we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) = \frac{1}{2}(\Phi((v + \alpha d_x)) + \Phi((v + \alpha d_s))) - \Phi(v). \tag{26}$$

Obviously, $f(0) = f_1(0) = 0$. Taking the first two derivatives of $f_1(\alpha)$ with respect to α, we have

$$f'_1(\alpha) = \sum_{i=1}^{n} (\psi'(v_i + \alpha d_{x_i}) d_{x_i} + \psi'(v_i + \alpha d_{s_i}) d_{s_i})\,,$$

$$f''_1(\alpha) = \sum_{i=1}^{n} (\psi''(v_i + \alpha d_{x_i}) d_{x_i}^2 + \psi''(v_i + \alpha d_{s_i}) d_{s_i}^2)\,.$$

Using (6) and (11), we have

$$f'_1(0) = \frac{1}{2}\langle \nabla\Phi(v), (d_x + d_s)\rangle = -\frac{1}{2}\langle \nabla\Phi(v), \nabla\Phi(v)\rangle = -2\delta(v)^2.$$

For convenience, we denote $v_1 = \min(v)$, $\delta = \delta(v)$, $\Phi = \Phi(v)$.

Remark 1. Throughout the paper, we assume that $\tau \geq 1$. Using Lemma 2 and the assumption that $\Phi(v) \geq \tau$, we have $\delta(v) \geq 1$.

From Lemmas 4.1–4.4 in [2], we have the following Lemmas 3–6, because $\psi(t)$ is kernel function and $\psi''(t)$ is monotonically decreasing.

Lemma 3 (Bai et al. [2]). *Let $f_1(\alpha)$ be as defined in (26) and $\delta(v)$ be as defined in (11). Then we have $f''_1(\alpha) \leq 2\delta^2\psi''(v_{\min} - 2\alpha\delta)$. Since $f_1(\alpha)$ is convex, we will have $f'_1(\alpha) \leq 0$ for all α less than or equal to the value where $f_1(\alpha)$ is minimal, and vice versa.*

The previous Lemma leads to the following three Lemmas:

Lemma 4 (Bai et al. [2]). *$f'_1(\alpha) \leq 0$ certainly holds if α satisfies the inequality*

$$\psi'(v_{\min}) - \psi'(v_{\min} - 2\alpha\delta) \leq 2\delta\,. \tag{27}$$

Lemma 5 (Bai et al. [2]). *The largest step size $\overline{\alpha}$ holding (27) is given by*

$$\overline{\alpha} = \frac{\rho(\delta) - \rho(2\delta)}{2\delta}\,.$$

Lemma 6 (Bai et al. [2]). *Let $\overline{\alpha}$ be as defined in Lemma 5. Then*

$$\overline{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}.$$

Now, we are in position to prove the following Lemma

Lemma 7. *Let ρ and $\overline{\alpha}$ be as defined in Lemma 6. If $\Phi(v) \geq \tau \geq 1$, then we have*

$$\overline{\alpha} \geq \frac{1}{2 + [1 + \frac{1}{m}\log(4\sqrt{\Phi(v)} + 1)]^2[1 + (m+2)(4\sqrt{\Phi(v)} + 1)]}.$$

Proof. Using Lemma 6, (14), (24), and (25) we have

$$\begin{aligned}
\overline{\alpha} &\geq \frac{1}{\psi''(\rho(2\delta))} \\
&= \frac{1}{2 + [\rho(2\delta)]^{-2} + (m + 2\rho(2\delta))[\rho(2\delta)]^{-4}\mathrm{e}^{m(\frac{1}{\rho(2\delta)}-1)}} \\
&\geq \frac{1}{2 + [1 + \frac{1}{m}\log(4\delta+1)]^2 + (m+2)[\rho(2\delta)]^{-2}[\rho(2\delta)]^{-2}\mathrm{e}^{m(\frac{1}{\rho(2\delta)}-1)}} \\
&\geq \frac{1}{2 + [1 + \frac{1}{m}\log(4\delta+1)]^2 + (m+2)[1 + \frac{1}{m}\log(4\delta+1)]^2[4\delta+1]} \\
&\geq \frac{1}{2 + [1 + \frac{1}{m}\log(4\delta+1)]^2[1 + (m+2)(4\delta+1)]} \\
&\geq \frac{1}{2 + [1 + \frac{1}{m}\log(4\sqrt{\Phi(v)}+1)]^2[1 + (m+2)(4\sqrt{\Phi(v)}+1)]}
\end{aligned}$$

This completes the proof. □

Denoting

$$\widetilde{\alpha} = \frac{1}{2 + [1 + \frac{1}{m}\log(4\sqrt{\Phi(v)} + 1)]^2[1 + (m+2)(4\sqrt{\Phi(v)} + 1)]}, \tag{28}$$

we have that $\widetilde{\alpha}$ is the default step size and that $\widetilde{\alpha} \leq \overline{\alpha}$.

Step 4.

Finding a positive constants κ and γ.

Lemma 8 (Lemma 4.5 in [2]). *If the step size α satisfies $\alpha \leq \overline{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

Proposition 3. *Let $\Phi_0 \geq \Phi(v) \geq 1$ and let $\widetilde{\alpha}$ be the default step size as defined in (28). Then, we have*

$$f(\widetilde{\alpha}) \leq -\kappa[(\Phi)_0]^{1-\gamma} \tag{29}$$

with $\kappa = \dfrac{1}{18m[1 + \frac{1}{m}\log(4\sqrt{\Phi_0} + 1)]^2}$ *and* $\gamma = \frac{1}{2}$.

Proof. From (25), (28) and by using Lemma 8 (Lemma 4.5 in [2]) with $\alpha = \widetilde{\alpha}$ we have

$$\begin{aligned}
\widetilde{\alpha}\delta^2 &= \frac{\delta^2}{2 + [1 + \frac{1}{m}\log(4\sqrt{\Phi(v)}+1)]^2[1+(m+2)(4\sqrt{\Phi(v)}+1)]}\\
&\geq \frac{\Phi(v)}{2 + [1 + \frac{1}{m}\log(4\sqrt{\Phi(v)}+1)]^2[1+(m+2)(4\sqrt{\Phi(v)}+1)]}\\
&\geq \frac{\Phi(v)}{2\sqrt{\Phi(v)} + [1 + \frac{1}{m}\log(4\sqrt{\Phi(v)}+1)]^2[1+(m+2)5]\sqrt{\Phi(v)}}\\
&\geq \frac{\sqrt{\Phi(v)}}{2 + [1 + \frac{1}{m}\log(4\sqrt{\Phi_0}+1)]^2[1+(m+2)5]}\\
&\geq \frac{\sqrt{\Phi(v)}}{18m[1 + \frac{1}{m}\log(4\sqrt{\Phi_0}+1)]^2}
\end{aligned}$$

This completes the proof. □

Step 5.

Calculate the uniform upper bound $(\Phi)_0$ for $\Phi(v)$.

Lemma 9. *Let $\sigma\colon [0,\infty) \to [1,+\infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$. Then we have*

$$\Phi(\beta v) \leq n\psi\left(\beta\sigma\left(\frac{\Phi(v)}{n}\right)\right), \quad v \in \mathbb{R}^*, \beta \geq 1.$$

Proof. Using (17) and (18), and Lemma 2.4 in [2], we can get the result. This completes the proof. □

Proposition 4. *Let $0 \leq \theta < 1$, $v_+ = \dfrac{v}{\sqrt{1-\theta}}$. If $\Phi(v) \leq \tau$, then we have*

$$\Phi(v_+) \leq \frac{2\sqrt{2\tau n} + 2\tau + \theta n}{(1-\theta)}.$$

Proof. Since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\sigma\left(\frac{\Phi(v)}{n}\right) \geq 1$, then $\frac{\sigma\left(\frac{\Phi(v)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. And for $t \geq 1$, we have $\psi(t) \leq t^2 - 1$.

Using Lemma 9 with $\beta = \frac{1}{\sqrt{1-\theta}}$, (23), and $\Phi(v) \leq \tau$, we have

$$\begin{aligned}
\Phi(v_+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Phi(v)}{n}\right)\right)\\
&\leq n\left(\left(\frac{\sigma\left(\frac{\Phi(v)}{n}\right)}{\sqrt{1-\theta}}\right)^2 - 1\right) = \frac{n}{(1-\theta)}\left(\left(\sigma\left(\frac{\Phi(v)}{n}\right)\right)^2 - (1-\theta)\right)\\
&\leq \frac{n}{(1-\theta)}\left(\left(1+\sqrt{2\frac{\Phi(v)}{n}}\right)^2 - (1-\theta)\right)\\
&\leq \frac{n}{(1-\theta)}\left(2\sqrt{\frac{2\tau}{n}} + \frac{2\tau}{n} + \theta\right) = \frac{2\sqrt{2\tau n} + 2\tau + \theta n}{(1-\theta)}.
\end{aligned}$$

This completes the proof. □

Denote

$$(\Phi)_0 = \frac{2\sqrt{2\tau n} + 2\tau + \theta n}{(1-\theta)} = L(n, \theta, \tau), \tag{30}$$

then $(\Phi)_0$ is an upper bound for $\Phi(v_+)$ during the process of the algorithm.

Step 6.

(An upper bound for the total iteration bound.)

Lemma 10. *Let K be the total number of inner iterations in the outer iteration. Then we have*

$$K \leq 36m\Big[1 + \frac{1}{m}\log\big(4\sqrt{\Phi_0} + 1\big)\Big]^2 (\Phi_0)^{\frac{1}{2}}$$

Proof. By Lemma 1.3.2 in [11], we have:

$$K \leq \frac{[(\Phi)_0]^\gamma}{\kappa\gamma} = 36m\Big[1 + \frac{1}{m}\log\big(4\sqrt{\Phi_0} + 1\big)\Big]^2 (\Phi_0)^{\frac{1}{2}}.$$

This completes the proof. □

The number of outer iterations is bounded above by $\dfrac{\log(\frac{n}{\varepsilon})}{\theta}$ (see [12] Lemma II.17, page 116). By multiplying the number of outer iterations by the number of inner iterations, we get an upper bound for the total number of iterations, namely,

$$36m\Big[1 + \frac{1}{m}\log\big(4\sqrt{\Phi_0} + 1\big)\Big]^2 (\Phi_0)^{\frac{1}{2}} \frac{\log(\frac{n}{\varepsilon})}{\theta}. \tag{31}$$

Step 7.

For large-update methods with $\tau = O(n)$ and $\theta = \Theta(1)$, we get

$$\Phi_0 = O(n)$$

and by choosing $m = \log(n)$ the iteration bound becomes

$$O\Big(\sqrt{n}\log(n)\log\Big(\frac{n}{\varepsilon}\Big)\Big)$$

iterations complexity.

In case of a small-update methods, we have $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$. Substitution of these values into (31) does not give the best possible bound. A better bound is obtained as follows.

By (22), (23) with $\psi(t) \leq \frac{1}{2}[m+5](t-1)^2$, $t > 1$. We have

$$\begin{aligned}
\Phi(v_+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Phi(v)}{n}\right)\right) \\
&\leq \frac{n(m+5)}{2}\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Phi(v)}{n}\right)-1\right)^2 \\
&= \frac{n(m+5)}{2(1-\theta)}\left(\sigma\left(\frac{\Phi(v)}{n}\right)-\sqrt{1-\theta}\right)^2 \\
&\leq \frac{n(m+5)}{2(1-\theta)}\left(1+\sqrt{\frac{\Phi(v)}{n}}-\sqrt{1-\theta}\right)^2 \\
&\leq \frac{(m+5)}{2(1-\theta)}(\theta\sqrt{n}+\sqrt{\tau})^2
\end{aligned}$$

where we also used that $1-\sqrt{1-\theta} = \dfrac{\theta}{1+\theta} \leq \theta$ and $\Phi(v) \leq \tau$, using this upper bound for $(\Phi)_0$, we get

$$\Phi_0 = O(m)$$

and the iteration bound becomes

$$O\left(m^{\frac{3}{2}}\sqrt{n}\log\left(\frac{n}{\varepsilon}\right)\right)$$

iterations complexity.

6 Numerical Results

In this section, we deal with the numerical implementation of this algorithm applied to the large dimension problem. Here we used Iter which means the iterations number produced by the algorithm. The implementation is manipulated in Matlab. Our tolerance is $\varepsilon = 10^{-4}$. For our kernel we take $m = \log(n)$.

Example 1. We consider the following (LO) problem (see [4])

$$n = 2k, \quad A(i,j) = \begin{cases} 0 & \text{if } i \neq j \text{ and } j \neq i+k \\ 1 & \text{if } i = j \text{ or } j = i+k \end{cases}$$

$c(i) = -1$, $c(i+k) = 0$, $b(i) = 2$, and the interior point condition (IPC), $x^0(i) = x^0(i+k) = 1$, $y^0(i) = -2$, $s^0(i) = 1$, $s^0(i+k) = 2$ for $i = 1, \ldots, k$. To prove the effectiveness of our new kernel function ψ and evaluate its effect on the behavior of the algorithm, we conducted comparative numerical tests between it and Elghami's kernel [2], $\psi_{Gh} = \frac{1}{2}(t^2-1)-\log(t)$. We summarize this numerical study in Tables 3, 4 and 5.

Kernel functions	Large update	Outer It.	Inner It.	Time(s)
ψ_{Gh}	$O(n\log(\frac{n}{\varepsilon}))$	5	6219	0.4998
ψ	$O(\sqrt{n}\log(n)\log(\frac{n}{\varepsilon}))$	5	3943	0.4785

Table 3: Comparison for $k = 25$, $n = 50$.

Kernel functions	Large update	Outer It.	Inner It.	Time(s)
ψ_{Gh}	$O(n\log(\frac{n}{\varepsilon}))$	5	11977	2.9003
ψ	$O(\sqrt{n}\log(n)\log(\frac{n}{\varepsilon}))$	5	5830	1.9039

Table 4: Comparison for $k = 50$, $n = 100$.

Kernel functions	Large update	Outer It.	Inner It.	Time(s)
ψ_{Gh}	$O(n\log(\frac{n}{\varepsilon}))$	5	17675	15.4944
ψ	$O(\sqrt{n}\log(n)\log(\frac{n}{\varepsilon}))$	5	7355	8.2340

Table 5: Comparison for $k = 75$, $n = 150$.

7 Conclusion

In this paper, we propose a new double barrier function and primal-dual interior point algorithms for LO and analyze the large-update and small-update versions of the primal-dual interior point algorithm described in Figure 1 that are based on the parameterized kernel function (12) with a logarithmic-exponential barier term. Another interesting choice is m dependent with n, which minimizes the iteration complexity bound. In fact, if we take $m = \log n$, we obtain the best known complexity bound for large-update methods namely $O(\sqrt{n}\log(n)\log(\frac{n}{\varepsilon}))$. This bound improves the so far obtained complexity results for large-update methods based on a logarithmic kernel function given by El Ghami et al. [2].

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